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A Data-Driven Online Stability Monitoring Method for Unknown Discrete-Time Nonlinear Systems

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Abstract— **This paper proposes a data-driven stability criterion based on the geometric interpretation of quadratic Lyapunov functions, which can be used for online stability assessment of unknown discrete-time nonlinear systems. The paper shows that the existence of a Quadratic Lyapunov Function can only be guaranteed if the intersection of the positive real space and the convex cone determined by the data set transformed from the measured states with a suitable orthogonal matrix is not empty, which can be numerically determined by solving a max-min problem. The stability judgment can be given according to the sign of the optimized value. The proposed method requires no system model but only the measurements of system states. Numerical examples are given to show the effectiveness of the proposed method.**

I. INTRODUCTION

Stability monitoring is an important subject in stability analysis and has been widely investigated in many engineering fields, such as in chemistry, vibrating structures, and especially electrical power systems [1]–[6]. The current stability monitoring techniques rely in general on a mathematical model of the system, such as transfer functions [3] or state-space equations [5], and can therefore be classified as model-based methods. However, it is reasonable to believe that the utilization of a mathematical model is not inevitably necessary, especially in cases when the model cannot be built analytically and needs to be established with system identification techniques. Since the identified model used for judging stability is obtained from the measured data, the stability of the concerned system should be able to be judged directly from the data set containing system trajectories, jumping over the step of system identification. From this point of view, the task of stability monitoring should be able to be fulfilled in a *data-driven* manner.

The term 'data-driven' is used to characterize the class of methods that use only measured data of the target system to solve system analysis and control problems [7]. Unfortunately, most discussions towards stability in the data-driven context concentrate on showing certain stability conditions for a specific data-driven control strategy [8]–[13], which can hardly be extended to assess the stability of an arbitrary dynamical systems. The representatives of other data-driven stability analysis approaches not linked to a specified controller design may include the data-space-based stability criterion in the form of a linear matrix inequality [14], the data-based stability test of assessing transient instability which is analog to the Nyquist criterion in frequency domain [15], and the stability condition formulated in terms of the H_∞-norm of a particular error function [16], all of which are suitable for stability judgment of LTI systems using measured data. Nevertheless, it is not shown that these methods can also be applied to real-time implementation and nonlinear systems. As a consequence, the known data-driven stability analysis methods cannot be directly used for stability monitoring of nonlinear systems.

Motivated by these concerns, this paper focuses on establishing a data-driven online stability monitoring method suitable for unknown discrete-time nonlinear systems. A necessary and sufficient condition for determining the existence of a Quadratic Lyapunov Function (QLF) for the currently measured system trajectory is proposed based on the geometrical links of QLFs with convex cones. The proposed stability condition shows that the existence of a QLF can only be guaranteed if the observed system trajectory can be mapped with one certain orthogonal matrix at every time instant into a negative halfspace, which is equivalent to the fact that the corresponding polar cone of the mapped data has a non-empty intersection with the positive real space. The geometric problem can be transformed into a max-min optimization problem according to computational geometry theory [17]. Correspondingly, the data-driven online stability monitoring can be realized by solving this optimization problem at every time instant. Considering its independence on mathematical models, the proposed data-driven method possesses the advantage of flexibility in dealing with the cases when the system models are hard to be identified, compared to model-based approaches.

The paper is organized as follows: at first, the problem definition of online stability monitoring is introduced in section II; secondly, the main results of this contribution, i.e., the stability condition and its numerical algorithm for implementation, are presented in section III; after that, in section IV numerical examples of the proposed method in monitoring the stability of unknown nonlinear systems are presented; finally, a conclusion of this contribution is given in the last section.

II. PROBLEM FORMULATION

The autonomous discrete-time nonlinear system concerned in this paper has the form of

$$\boldsymbol{x}\left(k+1\right)=\boldsymbol{f}(\boldsymbol{x}\left(k\right)), \qquad (1)$$

with $\boldsymbol{f}(\cdot):\Omega\rightarrow\mathbb{R}^n$ a mapping from a compact set $\Omega\subset\mathbb{R}^n$ into $\mathbb{R}^n$, and with the system state vector $\boldsymbol{x}$ belonging to the

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region Ω.

Following the definition in [18], the quadratic stability for such systems can be stated as follows:

Definition 1 (Quadratic Stability): The discrete-time nonlinear system (1) is said to be quadratic stable if there exists a positive definite Hermitian matrix $\boldsymbol{P}$ such that the first-order difference of the function $V(\boldsymbol{x}(k)) = \boldsymbol{x}(k)^T \boldsymbol{P} \boldsymbol{x}(k)$ along the solution of system (1) satisfies

$$\begin{aligned} \Delta V(\boldsymbol{x}(k)) &= V(\boldsymbol{x}(k+1)) - V(\boldsymbol{x}(k)) \\ &= V(f(\boldsymbol{x}(k))) - V(\boldsymbol{x}(k)) < 0. \end{aligned} \quad (2)$$

Correspondingly, the function $V(\boldsymbol{x}(k)) = \boldsymbol{x}(k)^T \boldsymbol{P} \boldsymbol{x}(k)$ is defined as the Quadratic Lyapunov Function (QLF). If in addition $\boldsymbol{P}$ is diagonal, $V(\boldsymbol{x}(k))$ is defined the Diagonal Quadratic Lyapunov Function (DQLF) and the system (1) is diagonally quadratic stable.

In the data-driven context, the existence of a QLF cannot be determined by using the analytical form of $\boldsymbol{f}(\boldsymbol{x})$ because it is unknown. Suppose that the system (1) be fully observable and the system states be measured without noise. At the time instant $t = r$, the data set containing r consecutive measurements of system states can be denoted as

$$\mathcal{X}_r = \{\boldsymbol{x}(1), \ldots, \boldsymbol{x}(r)\} . \quad (3)$$

In this paper, the task of online stability monitoring is defined as to determine the existence of a QLF directly from the data set (3) instead of a mathematical description of $\boldsymbol{f}(\boldsymbol{x})$ at every time instant. The system is judged as stable if and only if a QLF can be found based on the measured data.

It may be argued that a finite data-set is merely a subset of the whole data space of the concerned system and the stability judgment merely based on part of the data-space seems not convincing. However, in stability monitoring only the currently running motion of the concerned system is taken into consideration, rather than all the possible motions of the system. This is exactly analogous to the commonly used model-based stability monitoring methods which build a linearized model of the concerned system with the measured data and then make stability judgment based on the identified model [2], where the stability judgment is necessarily local but still of great practical interest [19].

III. THE MAIN RESULTS

A. Necessary and Sufficient Condition for DQLF

Consider the nonlinear discrete-time system (1) with an equilibrium point at $\boldsymbol{x} = \boldsymbol{0}$. Define a transformation for every vector $\boldsymbol{x}(k) \in \Omega$ as

$$\boldsymbol{v}(k) = \boldsymbol{x}(k+1) \odot \boldsymbol{x}(k+1) - \boldsymbol{x}(k) \odot \boldsymbol{x}(k) , \quad (4)$$

where $\boldsymbol{v}(k)$ represents the corresponding transformed vector and the symbol $\odot$ represents an array multiplication defined by

$$\boldsymbol{x}(k) \odot \boldsymbol{x}(k) = [x_j^2(k)],\ j = 1, \ldots, n. \quad (5)$$

Denote the complete vector set of $\boldsymbol{v}(k)$, $k = 1, \ldots, \infty$, as $\mathcal{V}$, and the convex conic hull (the smallest convex cone) determined by $\mathcal{V}$ as $\mathbf{cone}\,\mathcal{V}$. Define the vector set whose elements have nonnegative inner products with one certain vector located in the positive real space $\mathbb{R}^n$ as the negative halfspace, which is denoted as $\mathcal{H}^-$ and satisfies $\mathbb{R}^n_- \subseteq \mathcal{H}^-$. The necessary and sufficient condition of existence of a DQLF can be given as the following theorem:

Theorem 1: There exists a DQLF $V_d(\boldsymbol{x})$ for the considered nonlinear discrete-time system (1) within the domain Ω if and only if for all the $\boldsymbol{x} \in \Omega$, the convex conic hull $\mathbf{cone}\,\mathcal{V}$ of the transformed vectors $\boldsymbol{v}(k)$ is located in a negative halfspace $\mathcal{H}^-$ of $\mathbb{R}^n$.

Proof:To prove sufficiency, suppose the convex conic hull $\mathbf{cone}\,\mathcal{V}$ lie in a negative halfspace $\mathcal{H}^-$ of $\mathbb{R}^n$. Obviously all the vectors $\boldsymbol{v}$ within the set $\mathbf{cone}\,\mathcal{V}$ also belong to $\mathcal{H}^-$, because $\mathbf{cone}\,\mathcal{V} \subset \mathcal{H}^- \subset \mathbb{R}^n$.

Thus, according to the definition of the negative halfspace, there must exist at least one vector located in $\mathbb{R}^n_+$, denoted as $\boldsymbol{d}$ and $\boldsymbol{d} \in \mathbb{R}^n_+$, which has negative inner products with any vector $\boldsymbol{v}(k)$ belonging to $\mathbf{cone}\,\mathcal{V}$, i.e.,

$$\langle \boldsymbol{v}(k), \boldsymbol{d} \rangle = \boldsymbol{v}(k)^T \boldsymbol{d} = \boldsymbol{d}^T \boldsymbol{v}(k) \ < \ 0,\ \boldsymbol{d} \in \mathbb{R}^n_+. \quad (6)$$

Using the definition of $\boldsymbol{v}(k)$ in (4), the inner product between $\boldsymbol{v}(k)$ and $\boldsymbol{d}$ can be represented as

$$\langle \boldsymbol{v}(k), \boldsymbol{d} \rangle = \boldsymbol{d}^T \left(\boldsymbol{x}(k+1) \odot \boldsymbol{x}(k+1) - \boldsymbol{x}(k) \odot \boldsymbol{x}(k)\right). \quad (7)$$

Define a diagonal matrix $\boldsymbol{D}$ as $\boldsymbol{D} = \mathbf{diag}\,[\boldsymbol{d}]$. Obviously $\boldsymbol{D}$ is positive definite because it is diagonal and its diagonal elements vector $\boldsymbol{d}$ belongs to $\mathbb{R}^n_+$. With notation that

$$\begin{aligned} \boldsymbol{x}(k+1) \odot \boldsymbol{x}(k+1) &= \mathbf{diag}\,[\boldsymbol{x}(k+1)]\boldsymbol{x}(k+1) , \\ \boldsymbol{d}^T \mathbf{diag}[\boldsymbol{x}(k+1)] &= \boldsymbol{x}(k+1)^T\, \mathbf{diag}[\boldsymbol{d}] , \end{aligned} \quad (8)$$

and the similar relations for $\boldsymbol{d}$ and $\boldsymbol{x}(k)$, one can obtain the following equation by substituting (7) and (8) into the inequality (6), as

$$\begin{aligned} \langle \boldsymbol{v}(k), \boldsymbol{d} \rangle &= \boldsymbol{x}(k+1)^T\, \mathbf{diag}\,[\boldsymbol{d}]\, \boldsymbol{x}(k+1) \\ &\quad - \boldsymbol{x}(k)^T\, \mathbf{diag}\,[\boldsymbol{d}]\, \boldsymbol{x}(k) \\ &= \boldsymbol{x}(k+1)^T\, \boldsymbol{D}\, \boldsymbol{x}(k+1) \\ &\quad - \boldsymbol{x}(k)^T\, \boldsymbol{D}\, \boldsymbol{x}(k) < 0 . \end{aligned} \quad (9)$$

According to the definition of DQLF it can be seen that the function $V_d(\boldsymbol{x}(k)) = \boldsymbol{x}(k)^T \boldsymbol{D}\, \boldsymbol{x}(k)$ is a DQLF for the concerned nonlinear discrete-time system, because $\boldsymbol{D}$ is a diagonal positive definite matrix and $\Delta V_d(\boldsymbol{x}(k)) = \boldsymbol{x}(k+1)^T \boldsymbol{D}\, \boldsymbol{x}(k+1) - \boldsymbol{x}(k)^T \boldsymbol{D}\, \boldsymbol{x}(k) < 0$. This proves the sufficiency of the proposed theorem.

To prove the necessity, suppose there exist a diagonal quadratic Lyapunov function within the domain Ω, denoted as $V_d(\boldsymbol{x}(k)) = \boldsymbol{x}(k)^T \hat{\boldsymbol{D}}\, \boldsymbol{x}(k)$ with $\hat{\boldsymbol{D}} = \mathbf{diag}\,[\hat{\boldsymbol{d}}]$ and $\hat{\boldsymbol{d}} \in \mathbb{R}^n_+$. Because $\boldsymbol{x}(k+1)^T \hat{\boldsymbol{D}}\, \boldsymbol{x}(k+1) = \hat{\boldsymbol{d}}^T \mathbf{diag}\,[\boldsymbol{x}(k+1)]\boldsymbol{x}(k+1)$ and $\boldsymbol{x}(k)^T \hat{\boldsymbol{D}}\, \boldsymbol{x}(k) = \hat{\boldsymbol{d}}^T \mathbf{diag}\,[\boldsymbol{x}(k)]\boldsymbol{x}(k)$, the difference of $V_d(\boldsymbol{x}(k))$ can be expressed as

$$\begin{aligned}\Delta V_d(\boldsymbol{x}(k)) &= \boldsymbol{x}(k+1)^T\,\tilde{\boldsymbol{D}}\,\boldsymbol{x}(k+1)\\ &\quad -\boldsymbol{x}(k)^T\,\tilde{\boldsymbol{D}}\,\boldsymbol{x}(k)\\ &= \hat{\boldsymbol{d}}^T\left(\mathbf{diag}\,[\boldsymbol{x}(k+1)]\boldsymbol{x}(k+1)\right.\\ &\quad \left.-\mathbf{diag}\,[\boldsymbol{x}(k)]\boldsymbol{x}(k)\right)\\ &= \hat{\boldsymbol{d}}^T\boldsymbol{v}(k) < 0, \end{aligned} \tag{10}$$

with $\boldsymbol{x}(k+1)\odot\boldsymbol{x}(k+1) = \mathbf{diag}\,[\boldsymbol{x}(k+1)]\boldsymbol{x}(k+1)$ and $\boldsymbol{x}(k)\odot\boldsymbol{x}(k) = \mathbf{diag}\,[\boldsymbol{x}(k)]\boldsymbol{x}(k)$.

Equation (10) shows that the inner product of the vector $\boldsymbol{v}(k)$ with a vector $\hat{\boldsymbol{d}}$, $\hat{\boldsymbol{d}}\in\mathbb{R}^n_+$, is always less than zero. Therefore, all the transformed vectors $\boldsymbol{v}(k)$ are located within one negative halfspace. Denoting this negative halfspace as $\mathcal{H}^-_{\hat{d}}$, it can be concluded that the convex conic hull $\mathbf{cone}\,\mathcal{V}\subset\mathcal{H}^-_{\hat{d}}$. This completes the proof to the theorem. ■

B. Necessary and Sufficient Condition for QLF

Assume the discrete-time system $\boldsymbol{x}\,(k+1) = \boldsymbol{f}(\boldsymbol{x}\,(k))$ is quadratically stable within the domain Ω and there exists a QLF $V(\boldsymbol{x}) = \boldsymbol{x}\,(k)^T\,\boldsymbol{P}\,\boldsymbol{x}\,(k)$ proving its quadratic stability. Because the matrix $\boldsymbol{P}$ of a QLF is a positive definite matrix, there exists an orthogonal matrix $\boldsymbol{\Phi}$ such that $\boldsymbol{P}$ can be transformed into the diagonal form with positive diagonal elements, as

$$\boldsymbol{D} = \boldsymbol{\Phi}\,\boldsymbol{P}\,\boldsymbol{\Phi}^T, \tag{11}$$

where $\boldsymbol{D}$ is a diagonal matrix defined as $\boldsymbol{D} = \mathbf{diag}[\boldsymbol{d}]$, $\boldsymbol{d}\in\mathbb{R}^n_+$.

By left multiplying an orthogonal matrix $\boldsymbol{\Phi}$ to the both side of (1), the concerned discrete-time system can be transformed as

$$\boldsymbol{z}\,(k+1) = g(\boldsymbol{z}\,(k)), \tag{12}$$

with $\boldsymbol{z}\,(k) = \boldsymbol{\Phi}\,\boldsymbol{x}\,(k)$ and $g(\boldsymbol{z}\,(k)) = \boldsymbol{\Phi}\,f(\boldsymbol{x}\,(k))$. As it is well-known that the orthogonal transformation preserves stability characteristics, the transformed system (12) is also quadratic stable and this conclusion is stated in the following lemma.

Lemma 1: If the nonlinear discrete-time system (1) has a QLF $V(x\,(k)) = x\,(k)^T\,\boldsymbol{P}\,x\,(k)$, then there exists an orthogonal matrix $\boldsymbol{\Phi}$ such that the transformed system (12) possesses a DQLF as $V_d(\boldsymbol{z}\,(k)) = \boldsymbol{z}\,(k)^T\,\boldsymbol{D}\,\boldsymbol{z}\,(k)$, where $\boldsymbol{D} = \boldsymbol{\Phi}\,\boldsymbol{P}\,\boldsymbol{\Phi}^T$ and $\boldsymbol{z}\,(k) = \boldsymbol{\Phi}\,\boldsymbol{x}\,(k)$.

Lemma 1 shows that if there exists a DQLF for system (12) that is transformed from (1) with an orthogonal matrix, the system (1) also owns a QLF, and vice versa.

In accordance with the topological structure of QLF given in [20], the diagonal vector $\boldsymbol{d}$ of $\boldsymbol{D}$ in (11) belongs to the conventional topology of $\mathbb{R}^n_+$, the orthogonal matrix $\boldsymbol{\Phi}$ belongs to the special orthogonal group $SO(n,\mathbb{R})$, and the matrix $\boldsymbol{P}$ can be obtained from the following mapping as

$$\boldsymbol{P} = \{\boldsymbol{\Phi}^T\boldsymbol{D}\boldsymbol{\Phi},\ \boldsymbol{D}{=}\mathbf{diag}[\boldsymbol{d}],\ \boldsymbol{d}\in\mathbb{R}^n_+,\ \boldsymbol{\Phi}\in SO(n,\mathbb{R})\}. \tag{13}$$

The mapping (13) is proven in [20] to be a surjective mapping, which indicates that determining the existence of a QLF for a discrete-time system is equivalent to determining the existence of a DQLF for the concerned system using one certain orthogonal transformation. Therefore, considering Lemma 1, it can be concluded that the existence of a QLF for system (1) can be determined by examining the existence of a DQLF for system (12) transformed by any orthogonal matrix $\boldsymbol{\Phi}$ belonging to the special orthogonal group $SO(n,\mathbb{R})$.

Based on this fact, theorem 1 can be extended to the sufficient and necessary condition for the existence of a QLF, which is the theoretical fundamental of this paper and stated in the following theorem.

Theorem 2: Consider the nonlinear discrete-time system (1) with an equilibrium point at $\boldsymbol{x} = \boldsymbol{0}$. For every vector $\boldsymbol{x}(k)\in\Omega$, a new vector $\tilde{\boldsymbol{v}}(k)$ can be generated using the following calculation

$$\tilde{\boldsymbol{v}}(k) = \tilde{\boldsymbol{x}}(k+1)\odot\tilde{\boldsymbol{x}}(k+1) - \tilde{\boldsymbol{x}}(k)\odot\tilde{\boldsymbol{x}}(k)\,, \tag{14}$$

where $\tilde{\boldsymbol{x}}(k) = \boldsymbol{\Phi}\boldsymbol{x}(k)$ and $\boldsymbol{\Phi}$ is an orthogonal matrix. Let $\tilde{\mathcal{V}}$ represent the complete vector set of $\tilde{\boldsymbol{v}}(k)$, $k = 1\,,\,\ldots\,,\,\infty$, and the symbol $\mathbf{cone}\,\tilde{\mathcal{V}}$ represent the convex conic hull (the smallest convex cone) for $\tilde{\mathcal{V}}$. There exists a QLF for system (1), if and only if there exists at least one orthogonal matrix $\boldsymbol{\Phi}$ such that $\mathbf{cone}\,\tilde{\mathcal{V}}$ is located in a negative halfspace $\mathcal{H}^-$ of R^n.

Theorem 2 does not require explicitly an analytical form of the nonlinear function $\boldsymbol{f}(\cdot)$ in system (1), but the complete time histories of system states. This fact makes it possible to apply the above theorem in the data-driven context to judge quadratic stability.

C. Criterion Used for Data-Driven Stability Monitoring

At the time instant $t = r$, every system vector $\boldsymbol{x}(k)\in\mathbb{R}^n$, $k = 1,\,\ldots\,,r-1$, in the data set $\mathcal{X}_r$ can be transformed with one certain orthogonal matrix $\boldsymbol{\Phi}$ into a new vector $\tilde{\boldsymbol{v}}(k)$ according to the mapping defined in (14). Correspondingly, all the elements in the data set $\mathcal{X}_r$ mapped by (14) forms a new vector set $\tilde{\mathcal{V}}_{r-1} = \{\tilde{\boldsymbol{v}}(k)\}$, $k = 1,\,\ldots\,,r-1$.

It should be noted that the vector set $\tilde{\mathcal{V}}_{r-1}$ is different from the set $\tilde{\mathcal{V}}$ in theorem 2. The reason is that the set $\tilde{\mathcal{V}}$ is obtained by transforming *all* the states vectors $x(k)$ within Ω, but due to the finiteness of the measured data, the vector set $\mathcal{X}_r$ is only a subset of Ω where the nonlinear mapping $\boldsymbol{f}(\cdot)$ is defined. Thus $\tilde{\mathcal{V}}_{r-1} = \tilde{\mathcal{V}}$ is true only if $r\to\infty$.

According to theorem 2, the concerned system is quadratic stable if and only if there exist a suitable orthogonal matrix $\boldsymbol{\Phi}$ and a vector $\boldsymbol{d}\in\mathbb{R}^n_+$ so that for all the transformed vectors in the data set $\tilde{\mathcal{V}}_{r-1}$, $r\to\infty$, the following condition holds

$$\langle\boldsymbol{d},\,\tilde{\boldsymbol{v}}(k)\rangle < 0,\ \boldsymbol{d}\in\mathbb{R}^n_+,\ k = 1,\,\ldots\,,r-1,\text{and}\,r\to\infty. \tag{15}$$

On the other hand, the polar cone of $\mathbf{cone}\,\tilde{\mathcal{V}}_{r-1}$, denoted as $\mathbf{cone}\,\tilde{\mathcal{V}}^o_{r-1}$, can be represented as

$$\mathbf{cone}\,\tilde{\mathcal{V}}^o_{r-1} = \{\boldsymbol{y}|\boldsymbol{v}^T\boldsymbol{y}\le 0, \boldsymbol{v}\in\mathbf{cone}\,\tilde{\mathcal{V}}_{r-1},\ \boldsymbol{y}\in\mathbb{R}^n\}. \tag{16}$$

By comparing (16) with (15), it can be found that the inequalities in (15) are identical to the definition of $\mathbf{cone}\,\tilde{\mathcal{V}}^o_{r-1}$ except that $\boldsymbol{d}$ is defined within $\mathbb{R}^n_+$, while $\boldsymbol{y}\in\mathbb{R}^n$. In fact, because $\boldsymbol{d}$ is an arbitrary vector located in $\mathbb{R}^n_+$ and $\mathbb{R}^n_+$ is also a polyhedral cone, the geometrical meaning of (15) can be interpreted as the intersection of two polyhedral cones: one is the polar cone $\mathbf{cone}\,\tilde{\mathcal{V}}^o_{r-1}$; the other is the positive real space $\mathbb{R}^n_+$. Based on this fact, the quadratic stability condition can be given for the data-driven context into theorem 3.

Theorem 3: The nonlinear discrete-time system (1) is quadratic stable if and only if there exists an orthogonal matrix $\mathbf{\Phi}$ such that at every time instant $t = r$, $r \to \infty$, the polyhedral cone $\mathbf{cone}\,\tilde{\mathcal{V}}^o_{r-1}$ constructed by the data set $\mathcal{X}_r$ and the matrix $\mathbf{\Phi}$ follows the relationship

$$\mathbf{cone}\,\tilde{\mathcal{V}}^o_{r-1} \cap \mathbb{R}^n_+ \neq \varnothing\,. \tag{17}$$

Theorem 3 states that if the intersection between the set $\mathbf{cone}\,\tilde{\mathcal{V}}^o_{r-1}$ and $\mathbb{R}^n_+$ is not empty at every time instant, the system is quadratic stable. Furthermore, letting $\tilde{\boldsymbol{d}}$ be any vector located within $\mathbf{cone}\,\tilde{\mathcal{V}}^o_{r-1} \cap \mathbb{R}^n_+$, $r = \infty$, the QLF for this system can be expressed as

$$V(\boldsymbol{x}) = \boldsymbol{x^T}\mathbf{\Phi}\,\mathbf{diag}\,[\tilde{\boldsymbol{d}}]\,\mathbf{\Phi}^T\boldsymbol{x}\,. \tag{18}$$

The condition (17) must be satisfied at every time instant for a quadratic stable system. Thus, to give a correct stability judgment, this criterion has to be implemented online and checked at every time instant. If at some time instant this condition is not satisfied, the observed motion is judged as not quadratic stable.

It should be pointed out that the initial time of measurements is irrelevant to the final result of the stability judgment. By examining (16), it can be seen that $\mathbf{cone}\,\tilde{\mathcal{V}}^o_{r-1}$ is a convex cone by adding an inequality constraints $-\tilde{\boldsymbol{v}}(r-1)^T\,\boldsymbol{d} > 0$ to $\mathbf{cone}\,\tilde{\mathcal{V}}^o_{r-2}$, which implies that

$$\mathbf{cone}\,\tilde{\mathcal{V}}^o_{r-1} = \bigcap_{l=1}^{r-1}\mathbf{cone}\,\tilde{\mathcal{V}}^o_{l}\,. \tag{19}$$

Equation (19) indicates that if $\mathbf{cone}\,\tilde{\mathcal{V}}^o_{r-1} \cap \mathbb{R}^n_+ \neq \varnothing$ at $t = r$, the intersections between $\mathbb{R}^n_+$ and any of the cones $\mathbf{cone}\,\tilde{\mathcal{V}}^o_{l}$ formulated at former time instants $1 \leq l < r-1$, is inherently nonempty, which proves the irrelevance of initial judging time to the final result.

Theorem 3 indicates that the task of online stability monitoring is identical to determine whether there exists an orthogonal matrix $\mathbf{\Phi}$ so that the stability condition (17) can be satisfied at every time instant. If such a matrix exists, the observed motion of the concerned system is stable at the present time; and vice versa. In the next subsection, it is shown that this task can be transformed into an max-min problem.

D. Algorithm for Implementation

Suppose at first that an orthogonal matrix $\mathbf{\Phi}$ be available. As stated before, the original data set $\mathcal{X}_r$ can be transformed with $\mathbf{\Phi}$ into a new set $\tilde{\mathcal{V}}_{r-1}$ and the corresponding polar cone $\mathbf{cone}\,\tilde{\mathcal{V}}^o_{r-1}$ can be determined. The objective now is to examine whether the condition (17) can be satisfied.

The positive real space $\mathbb{R}^n_+$ in (17) can be represented by using the unity matrix $\boldsymbol{I} \in \mathbb{R}^{n\times n}$ as

$$\mathbb{R}^n_+ = \mathbf{cone}(\boldsymbol{I}) = \{\boldsymbol{y}\,|\,\boldsymbol{I}\,\boldsymbol{y} > \mathbf{0},\ \boldsymbol{y} \in \mathbb{R}^n\}. \tag{20}$$

In order to reduce the data scale, the polar cone $\mathbf{cone}\,\tilde{\mathcal{V}}^o_{r-1}$ is represented by the extreme rays of $\mathbf{cone}\,\tilde{\mathcal{V}}_{r-1}$. Suppose that the cone $\mathbf{cone}\,\tilde{\mathcal{V}}_{r-1}$ have p extreme rays, which are denoted as $\{\tilde{\boldsymbol{v}}^{\mathrm{X}}_1,\, \tilde{\boldsymbol{v}}^{\mathrm{X}}_2,\, \dots,\, \tilde{\boldsymbol{v}}^{\mathrm{X}}_p\}$. By using the matrix $\tilde{\boldsymbol{V}}_{r-1} \in \mathbb{R}^{p\times n}$ defined as $\tilde{\boldsymbol{V}}_{r-1} = [-\tilde{\boldsymbol{v}}^{\mathrm{X}}_i]^T$, $i = 1, \dots p$, the polar cone $\mathbf{cone}\,\tilde{\mathcal{V}}^o_{r-1}$ can also be reformulated as $\mathbf{cone}(\tilde{\boldsymbol{V}}_{r-1})$ in the form of matrix inequalities, i.e.

$$\begin{aligned}\mathbf{cone}\,\tilde{\mathcal{V}}^o_{r-1} &= \mathbf{cone}(\tilde{\boldsymbol{V}}_{r-1}) \\ &= \{\boldsymbol{y}\,|\,\tilde{\boldsymbol{V}}_{r-1}\,\boldsymbol{y} > \mathbf{0},\ \boldsymbol{y} \in \mathbb{R}^n\}\,.\end{aligned} \tag{21}$$

Because both $\mathbb{R}^n_+$ and $\mathbf{cone}\,\tilde{\mathcal{V}}^o_{r-1}$ are polyhedral cones, their intersection is also a polyhedral cone, which can be represented by taking advantage of the new forms of $\mathbb{R}^n_+$ and $\mathbf{cone}\,\tilde{\mathcal{V}}^o_{r-1}$ defined in (20) and (21). Define a new matrix $\boldsymbol{B} \in \mathbb{R}^{(n+p)\times n}$ as

$$\boldsymbol{B} = \begin{bmatrix} \tilde{\boldsymbol{V}}_{r-1} \\ \boldsymbol{I} \end{bmatrix}, \tag{22}$$

the intersection set between $\mathbb{R}^n_+$ and $\mathbf{cone}\,\tilde{\mathcal{V}}^o_{r-1}$ can be represented as the polyhedral cone determined by matrix $\boldsymbol{B}$, as

$$\mathcal{L}_{r-1} = \{\boldsymbol{y}\,|\,\boldsymbol{B}\boldsymbol{y} > \mathbf{0},\ \boldsymbol{y} \in \mathbb{R}^n\}. \tag{23}$$

Theorem 3 shows that the system is quadratic stable if the polyhedral cone $\mathcal{L}_{r-1}$ is not empty, i.e., $\mathcal{L}_{r-1} \neq \varnothing$, $r = 1, \dots, \infty$. According to the computational geometry theory, to determine the emptiness of a polyhedral cone can be dealt with by solving a quadratic programming problem established according to the famous Farkas' Lemma [17]. As far as the polyhedral cone $\mathcal{L}_{r-1}$ is concerned, the corresponding quadratic programming problem with respect to determining its emptiness can be detailed as the following optimization problem

$$\begin{aligned}\min_{\boldsymbol{\alpha}}.\quad & \varphi(\boldsymbol{\alpha}) = \boldsymbol{\alpha}^T\,\boldsymbol{M}\,\boldsymbol{\alpha} \\ s.t.\quad & \textstyle\sum_{i=1}^{n+p}\alpha_i = 1,\ \text{and}\,, \\ & \alpha_i \geq 0,\ i = 1, 2, \dots (n+p)\,,\end{aligned} \tag{24}$$

where $\boldsymbol{M}$ is determined by $\boldsymbol{M} = \boldsymbol{B}\boldsymbol{B}^T$ and $\boldsymbol{\alpha} \in \mathbb{R}^{n+p}$ is the vector of optimization variables. It is shown in [17] that the polyhedral set $\mathcal{L}_{r-1}$ is not empty if and only if the optimized value of $\varphi(\boldsymbol{\alpha})$, denoted as φ^*, is greater than zero.

Denoting the solution to the above optimization problem as $\boldsymbol{\alpha}^*$, it is also proven in [17] that if $\varphi(\boldsymbol{\alpha}^*) > 0$, the vector $\boldsymbol{y}^*$, $\boldsymbol{y}^* = \boldsymbol{B}^T\boldsymbol{\alpha}^*$, is located inside the interior of $\mathcal{L}_{r-1}$. If for $r = 1, \dots, \infty$, $\mathcal{L}_{r-1}$ shares the same vector $\boldsymbol{y}^*$, then $\tilde{\boldsymbol{d}} = \boldsymbol{y}^*$ and a corresponding QLF can be given according to (18).

Because the matrix $\boldsymbol{M}$ in the optimization problem (24) is dependent on the choice of the orthogonal matrix $\mathbf{\Phi}$, the value of φ^* also depends on $\mathbf{\Phi}$. Therefore, if the maximum of φ^* with respect to the complete set of orthogonal matrices is not greater than zero, no suitable $\mathbf{\Phi}$ can be found to fulfill the stability condition (17). This fact indicates that the stability condition (17) can be examined by solving a max-min problem: using different $\mathbf{\Phi}$ to maximize the minimum of $\varphi(\boldsymbol{\alpha})$ with respect to $\boldsymbol{\alpha}$. If the optimized value in this max-min problem has a positive sign at every time instant, the observed motion of the concerned system is quadratic stable, and vice versa.

The parametric representation of orthogonal matrices proposed in [21] taken to construct Φ in the max-min problem, because it is proven to be capable of covering *stochastically* the complete set of $n \times n$ orthogonal matrices. This parametric representation of orthogonal matrices is composed of $n(n-1)/2$ parameters which can vary within the interval $[0,\ 2\pi)$ respectively. Use $\boldsymbol{\theta}$ to represent the vector composed of $\theta_i \in [0,\ 2\pi)$, $i = 1, \ldots, n(n-1)/2$. By choosing each θ_i from the normal distribution within $[0,\ 2\pi)$ randomly, the matrix $\boldsymbol{\Phi}$ constructed by this method is capable of representing every the $n \times n$ orthogonal matrix.

Thus, taking $\boldsymbol{\Phi}$ as a function of $\boldsymbol{\theta}$, the max-min optimization can be written by substituting $\boldsymbol{\theta}$ as

$$\begin{aligned} \max_{\boldsymbol{\theta}}.\min_{\boldsymbol{\alpha}}.\quad & \varphi(\boldsymbol{\alpha}, \boldsymbol{\theta}) = \boldsymbol{\alpha}^T \boldsymbol{M}(\boldsymbol{\Phi}(\boldsymbol{\theta}))\, \boldsymbol{\alpha} \\ s.t.\quad & \theta_i \in [0,\ 2\pi),\ i = 1, 2, \ldots, n(n-1)/2, \\ & \textstyle\sum_{j=1}^{n+p} \alpha_j = 1, \text{ and} \\ & \alpha_j \geq 0,\ j = 1, 2, \ldots, (n+p). \end{aligned} \tag{25}$$

Because the optimization variable $\boldsymbol{\theta}$ in (25) must be chosen stochastically, the gradient of the maximization problem cannot be obtained. Hence, the max-min problem developed above should correspondingly be solved with utilization of random optimization algorithms. Denoting the solution to the max-min optimization problem as $\{\boldsymbol{\Phi}(\boldsymbol{\theta}^*),\ \boldsymbol{\alpha}^*\}$ and the optimized value as $\boldsymbol{\varphi}^{**}$, the system is quadratic stable if and only if $\boldsymbol{\varphi}^{**} > 0$ at every time instant.

For the implementation in the stability monitoring problem, the aforementioned max-min problem should be solved online at every time instant. If at any time instant, say, $t = r$, the observed motion is judged as stable according to the optimized value of the max-min problem, it can only be concluded that the observed motion from $t = 0$ until $t = r$ is quadratic stable. No prediction about stability in the future can be made. On the other hand, if at $t = r$ the observed motion is judged as unstable, the observed motion must be quadratic unstable with respect to the discussion about (19).

IV. NUMERICAL EXAMPLES

Consider a 3-dimensional nonlinear system [23]. The dynamical behavior of this system can be characterized by the existence of an unstable limit cycle oscillation. The mathematical description of this 3-dimensional system is expressed below as

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= x_3, \\ \dot{x}_3 &= -x_3 - x_2 - 2x_1 - x_3 - x_2^2. \end{aligned} \tag{26}$$

The proposed method are executed to monitor the stability of system (26) under two different initial conditions, respectively. In both cases, the sampling time for measuring the system states are set to be 0.01s.

The initial condition of the first trajectory, $\boldsymbol{x}(0) = [-0.8,\ -0.4,\ 1.5]^T$, is inside the unstable limit cycle. Clearly this trajectory owns an stable equilibrium at the origin. This stable trajectory obtained at $t = 10$s is shown in the phase plane in Fig. 1.

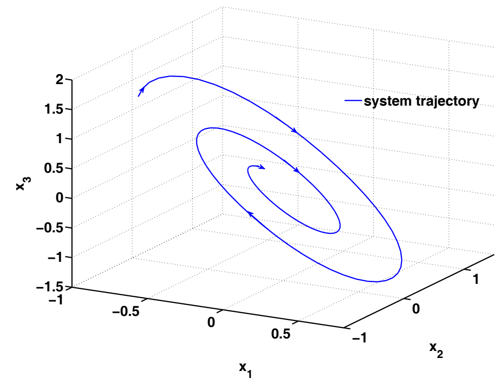

Fig. 1. Observed trajectory of system (26) with initial point inside limit cycle

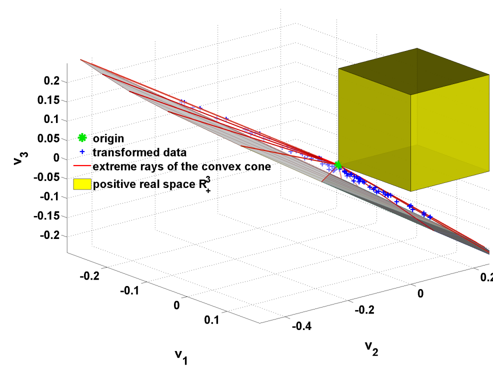

Fig. 2. Convex hull constructed by the trajectory of system (26) with initial point inside limit cycle

Using the data of this trajectory, the proposed data-driven method is applied to judge the stability of the motion of system (26) with the corresponding initial condition at $t = 10$s. The max-min problem is solved by genetic algorithm with binary encoding technique [22]. The reason for choosing the genetic algorithm is because the orthogonal matrix $\boldsymbol{\Phi}$ is produced randomly, and the genetic algorithm is inherently a random optimization solver which is able to converge to the optimal solution with a probability approaching to 1. The optimization results show that when $\boldsymbol{\theta} = [37.5561°,\ 0.6819°,\ 4.8378°]^T$, the optimized value $\varphi^{**} = 6.4014 \times 10^{-5} > 0$. The transformed vector set $\tilde{\mathcal{V}}_{r-1}$ and the corresponding convex conic hull are shown in Fig. 2. It can be seen in Fig. 2 that the convex conic hull (smallest convex cone) determined by the transformed data-set $\tilde{\mathcal{V}}_{r-1}$ are located in an open negative halfspace, indicating that the observed motion at $t = 10$s can be judged as quadratic stable.

The second trajectory with the initial conditions $\boldsymbol{x}(0) = [1.8,\ 0.4,\ 1.5]^T$ is outside the limit cycle, which implies when time goes to infinity, the trajectory will also approach

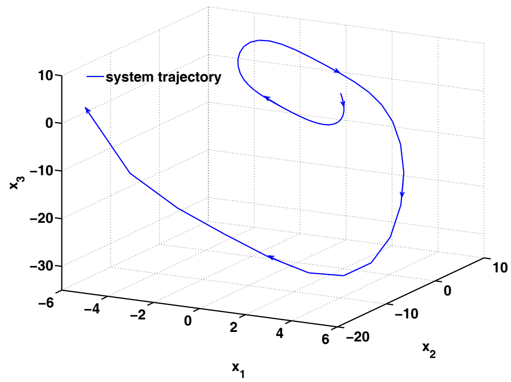

Fig. 3. Observed trajectory of system (26) with initial point outside limit cycle

to infinity. This trajectory at $t = 5$s is shown in Fig. 3 and the corresponding data-set is given to the proposed method for stability judge. The optimized value φ^{**} equals to 1.4131×10^{-28} for all the different values of $\boldsymbol{\theta}$, which can be treated as zero and indicates the system is not quadratic stable according to theorem 3. In fact, the origin is contained in the convex hull determined the transformed data-set, no matter which orthogonal matrix is used for the transformation. This means every hyperplane passing the origin will separate the transformed-data set into two parts. Thus, no negative halfspace containing the transformed data-set will exist and this motion is accordingly classified as not quadratic stable.

From these examples it can be seen that the stability judgments given by the proposed data-driven stability judgment method are consistent with the real stability behaviors of the concerned system, showing the effectiveness of the proposed method.

V. CONCLUSION

This paper proposes a quadratic stability criterion which can be used in online data-driven stability monitoring method of unknown nonlinear discrete-time systems. The paper shows that the existence of a QLF is identical to the existence of a suitable orthogonal matrix with which all the system states can be mapped into a negative halfspace. This geometric problem can be further converted into determining the emptiness of the intersection between the n-dimensional real positive space and the convex cone generated by the data set transformed with an orthogonal matrix from the measured systems states, which can be coped with by solving a max-min problem according to computational geometry theory. The stability judgment can be given according to the sign of the optimized value of the max-min problem. From the simulation results it can be seen that the online quadratic stability monitoring of unknown discrete-time nonlinear systems can be realized by the proposed method.

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