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A Subspace Algorithm for Identifying 2-D CRSD Systems with Deterministic Inputs

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***Abstract*— In this paper, the class of subspace system identification algorithms is used to derive a new identification algorithm for 2-D causal, recursive, and separable-in-denominator $(\mathbf{CRSD})$ state space systems in the Roesser model form. The algorithm take a given deterministic input-output pair of 2-D signals and computes the system order (n) and system parameter matrices $\{A, B, C, D\}$. Since the CRSD model can be treated as two 1-D systems, the proposed algorithm first separates the vertical component from the state and output equations and then formulates an equivalent set of 1-D horizontal subspace equations. The solution to the horizontal subspace identification subproblem contains all the information necessary to compute the system order and parameter matrices, including those from the vertical subsystem.**

I. Introduction

Two-dimensional systems refer to those described by two independent variables, whether they are time/space or space/space. They usually appear, for example, in image processing [10], 2-D control systems [21], iterative learning control [12], in texture synthesis and classification [3], and the discretization of partial differential equations [2]. Discrete 2-D systems can be represented by either transfer function models using the 2-D z-transform, or by using state space models. A number of 2-D state space models can be found in the literature, such as those proposed Roesser [15], Attasi [1], Fornasini and Marchesini [4], and Kurek [9]. It is now accepted that the Roesser model [15] can represent most 2-D causal systems of interest [2], [8], and hence in this paper, this model will be adopted.

It is well recognized that the study of 2-D systems is much more challenging than 1-D systems, as there are many results for 1-D systems that have no counterpart in 2-D systems [6]. However, a subclass of 2-D state space models which has many features similar to 1-D systems is the *causal, recursive, and separable-in-denominator* (CRSD) system. The corresponding transfer function for this model has a denominator that can be factored into two polynomials, and each is a function of only one shift operator. The CRSD Roesser model has the advantage that it can be treated as two 1-D systems. Furthermore, it can approximate a general (i.e., nonseparable) transfer function at the expense of a higher dimensional state space dimension.

Lashgari, et al. [10] showed that a CRSD model can be characterized by two 1-D systems and proposed a method for the approximation of a general 2-D filter by a CRSD model which uses traditional 1-D Hankel-based approximation methods. Xiao, et al. [20] defined the extended impulse response Gramians and used these to identify a 2-D balanced CRSD model from the impulse response data. Similarly, Treasure, et al. [16] defined a pair of horizontal/vertical impulse response Hankel matrices and from these identified a 2-D balanced realization.

In the area of system identification theory, subspace methods [7], [17], [18] have shown to have marked advantages from an implementation point of view. The extension of subspace system identification methods to 2-D state space systems is, therefore, of significant interest. However, due to the coupled structure of the horizontal and vertical states in 2-D systems, the applicability of subspace system identification methods to 2-D systems has been very limited [5], [13]. Ramos [13] was the first to apply the intersection subspace method of [11] to the identification of 2-D CRSD systems from given input-output data. Unlike the algorithms of [16], [20], this algorithm finds a 2-D balanced realization directly from the input-output data. However, one drawback is that only one column of the 2-D output data is used in the identification procedure, which may lead to inaccurate results when using a real data set. That is, some of the vertical dynamics prevalent in the system may not be properly captured. Unlike in [13], the proposed approach uses all the input-output data in the identification process, thus leading to a more mathematically sound algorithm in terms of asymptotic properties. To the authors knowledge, the only known work that has been able to solve the general 2-D state space system identification problem is [19], and it is based on a neural network approach.

In this paper, the problem of subspace system identification of the CRSD model is revisited along the lines of [7], [13], [17], [18], and an algorithm that makes use of all the input-output data is introduced. In this new framework we adopt notions of current subspace system identification methods, which are expected to shed light on solving the general 2-D state space system identification problem. The rest of the paper is organized as follows. In Section 2 the problem is briefly introduced. In Section 3 the 2-D subspace system identification algorithm is presented for the CRSD model. In Section 4 we use the proposed algorithm to identify a 2-D SISO system. Finally, conclusions are drawn in Section 5.

II. Problem Formulation

Consider the 2-D CRSD Roesser model given by

$$x^h_{r+1,s} = A_1 x^h_{r,s} + A_2 x^v_{r,s} + B_1 u_{r,s} \quad (1)$$
$$x^v_{r,s+1} = A_4 x^v_{r,s} + B_2 u_{r,s} \quad (2)$$
$$y_{r,s} = C_1 x^h_{r,s} + C_2 x^v_{r,s} + D u_{r,s}, \quad (3)$$

for $r \in [0, N]$ and $s \in [0, M]$. In the above model the vertical states $(x^v_{r,s})$ are decoupled from the horizontal states $(x^h_{r,s})$, a property that will be exploited in the derivation of the algorithms. The problem is now stated as follows:

Definition 1: 2-D CRSD System Identification Problem: Given a pair of 2-D input-output data signals $\{u_{r,s}, y_{r,s}\}$, for $r \in [0, N]$ and $s \in [0, M]$, find the system order $n = n_h + n_v$ and system matrices $A \in \mathbb{R}^{n\times n}$,$B \in \mathbb{R}^{n\times m}$, $C \in \mathbb{R}^{\ell\times n}$, and $D \in \mathbb{R}^{\ell\times m}$, where

$$\begin{aligned}
A &\triangleq \begin{array}{cc} \left[\begin{array}{c|c} A_1 & A_2 \\ \hline 0_{n_v\times n_h} & A_4 \end{array}\right] & \begin{array}{c} n_h \\ n_v \end{array} \\ \begin{array}{cc} \quad n_h & \quad n_v \end{array} & \end{array} \\
B &\triangleq \begin{array}{cc} \left[\begin{array}{c} B_1 \\ B_2 \end{array}\right] & \begin{array}{c} n_h \\ n_v \end{array} \\ m & \end{array} \\
C &\triangleq \begin{array}{cc} \left[\begin{array}{c|c} C_1 & C_2 \end{array}\right] & \ell \\ \begin{array}{cc} n_h & n_v \end{array} & \end{array} \\
D &\triangleq \begin{array}{cc} \left[\begin{array}{c} D \end{array}\right] & \ell \\ m & \end{array} ,
\end{aligned} \quad (4)$$

with $0_{n_1\times n_2}$ being a zero matrix of size $(n_1 \times n_2)$. Here $u_{r,s} \in \mathbb{R}^m$, $y_{r,s} \in \mathbb{R}^\ell$, $x^h_{r,s} \in \mathbb{R}^{n_h}$, and $x^v_{r,s} \in \mathbb{R}^{n_v}$ are, respectively, the input, output, local horizontal state, and local vertical state vectors at the spatial location $\{r, s\}$ of a finite field $\mathcal{D}$.

A. Horizontal Data Processing

Let us now define the past and future state matrices for $k = 0, 1, \ldots, M$ and $N = 2i + j - 2$ as follows:

$$X^h_p(k) \triangleq \begin{bmatrix} x^h_{0,k} & x^h_{1,k} & \cdots & x^h_{j-1,k} \end{bmatrix} \quad (5)$$
$$X^h_f(k) \triangleq \begin{bmatrix} x^h_{i,k} & x^h_{i+1,k} & \cdots & x^h_{i+j-1,k} \end{bmatrix} \quad (6)$$
$$X^v_p(k) \triangleq \begin{bmatrix} x^v_{0,k} & x^v_{1,k} & \cdots & x^v_{j-1,k} \\ x^v_{1,k} & x^v_{2,k} & \cdots & x^v_{j,k} \\ \vdots & \vdots & \ddots & \vdots \\ x^v_{i-1,k} & x^v_{i,k} & \cdots & x^v_{i+j-2,k} \end{bmatrix} \quad (7)$$
$$X^v_f(k) \triangleq \begin{bmatrix} x^v_{i,k} & x^v_{i+1,k} & \cdots & x^v_{i+j-1,k} \\ x^v_{i+1,k} & x^v_{i+2,k} & \cdots & x^v_{i+j,k} \\ \vdots & \vdots & \ddots & \vdots \\ x^v_{2i-1,k} & x^v_{2i,k} & \cdots & x^v_{2i+j-2,k} \end{bmatrix}, \quad (8)$$

where throughout the sequel, subscripts p and f denote *past* and *future*, respectively, superscripts h and v denote *horizontal* and *vertical*, respectively, and i and j are fixed integer constants such that $j \gg \max(mi, \ell i)$ and $\ell i \gg n_h$. Likewise, we define the past and future input-output data matrices for $k = 0, 1, \ldots, M$ and $N = 2i+j-2$ as follows:

$$U^h_p(k) \triangleq \begin{bmatrix} u_{0,k} & u_{1,k} & \cdots & u_{j-1,k} \\ u_{1,k} & u_{2,k} & \cdots & u_{j,k} \\ \vdots & \vdots & \ddots & \vdots \\ u_{i-1,k} & u_{i,k} & \cdots & u_{i+j-2,k} \end{bmatrix} \quad (9)$$
$$Y^h_p(k) \triangleq \begin{bmatrix} y_{0,k} & y_{1,k} & \cdots & y_{j-1,k} \\ y_{1,k} & y_{2,k} & \cdots & y_{j,k} \\ \vdots & \vdots & \ddots & \vdots \\ y_{i-1,k} & y_{i,k} & \cdots & y_{i+j-2,k} \end{bmatrix} \quad (10)$$

$$U_f^h(k) \triangleq \begin{bmatrix} u_{i,k} & u_{i+1,k} & \cdots & u_{i+j-1,k} \\ u_{i+1,k} & u_{i+2,k} & \cdots & u_{i+j,k} \\ \vdots & \vdots & \ddots & \vdots \\ u_{2i-1,k} & u_{2i,k} & \cdots & u_{2i+j-2,k} \end{bmatrix} \quad (11)$$

$$Y_f^h(k) \triangleq \begin{bmatrix} y_{i,k} & y_{i+1,k} & \cdots & y_{i+j-1,k} \\ y_{i+1,k} & y_{i+2,k} & \cdots & y_{i+j,k} \\ \vdots & \vdots & \ddots & \vdots \\ y_{2i-1,k} & y_{2i,k} & \cdots & y_{2i+j-2,k} \end{bmatrix}. \quad (12)$$

These last four matrices, when evaluated at $k = 0, 1, \ldots, M$, represent the data to be used in the algorithms.

B. Computing the Vertical States

In order to decouple the vertical states from the horizontal system, we need to solve (2) recursively. That is,

$$\begin{aligned} x_{r,s}^v &= A_4 x_{r,s-1}^v + B_2 u_{r,s-1} \\ &= A_4^s x_{r,0}^v + \sum_{i=1}^{s} A_4^{i-1} B_2 u_{r,s-i}. \end{aligned} \quad (13)$$

Since (7) and (8) are Hankel matrices, we need to convert (13) to a Hankel type matrix equation. For this we need to define the complete past and future vertical state and input matrices, along with certain vertical controllability-like matrices. First we need to introduce the following simplifying dimensions $\bar{m} \triangleq m(M+1)$ and $\bar{j} \triangleq j(M+1)$. These will be used extensively from this point forward.

$$\boldsymbol{X}_p^v \triangleq \left[\, X_p^v(0) \mid X_p^v(1) \mid \cdots \mid X_p^v(M) \,\right] \quad (14)$$

$$\boldsymbol{X}_f^v \triangleq \left[\, X_f^v(0) \mid X_f^v(1) \mid \cdots \mid X_f^v(M) \,\right] \quad (15)$$

$$\boldsymbol{U}_p^h \triangleq \begin{bmatrix} U_p^h(0) & U_p^h(1) & \cdots & U_p^h(M) \\ & U_p^h(0) & \cdots & U_p^h(M-1) \\ & & \ddots & \vdots \\ & & & U_p^h(0) \end{bmatrix} \quad (16)$$

$$\boldsymbol{U}_f^h \triangleq \begin{bmatrix} U_f^h(0) & U_f^h(1) & \cdots & U_f^h(M) \\ & U_f^h(0) & \cdots & U_f^h(M-1) \\ & & \ddots & \vdots \\ & & & U_f^h(0) \end{bmatrix} \quad (17)$$

$$\begin{aligned} \mathcal{C}^v &\triangleq \left[\, {}_iB_2 \mid {}_i(A_4B_2) \mid \cdots \mid {}_i(A_4^{M-1}B_2) \,\right] \\ \Delta_{x_p^v(0)} &\triangleq \left[\, X_p^v(0) \mid {}_iA_4X_p^v(0) \mid \cdots \mid {}_iA_4^MX_p^v(0) \,\right] \\ \Delta_{x_f^v(0)} &\triangleq \left[\, X_f^v(0) \mid {}_iA_4X_f^v(0) \mid \cdots \mid {}_iA_4^MX_f^v(0) \,\right], \end{aligned}$$

where ${}_iB_2 = (I_i \otimes B_2)$, ${}_i(A_4^kB_2) = (I_i \otimes A_4^kB_2)$, ${}_iA_4^k = (I_i \otimes A_4^k)$, I_i denotes an $(i \times i)$ identity matrix, and $\otimes$ denotes the Kronecker matrix product. It can now be shown that the vertical states satisfy a pair of Hankel matrix equations similar to (13), i.e., $\boldsymbol{X}_p^v = \Delta_{x_p^v(0)} + \left[\, 0_{n_vi\times mi} \mid \mathcal{C}^v \,\right]\boldsymbol{U}_p^h$ and $\boldsymbol{X}_f^v = \Delta_{x_f^v(0)} + \left[\, 0_{n_vi\times mi} \mid \mathcal{C}^v \,\right]\boldsymbol{U}_f^h$. If we now assume that $X_p^v(0) = X_f^v(0) = 0_{n_vi\times j}$, then $\Delta_{x_p^v(0)} = 0_{n_vi\times\bar{j}}$ and $\Delta_{x_f^v(0)} = 0_{n_vi\times\bar{j}}$. This means we can assign the vertical state dependence through the inputs, thus completely removing any vertical state variable dependence from the horizontal model. This will become apparent in the formation of equations (34) – (36). We then obtain the final expressions for $\boldsymbol{X}_p^v$ and $\boldsymbol{X}_f^v$ as

$$\boldsymbol{X}_p^v = \left[\, 0_{n_vi\times mi} \mid \mathcal{C}^v \,\right]\boldsymbol{U}_p^h \quad (18)$$

$$\boldsymbol{X}_f^v = \left[\, 0_{n_vi\times mi} \mid \mathcal{C}^v \,\right]\boldsymbol{U}_f^h. \quad (19)$$

C. Horizontal Subspace Equations

Since we have found expressions for $\boldsymbol{X}_p^v$ and $\boldsymbol{X}_f^v$ in terms of the input data, for the purpose of horizontal data processing we can work with the equivalent horizontal subsystem

$$x_{r+1,s}^h = A_1x_{r,s}^h + A_2x_{r,s}^v + B_1u_{r,s} \quad (20)$$

$$y_{r,s} = C_1x_{r,s}^h + C_2x_{r,s}^v + Du_{r,s}, \quad (21)$$

for $r = 0, 1, \ldots, N$ and $s = 0, 1, \ldots, M$. Let us now define the complete past and future horizontal state and output data matrices as

$$\boldsymbol{X}_p^h \triangleq \left[\, X_p^h(0) \mid X_p^h(1) \mid \cdots \mid X_p^h(M) \,\right] \quad (22)$$

$$\boldsymbol{X}_f^h \triangleq \left[\, X_f^h(0) \mid X_f^h(1) \mid \cdots \mid X_f^h(M) \,\right] \quad (23)$$

$$\boldsymbol{Y}_p^h \triangleq \left[\, Y_p^h(0) \mid Y_p^h(1) \mid \cdots \mid Y_p^h(M) \,\right] \quad (24)$$

$$\boldsymbol{Y}_f^h \triangleq \left[\, Y_f^h(0) \mid Y_f^h(1) \mid \cdots \mid Y_f^h(M) \,\right]. \quad (25)$$

Likewise, we define the horizontal observability matrix Γ_i^h and controllability-like matrices Φ_i^{hv} and $\mathcal{C}_i^h$ as

$$\Gamma_i^h \triangleq \left[\, C_1^T \mid (C_1A_1)^T \mid \cdots \mid (C_1A_1^{i-1})^T \,\right]^T \quad (26)$$

$$\Phi_i^{hv} \triangleq \left[\, A_1^{i-1}A_2 \mid A_1^{i-2}A_2 \mid \cdots \mid A_2 \,\right] \quad (27)$$

$$\mathcal{C}_i^h \triangleq \left[\, A_1^{i-1}B_1 \mid A_1^{i-2}B_1 \mid \cdots \mid B_1 \,\right]. \quad (28)$$

Finally, we define the lower block triangular Toeplitz matrices G_T^{hv} and H_T^h as

$$G_T^{hv} \triangleq \begin{bmatrix} C_2 & & & \\ C_1A_2 & C_2 & & \\ \vdots & \vdots & \ddots & \\ C_1A_1^{i-2}A_2 & C_1A_1^{i-3}A_2 & \cdots & C_2 \end{bmatrix} \quad (29)$$

$$H_T^h \triangleq \begin{bmatrix} D & & & \\ C_1B_1 & D & & \\ \vdots & \vdots & \ddots & \\ C_1A_1^{i-2}B_1 & C_1A_1^{i-3}B_1 & \cdots & D \end{bmatrix}. \quad (30)$$

One can now show that the following equations are satisfied for $k = 0, 1, \ldots, M$

$$Y_p^h(k) = \Gamma_i^hX_p^h(k) + G_T^{hv}X_p^v(k) + H_T^hU_p^h(k) \quad (31)$$

$$Y_f^h(k) = \Gamma_i^hX_f^h(k) + G_T^{hv}X_f^v(k) + H_T^hU_f^h(k) \quad (32)$$

$$X_f^h(k) = A_1^iX_p^h(k) + \Phi_i^{hv}X_p^v(k) + \mathcal{C}_i^hU_p^h(k). \quad (33)$$

Now using (16), (18), (22), and (24) to write (31) simultaneously for all $k=0,1,\ldots,M$, we obtain

$$\begin{aligned}\boldsymbol{Y}_p^h &= \Gamma_i^h \boldsymbol{X}_p^h + G_T^{hv}\boldsymbol{X}_p^v + H_T^h \boldsymbol{U}_p^h(1:mi,:)\\ &= \Gamma_i^h \boldsymbol{X}_p^h + G_T^{hv}\left[\begin{array}{c|c} 0_{n_v i\times mi} & \mathcal{C}^v\end{array}\right]\boldsymbol{U}_p^h\\ &\quad+\left[\begin{array}{c|c} H_T^h & 0_{\ell i\times Mmi}\end{array}\right]\boldsymbol{U}_p^h\\ &= \Gamma_i^h \boldsymbol{X}_p^h + \underbrace{\left[\begin{array}{c|c} H_T^h & G_T^{hv}\mathcal{C}^v\end{array}\right]}_{H_i^h}\boldsymbol{U}_p^h,\end{aligned}$$

where we used the MATLAB[1] notation $M(1:mi,:)$ to denote the first mi rows of M and all its columns. Operating similarly on $\boldsymbol{Y}_f^h$, this time using (17), (19), (23), and (25) to write (32) simultaneously for all $k=0,1,\ldots,M$, we obtain $\boldsymbol{Y}_f^h=\Gamma_i^h\boldsymbol{X}_f^h+H_i^h\boldsymbol{U}_f^h$. Finally, operating on $\boldsymbol{X}_f^h$, using (16), (18), and (22) to write (33) simultaneously for all $k=0,1,\ldots,M$, we obtain

$$\begin{aligned}\boldsymbol{X}_f^h &= A_1^i \boldsymbol{X}_p^h + \Phi_i^{hv}\boldsymbol{X}_p^v + \mathcal{C}_i^h \boldsymbol{U}_p^h(1:mi,:)\\ &= A_1^i \boldsymbol{X}_p^h + \Phi_i^{hv}\left[\begin{array}{c|c} 0_{n_v i\times mi} & \mathcal{C}^v\end{array}\right]\boldsymbol{U}_p^h\\ &\quad+\left[\begin{array}{c|c} \mathcal{C}_i^h & 0_{n_v i\times Mmi}\end{array}\right]\boldsymbol{U}_p^h\\ &= A_1^i \boldsymbol{X}_p^h + \underbrace{\left[\begin{array}{c|c} \mathcal{C}_i^h & \Phi_i^{hv}\mathcal{C}^v\end{array}\right]}_{\Delta_i^h}\boldsymbol{U}_p^h\\ &= A_1^i \boldsymbol{X}_p^h + \Delta_i^h \boldsymbol{U}_p^h.\end{aligned}$$

From the above results we obtain the following subspace equations in the horizontal direction:

$$\boldsymbol{Y}_p^h = \Gamma_i^h\boldsymbol{X}_p^h + H_i^h\boldsymbol{U}_p^h \tag{34}$$
$$\boldsymbol{Y}_f^h = \Gamma_i^h\boldsymbol{X}_f^h + H_i^h\boldsymbol{U}_f^h \tag{35}$$
$$\boldsymbol{X}_f^h = A_1^i\boldsymbol{X}_p^h + \Delta_i^h\boldsymbol{U}_p^h. \tag{36}$$

D. Assumptions

The 2-D subspace system identification algorithms developed here depend on two fundamental rank conditions from which the horizontal and vertical system orders can be determined. The main assumptions leading to these fundamental rank conditions are now presented.

A1. All system modes are sufficiently excited. That is, $\text{rank}\{\boldsymbol{X}_p^h\}=\text{rank}\{\boldsymbol{X}_f^h\}=n_h$.

A2. The input matrices $\boldsymbol{U}_p^h$ and $\boldsymbol{U}_f^h$ are persistently exciting of order i. That is, $\text{rank}\{\boldsymbol{U}_p^h\}=\text{rank}\{\boldsymbol{U}_f^h\}=\bar{m}i$, where $i>n_h$. Furthermore, the concatenation of both input matrices is persistently exciting of order $2i$. That is, $\text{rank}\left\{\begin{bmatrix}\boldsymbol{U}_p^h\\ \boldsymbol{U}_f^h\end{bmatrix}\right\}=2\bar{m}i$.

A3. There is no linear feedback from the states to the inputs. That is, $\text{span}_{\text{row}}\{\boldsymbol{X}_p^h\}\cap\text{span}_{\text{row}}\{\boldsymbol{U}_p^h\}=\{0\}$ and $\text{span}_{\text{row}}\{\boldsymbol{X}_f^h\}\cap\text{span}_{\text{row}}\{\boldsymbol{U}_f^h\}=\{0\}$, where $\text{span}_{\text{row}}\{M\}$ denotes the row span of M. Because of the fact that $\boldsymbol{X}_f^h$ is a function of $\boldsymbol{U}_p^h$ and $\boldsymbol{X}_p^h$, we also need $\text{span}_{\text{row}}\{\boldsymbol{X}_p^h\}\cap\text{span}_{\text{row}}\{\boldsymbol{U}_f^h\}=\{0\}$, see [7].

A4. A_1 is a stable matrix and $\text{rank}\{\Gamma_i^h\}=n_h$ and $\text{rank}\{\Delta_i^h\}=n_h$.

A5. A_4 is a stable matrix and $\text{rank}\{\Gamma_i^v\}=n_v$ and $\text{rank}\{\mathcal{C}_M^v\}=n_v$, where

$$\Gamma_i^v \triangleq \left[\begin{array}{c|c|c|c} C_2^T & (C_2A_4)^T & \cdots & (C_2A_4^{i-1})^T\end{array}\right]^T \tag{37}$$
$$\mathcal{C}_M^v \triangleq \left[\begin{array}{c|c|c|c} B_2 & A_4B_2 & \cdots & A_4^{M-1}B_2\end{array}\right]. \tag{38}$$

E. Main Rank and Dimension Results

We now informally present the main rank and dimension results that will lead to determining the system order, n_h (see [14] for details).

R1. The rank of $\boldsymbol{W}_p^h=\begin{bmatrix}\boldsymbol{U}_p^h\\ \boldsymbol{Y}_p^h\end{bmatrix}$ and $\boldsymbol{W}_f^h=\begin{bmatrix}\boldsymbol{U}_f^h\\ \boldsymbol{Y}_f^h\end{bmatrix}$ are both equal to $\bar{m}i+n_h$.

R2. The rank of $\boldsymbol{W}^h=\begin{bmatrix}\boldsymbol{W}_p^h\\ \boldsymbol{W}_f^h\end{bmatrix}$ is $2\bar{m}i+n_h$.

R3. $\text{span}_{\text{row}}\{\boldsymbol{X}_f^h\}=\text{span}_{\text{row}}\left\{\boldsymbol{W}_p^h\right\}\cap\text{span}_{\text{row}}\left\{\boldsymbol{W}_f^h\right\}$.

R4. The $\dim\left(\boldsymbol{W}_p^h\cap\boldsymbol{W}_f^h\right)=n_h$.

III. The 2-D N4SID Algorithm for CRSD Models

This algorithm is a 2-D extension of the 1-D time domain algorithm presented in [17], named N4SID, which stands for Numerical algorithms for State Space Subspace System IDentification.

The heart of the N4SID algorithm is an LQ decomposition of a concatenated past/future Hankel data matrix as

$$\begin{bmatrix}\boldsymbol{U}_f^h\\ \boldsymbol{W}_p^h\\ \boldsymbol{Y}_f^h\end{bmatrix}=\underbrace{\begin{bmatrix}\boxed{L_{11}} & \times & \times\\ \boxed{L_{21}} & \boxed{L_{22}} & \times\\ \boxed{L_{31}} & \boxed{L_{32}} & \boxed{L_{33}}\end{bmatrix}}_{L}\underbrace{\begin{bmatrix}\boxed{Q_1^T}\\ \boxed{Q_2^T}\\ \boxed{Q_3^T}\end{bmatrix}}_{Q^T} \tag{39}$$

where L is a square lower block triangular matrix whose block dimensions can be easily obtained by inspection and Q is an orthogonal matrix. We should point out that when the system is deterministic as is the case here, then $L_{33}=0_{\ell i\times\ell i}$ [7].

1) Compute $\boldsymbol{Y}_f^h/_{\boldsymbol{U}_f^h}\boldsymbol{W}_p^h=\mathcal{O}_i^h$, the oblique projection of the row space of $\boldsymbol{Y}_f^h$ along the row space of the future inputs $\boldsymbol{U}_f^h$ on the row space of the past $\boldsymbol{W}_p^h$ input-output data, as defined in [17], i.e.,

$$\begin{aligned}\mathcal{O}_i^h &= \boldsymbol{Y}_f^h/_{\boldsymbol{U}_f^h}\boldsymbol{W}_p^h\\ &= \Gamma_i^h T_h^{-1}\cdot T_h\boldsymbol{X}_f^h\\ &= L_{32}L_{22}^{\dagger}\boldsymbol{W}_p^h.\end{aligned} \tag{40}$$

2) Compute the SVD of $\mathcal{O}_i^h$, obtain the horizontal system order n_h from the nonzero singular values

[1]MATLAB is a trademark of The Mathworks, Inc.

of $\mathcal{O}_i^h$. Compute $\bar{\Gamma}_i^h$, and the parameter matrices $\{\bar{A}_1, \bar{C}_1\}$.

$$\begin{aligned}\mathcal{O}_i^h &= \left[\begin{array}{c|c} U_h & U_h^{\perp}\end{array}\right]\begin{bmatrix} S_h & \times \\ \times & \times\end{bmatrix}\begin{bmatrix} V_h^T \\ (V_h^{\perp})^T\end{bmatrix} \\ &= U_h S_h^{\frac{1}{2}} \cdot S_h^{\frac{1}{2}} V_h^T\end{aligned}$$

where $\bar{\Gamma}_i^h = \Gamma_i^h T_h^{-1} = U_h S_h^{\frac{1}{2}}$, $(\bar{\Gamma}_i^h)^{\perp} = U_h^{\perp}$, n_h is the number of nonzero singular values of $\mathcal{O}_i^h$, and $S_h = \text{diag}\{s_1^h, s_2^h, \ldots, s_{n_h}^h\} \in \mathbb{R}^{n_h \times n_h}$. We can compute $\{\bar{A}_1, \bar{C}_1\}$ from

$$\bar{C}_1 = \text{first } \ell \text{ rows of } \bar{\Gamma}_i^h \tag{41}$$

$$\bar{A}_1 = \left(\bar{\Gamma}_i^h\right)_b^{\dagger} \left(\bar{\Gamma}_i^h\right)_t, \tag{42}$$

where $\left(\bar{\Gamma}_i^h\right)_b$ is the $\bar{\Gamma}_i^h$ matrix with the bottom ℓ rows deleted and $\left(\bar{\Gamma}_i^h\right)_t$ has the top ℓ rows deleted.

3) Compute H_i^h, extract H_T^h, and compute $\{\bar{B}_1, \bar{D}\}$. One can show that H_i^h can be obtained from

$$H_i^h = \left[\begin{array}{c|c} H_T^h & G_T^{hv}\mathcal{C}^v\end{array}\right] = (L_{31} - L_{32}L_{22}^{\dagger}L_{21})L_{11}^{-1},$$

from which we can calculate the individual blocks H_T^h and $G_T^{hv}\mathcal{C}^v$. Now we need the orthogonal complement of $\bar{\Gamma}_i^h$, which was defined to be $(\bar{\Gamma}_i^h)^{\perp} = U_h^{\perp}$. Thus, we can use it to compute

$$\begin{aligned}\mathcal{M} = [\left(\bar{\Gamma}_i^h\right)^{\perp}]^T &\left[\boldsymbol{Y}_f^h - G_T^{hv}\mathcal{C}^v \boldsymbol{U}_f^h(mi+1 : \bar{m}i, :)\right] \\ &\times \left[\boldsymbol{U}_f^h(1 : mi, :)\right]^{\dagger}\end{aligned}$$

and $\mathcal{L} = [\left(\bar{\Gamma}_i^h\right)^{\perp}]^T$, to obtain $\mathcal{M} = \mathcal{L}H_T^h$. Let us further define $\mathcal{L} = \left[\begin{array}{c|c|c|c} \mathcal{L}_0 & \mathcal{L}_1 & \cdots & \mathcal{L}_{i-1}\end{array}\right]$ and $\mathcal{M} = \left[\begin{array}{c|c|c|c} \mathcal{M}_0 & \mathcal{M}_1 & \cdots & \mathcal{M}_{i-1}\end{array}\right]$, where $\mathcal{L}_k \in \mathbb{R}^{(\ell i - n_h)\times \ell}$ and $\mathcal{M}_k \in \mathbb{R}^{(\ell i - n_h)\times m}$, for $k = 0, 1, \ldots, i-1$. Then by enforcing the lower triangular Toeplitz matrix property of H_T^h, one can show that $\mathcal{M} = \mathcal{L}H_T^h$ can be rearranged into [7], [17], [18]

$$\begin{aligned}\begin{bmatrix} \mathcal{M}_0 \\ \mathcal{M}_1 \\ \vdots \\ \mathcal{M}_{i-1}\end{bmatrix} &= \begin{bmatrix} \mathcal{L}_0 & \mathcal{L}_1 & \cdots & \mathcal{L}_{i-2} & \mathcal{L}_{i-1} \\ \mathcal{L}_1 & \mathcal{L}_2 & \cdots & \mathcal{L}_{i-1} & \times \\ \mathcal{L}_2 & \mathcal{L}_3 & \cdots & \times & \times \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \mathcal{L}_{i-1} & \times & \cdots & \times & \times\end{bmatrix} \\ &\times \begin{bmatrix} I_\ell & 0_{\ell\times n_h} \\ 0_{\ell(i-1)\times \ell} & \bar{\Gamma}_{i-1}^h\end{bmatrix} \cdot \begin{bmatrix} \bar{D} \\ \bar{B}_1\end{bmatrix},\end{aligned}$$

where $\bar{\Gamma}_{i-1}^h$ is the $\bar{\Gamma}_i^h$ matrix with the bottom $\bar{C}_1\bar{A}_1^{i-1}$ block deleted. This last equation is linear in $\{\bar{B}_1, \bar{D}\}$ and can be solved in the least squares sense.

4) Extract $G_T^{hv}\mathcal{C}^v$ from H_i^h. Observing the matrix $G_T^{hv}\mathcal{C}^v \in \mathbb{R}^{\ell i \times mMi}$, whose block columns can be defined as $G_T^{hv}\mathcal{C}^v \triangleq \left[\begin{array}{c|c|c|c} G_1 & G_2 & \cdots & G_M\end{array}\right]$, where each $G_k \in \mathbb{R}^{\ell i\times mi}$ block, for $k = 1, 2, \ldots, M$, is a lower triangular Toeplitz matrix. Thus, we can extract the first m columns of each G_k and form the matrix $\boldsymbol{G}$ as,

$$\begin{aligned}\boldsymbol{G} &= \left[\begin{array}{c|c|c} G_1(:, 1:m) & \ldots & G_M(:, 1:m)\end{array}\right] \\ &= \left[\begin{array}{c|c|c|c|c} h_1^v & h_2^v & h_3^v & \cdots & h_M^v \\ \hline \multicolumn{5}{c}{\mathcal{F}_{i-1,M}^{hv}}\end{array}\right] \\ &= \left[\begin{array}{c|c|c|c|c} h_1^v & h_2^v & h_3^v & \cdots & h_M^v \\ \hline \multicolumn{5}{c}{\Gamma_{i-1}^h \cdot A_2 \cdot \mathcal{C}_M^v}\end{array}\right],\end{aligned}$$

where from the first ℓ rows of $\boldsymbol{G}$ we obtain the vertical Markov parameters $h_i^v = C_2 A_4^{i-1} B_2$, for $i = 1, 2, \ldots, M$, Γ_{i-1}^h is the Γ_i^h matrix with the bottom ℓ rows deleted, and $\mathcal{F}_{i-1,M}^{hv} \triangleq \Gamma_{i-1}^h \cdot A_2 \cdot \mathcal{C}_M^v$. We can now use the vertical Markov parameters to construct the vertical Hankel matrix

$$\begin{aligned}\mathcal{H}_{i,M}^v &\triangleq \begin{bmatrix} h_1^v & h_2^v & \cdots & h_{M-i+1}^v & \cdots & h_M^v \\ h_2^v & h_3^v & \cdots & h_{M-i+2}^v & \cdots & \\ h_3^v & h_4^v & \cdots & h_{M-i+3}^v & \cdot\cdot & \\ \vdots & \vdots & \ddots & \vdots & \cdot & \\ h_i^v & h_{i+1}^v & \cdots & h_M^v & & \end{bmatrix} \\ &= \left[\begin{array}{c|c} U_v & U_v^{\perp}\end{array}\right]\begin{bmatrix} S_v & \times \\ \times & \times\end{bmatrix}\begin{bmatrix} V_v^T \\ (V_v^{\perp})^T\end{bmatrix} \\ &= \underbrace{U_v S_v^{\frac{1}{2}}}_{\bar{\Gamma}_i^v} \cdot \underbrace{S_v^{\frac{1}{2}} V_v^T}_{\bar{\mathcal{C}}_M^v},\end{aligned}$$

where the singular value matrix, S_v, contains exactly n_v nonzero singular values, from which the vertical system order n_v can be obtained. Let

$$\bar{\Gamma}_i^v = \begin{bmatrix} \bar{C}_2 \\ \left(\bar{\Gamma}_i^v\right)_b \bar{A}_4\end{bmatrix} \tag{43}$$

$$\bar{\mathcal{C}}_M^v = \left[\begin{array}{c|c|c|c} \bar{B}_2 & \bar{A}_4\bar{B}_2 & \cdots & \bar{A}_4^{M-1}\bar{B}_2\end{array}\right], \tag{44}$$

where $\left(\bar{\Gamma}_i^v\right)_t$ denotes the $\bar{\Gamma}_i^v$ matrix with the first ℓ rows deleted and $\left(\bar{\Gamma}_i^v\right)_b$ denotes the $\bar{\Gamma}_i^v$ matrix with the bottom ℓ rows deleted. The computation of $\{\bar{A}_2, \bar{A}_4, \bar{B}_2, \bar{C}_2\}$ follows from

$$\bar{C}_2 = \text{first } \ell \text{ rows of } \bar{\Gamma}_i^v \tag{45}$$

$$\bar{A}_4 = \left(\bar{\Gamma}_i^v\right)_b^{\dagger} \left(\bar{\Gamma}_i^v\right)_t \tag{46}$$

$$\bar{A}_2 = \left(\bar{\Gamma}_{i-1}^h\right)^{\dagger} \mathcal{F}_{i-1,M}^{hv} \left(\bar{\mathcal{C}}_M^v\right)^{\dagger} \tag{47}$$

$$\bar{B}_2 = \text{first } m \text{ columns of } \bar{\mathcal{C}}_M^v, \tag{48}$$

where $\mathcal{F}_{i-1,M}^{hv}$ is extracted from the bottom $(\ell(i-1)\times mM)$ sub matrix of $\boldsymbol{G}$.

5) End N4SID Algorithm.

IV. Simulation Example

We consider a 2-D CRSD model with true parameters

$$A = \left[\begin{array}{ccc|cc} -0.258 & -0.499 & -0.258 & 0.396 & 0.123 \\ -0.452 & 0.294 & -0.061 & -0.045 & 0.285 \\ -0.073 & 0.082 & -0.085 & 0.339 & -0.061 \\ \hline 0 & 0 & 0 & -0.117 & -0.180 \\ 0 & 0 & 0 & 0.125 & 0.326\end{array}\right]$$

$$B = \left[\begin{array}{ccc|cc} 0.100 & 0.127 & -0.231 & -1.063 & -0.124\end{array}\right]^T$$

$$C = \left[\begin{array}{ccc|cc} 1.457 & 1.713 & 0.682 & 0.315 & 2.570 \end{array}\right]$$

$$D = \left[\begin{array}{c} -0.263 \end{array}\right].$$

The size of the input-output data fields were $N = M = 50$ and the tuning parameters corresponded to $i = 10$ and $j = 32$. Finally, $m = \ell = 1$, the system orders were $n_h = 3$ and $n_v = 2$, thus $n = n_h + n_v = 5$. The size of the Hankel matrix was $\boldsymbol{W}^h \in \mathbb{R}^{1040\times 1632}$. Note also that $j = 32 > i = 10$ and $\ell i = 10 > n_h = 3$. The horizontal system order was determined from rank$\{\mathcal{O}_i^h\} = n_h = 3$. Likewise, the vertical system order was obtained from rank$\{\mathcal{H}_{i,M}^v\} =$ rank$\{\bar{\Gamma}_i^v\} =$ rank$\{\bar{\mathcal{C}}_M^v\} = n_v = 2$. Finally, the identified system matrices for the N4SID algorithm were

$$\bar{A} = \left[\begin{array}{rrr|rr} -0.102 & 0.321 & -0.018 & -1.757 & -0.440 \\ 0.321 & 0.149 & 0.442 & 3.601 & 1.629 \\ -0.018 & 0.442 & -0.096 & -3.064 & -1.039 \\ \hline 0 & 0 & 0 & 0.126 & -0.049 \\ 0 & 0 & 0 & -0.524 & 0.082 \end{array}\right]$$

$$\bar{B} = \left[\begin{array}{ccc|cc} 0.454 & 0.013 & -0.008 & 1.158 & 0.079 \end{array}\right]^T$$

$$\bar{C} = \left[\begin{array}{ccc|cc} 0.454 & 0.013 & -0.008 & -0.602 & 0.525 \end{array}\right]$$

$$\bar{D} = \left[\begin{array}{c} -0.263 \end{array}\right].$$

As a check, we computed the Markov parameters of the system and all agreed with the true values, thus indicating that the N4SID algorithm computed the correct parameters. The eigenvalues of the A and $\bar{A}$ matrices were computed for the N4SID algorithm and are shown in Table 1 below.

Table 1. Eigenvalues of the A and $\bar{A}$ matrices.

k	1	2	3	4	5
$\lambda_k(A)$	0.579	-0.546	-0.083	0.267	-0.058
$\lambda_k(\bar{A})$	0.579	-0.546	-0.083	0.267	-0.058

Further details of this and other algorithms for the 2-D CRSD model can be found in [14].

V. Conclusion

In this paper we have presented a 2-D subspace system identification approach for CRSD models in the Roesser form. The main contribution of the paper is the formulation of the subspace equations utilizing all the data. This served as the basis for introducing the N4SID algorithm, which makes use of all the data available, contrary to the algorithm of [13], which uses only one column of the output data. Although the algorithm of [13] is less computationally intensive than the one proposed here, it will not give accurate results when using real data. The proposed algorithm is formulated in a completely new framework, which is more in line with the subspace family of algorithms available in the literature.

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