

Distributed Control of Positive Systems

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Abstract—Stabilization and optimal control is studied for state space systems with nonnegative coefficients (positive systems). In particular, we show that a stabilizing distributed feedback controller, when it exists, can be computed using linear programming. The same methods are also used to minimize the closed loop input-output gain. An example devoted to distributed control of a vehicle platoon is examined.

I. INTRODUCTION

The study of matrices with nonnegative coefficients has a long history dating back to the Perron-Frobenius Theorem in 1912. A classic book on the topic is [2]. The theory has been used in Leontief economics, where the states denote nonnegative quantities of commodities, in the study of Markov chains, where the states denote nonnegative probabilities and compartment models, where the states could denote quantities of chemical species in an organism.

Positive systems have received increasing attention in the control literature during the last decade. In particular, stabilization of positive linear systems was studied in [4], [7]. Basic control theory for monotone systems, the nonlinear counterpart of positive systems, was developed in [1]. The importance of nonnegative matrices for consensus algorithms has been widely recognized (see e.g. [6]) and can be traced back to the fact that stability of a nonnegative matrix can be verified using a *linear* Lyapunov function [5].

A recent remarkable result by [8] shows that decentralized controllers can be optimized for positive systems using semi-definite programming. The criterion is the closed loop H_∞ norm and the authors show that diagonal quadratic storage functions can be used without conservatism.

This paper can be viewed as an extension and alternative to [8]. First we demonstrate that stability analysis and synthesis of stabilizing controllers can be performed by distributed linear programming. Secondly, we show that several notions of system gain are equivalent for positive systems and can be optimized using the same methods that solved the stabilization problem.

II. NOTATION AND PRELIMINARIES

Let \mathbf{R}_+ denote the set of nonnegative real numbers. The inequality $X > 0$ ($X \geq 0$) means that all elements

of the matrix (or vector) X are positive (nonnegative). For a symmetric matrix X , the inequality $X > 0$ means that the matrix is positive definite. The matrix $A \in \mathbf{R}^{n \times n}$ is said to be *Schur* if all eigenvalues are strictly inside the unit circle. It is *Hurwitz* if all eigenvalues have positive real part. Finally, the matrix is said to be *Metzler* if all off-diagonal elements are nonnegative.

III. DISTRIBUTED STABILITY VERIFICATION

Proposition 1: Let $A \in \mathbf{R}_+^{n \times n}$. Then the following statements are equivalent:

- (i) The matrix A is Schur
- (ii) There is a $x \in \mathbf{R}_+^n$ such that $Ax < x$.
- (iii) There is a *diagonal* matrix $P > 0$ such that

$$A^T P A < P$$

Remark 1. If A is Schur, then the same is true for A^T , so the proposition guarantees existence of $p \in \mathbf{R}_+^n$ with $A^T p < p$. Hence the linear function $V(x) = p^T x$ is a Lyapunov function for the dynamics $x^+ = Ax$.

Proposition 1 is well known [2, Theorem 6.2.3], but we give a proof for completeness:

Proof. If A is Schur, take any vector $\xi > 0$ and define $x = (I - A)^{-1} \xi$. Then $x = \xi + A\xi + A^2\xi + A^3\xi + \dots > 0$ and $x - Ax = \xi > 0$, so (ii) holds. On the other hand, if (ii) holds, choose $\epsilon > 0$ such that $Ax < (1 - \epsilon)x$. Then for every $z \in \mathbf{R}^n$ with $0 < z < x$ we have

$$0 \leq A^t z \leq A^t x < (1 - \epsilon)^t x \quad \text{for } t = 1, 2, 3, \dots$$

In particular, $\lim_{t \rightarrow \infty} A^t z = 0$, so A must be Schur.

The implication from (iii) to (i) is standard, so it remains to prove the opposite direction. Hence, assuming that A (and therefore A^T) is Schur, we use (ii) to pick x and y in $\mathbf{R}_+^{n \times 1}$ with

$$Ax < x \quad A^T y < y$$

Define the matrix $P = \text{diag}\{p_1, \dots, p_n\}$ by $p_k = y_k/x_k$ and the vector $z \in \mathbf{R}^n$ by $z_k = \sqrt{x_k y_k}$ for $k = 1, \dots, n$. Then

$$\begin{aligned} P^{-1/2} A^T P A P^{-1/2} z &= P^{-1/2} A^T P A x \\ &< P^{-1/2} A^T P x = P^{-1/2} A^T y < P^{-1/2} y = z \end{aligned}$$

so the symmetric matrix $P^{-1/2} A^T P A P^{-1/2}$ is Schur. Hence

$$P^{-1/2} A^T P A P^{-1/2} < I$$

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Fig. 1. A one-dimensional graph of interconnected systems.

and (iii) follows. \square

The corresponding continuous time result can be stated as follows:

Proposition 2: Let $A \in \mathbf{R}^{n \times n}$ be Metzler. Then the following statements are equivalent:

- (i) The matrix A is Hurwitz
- (ii) There exists a $x \in \mathbf{R}_+^{n \times 1}$ such that $Ax < 0$.
- (iii) There is a *diagonal* matrix $P > 0$ such that

$$A^T P + PA < 0$$

Example 1 Consider a dynamical system interconnected according to the graph illustrated in Figure 1:

$$\begin{cases} \dot{x}_1 = \alpha_1 x_1 + \ell_{12}(x_2 - x_1) \\ \dot{x}_2 = \alpha_2 x_2 + \ell_{21}(x_1 - x_2) + \ell_{23}(x_3 - x_2) \\ \dot{x}_3 = \alpha_3 x_3 + \ell_{32}(x_2 - x_3) + \ell_{34}(x_4 - x_3) \\ \dot{x}_4 = \alpha_4 x_4 + \ell_{43}(x_3 - x_4) \end{cases} \quad (1)$$

The model could for example be used to describe a platoon of four vehicles using distance measurements for position adjustment. The parameters $\ell_{ij} \geq 0$ can be used stabilize the dynamics even in situations where some of the α_i -parameters are positive. Notice that the dynamics can be written as $\dot{x} = Mx$ where M is a Metzler matrix. Hence, by Proposition 2, stability is equivalent to existence of numbers $x_1, \dots, x_4 > 0$ such that

$$\begin{aligned} (\alpha_1 - \ell_{12})x_1 + \ell_{12}x_2 &< 0 \\ \ell_{21}x_1 + (\alpha_2 - \ell_{21} - \ell_{23})x_2 + \ell_{23}x_3 &< 0 \\ \ell_{32}x_2 + (\alpha_3 - \ell_{32} - \ell_{34})x_3 + \ell_{34}x_4 &< 0 \\ \ell_{43}x_3 + (\alpha_4 - \ell_{43})x_4 &< 0 \end{aligned}$$

This can be implemented as a distributed test where the first node verifies the first inequality, the second node verifies the second inequality and so on. \square

IV. DISTRIBUTED STABILIZATION BY LINEAR PROGRAMMING

Given the distributed stability test, the next step is to search for stabilizing feedback laws using distributed optimization. This can be done using the following theorem:

Theorem 1: Let $A \in \mathbf{R}^{n \times n}$, $E \in \mathbf{R}_+^{n \times m}$, $F \in \mathbf{R}^{m \times n}$ and let \mathcal{D} be the set of $m \times m$ diagonal matrices with entries in $[0, 1]$. Suppose that $A + ELF$ is non-negative for all $L \in \mathcal{D}$. Then, the following are equivalent:

- (i) There is $L \in \mathcal{D}$ such that $A + ELF$ is Schur.
- (ii) There exist $p \in \mathbf{R}_+^n$, $q \in \mathbf{R}_+^m$ with $q \leq E^T p$ and $A^T p + F^T q < 0$.

Proof. Suppose (i) holds. Let $A + ELF$ be Schur and define $p \in \mathbf{R}_+^n$ with $(A + ELF)^T p < 0$. Let $q = LE^T p$. Then $q \leq E^T p$ and $A^T p + F^T q = (A + ELF)^T p < 0$.

Conversely, suppose that (ii) holds. Choose $L \in \mathcal{D}$ to get $q = LF^T p$. Then

$$(A + ELF)^T p = A^T p + F^T q < 0$$

so $A + ELF$ is Schur. \square

A continuous time version can be stated as follows:

Theorem 2: Let $A \in \mathbf{R}^{n \times n}$, $E \in \mathbf{R}_+^{n \times m}$, $F \in \mathbf{R}^{m \times n}$ and let \mathcal{D} be the set of $m \times m$ diagonal matrices with entries in $[0, 1]$. Suppose that $A + ELF$ is Metzler for all $L \in \mathcal{D}$. Then, the following are equivalent:

- (i) There exists $L \in \mathcal{D}$ such that $A + ELF$ is Hurwitz.
- (ii) There exist $p \in \mathbf{R}_+^n$, $q \in \mathbf{R}_+^m$ with $q \leq E^T p$ and $A^T p + F^T q < 0$.

Remark 2. It is natural to compare the expression $A + ELF$ with the “state feedback” expression $A + BK$ of standard linear quadratic optimal control. A major difference is the matrix F , which makes the optimization of $A + ELF$ into a problem of “static output feedback” rather than state feedback. Another difference is the diagonally structured L instead of the full matrix K . The diagonal structure gives a much higher degree of flexibility, particularly in the specification of distributed controllers.

Remark 3. If the diagonal elements of \mathcal{D} are restricted to \mathbf{R}_+ instead of $[0, 1]$, then the condition $q \leq E^T p$ is replaced by $0 \leq E^T p$.

Remark 4. Each row of the inequalities $q \leq E^T p$ and $A^T p + F^T q < 0$ can be verified separately to get a distributed test.

Example 2 Given $\alpha_1, \dots, \alpha_4$ consider the system (1) and the problem to find feedback gains $\ell_{ij} \in [0, 1]$ that stabilize the system. The problem can be solved by applying distributed linear programming to condition (ii) in Theorem 2 with

$$A = \text{diag}\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$$

$$E = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$L = \text{diag}\{\ell_{12}, \ell_{21}, \ell_{23}, \ell_{32}, \ell_{34}, \ell_{43}\}$$

$$F = \begin{pmatrix} -1 & 1 & 0 & 0 & 0 & 0 \\ 1 & -1 & -1 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

$M = A + ELF$ is the state space matrix of (1). \square

V. DISTRIBUTED PERFORMANCE OPTIMIZATION

Given the formulas for stabilizing feedback found in the previous section, it is natural to move beyond stability and also optimize input-output performance. This can be done using the following theorem.

Theorem 3: Let $G(z) = C(zI - A)^{-1}B + D$ where $A \in \mathbf{R}_+^{n \times n}$, $B \in \mathbf{R}_+^{n \times 1}$, $C \in \mathbf{R}_+^{1 \times n}$ and $D \in \mathbf{R}_+$. Then the following two conditions are equivalent:

- (i) The matrix A is Schur and $\|G\|_\infty < \gamma$.
- (ii) The matrix $\begin{bmatrix} A & B \\ \gamma^{-1}C & \gamma^{-1}D \end{bmatrix}$ is Schur.

Moreover, $\|G\|_\infty = C(I - A)^{-1}B + D$.

Proof. First note that the maximum $\max_\omega |G(e^{i\omega})|$ must be attained at $\omega = 0$ since

$$\begin{aligned} |G(e^{i\omega})| &= \left| D + \sum_{t=1}^{\infty} C(e^{-i\omega}A)^{t-1}B \right| \\ &\leq |D| + \sum_t |CA^{t-1}B| = C(I - A)^{-1}B + D = G(1) \end{aligned}$$

Hence $\|G\|_\infty < \gamma$ may equivalently be written

$$C(I - A)^{-1}B + D < \gamma$$

Suppose that (i) holds. By Proposition 1 there exists $\xi \in \mathbf{R}_+^{1 \times n}$ such that $A\xi < \xi$. Define $x = \xi + (I - A)^{-1}B$. Then $x > 0$ and

$$Ax + B - x = A\xi - \xi < 0$$

If ξ is sufficiently small, we also get $Cx + D < \gamma$ so

$$\begin{bmatrix} A & B \\ \gamma^{-1}C & \gamma^{-1}D \end{bmatrix} \begin{bmatrix} x \\ 1 \end{bmatrix} < \begin{bmatrix} x \\ 1 \end{bmatrix}$$

and (ii) holds. Conversely, (ii) implies that $Ax < x$ and

$$\begin{aligned} (I - A)^{-1}B &< x \\ 0 \leq C(I - A)^{-1}B + D &< Cx + D < \gamma \end{aligned}$$

so (i) follows. \square

A continuous time version is given without proof:

Theorem 4: Suppose that $G(s) = C(sI - A)^{-1}B + D$ where $A \in \mathbf{R}^{n \times n}$ is Metzler, while $B \in \mathbf{R}_+^{n \times 1}$, $C \in \mathbf{R}_+^{1 \times n}$ and $D \in \mathbf{R}_+$. Then the following two conditions are equivalent:

- (i) The matrix A is Hurwitz and $\|G\|_\infty < \gamma$.
- (ii) The matrix $\begin{bmatrix} A & B \\ C & D - \gamma \end{bmatrix}$ is Hurwitz.

Moreover, $\|G\|_\infty = D - CA^{-1}B$.

Combining this theorem with Theorem 2 gives a linear programming formulation of the problem to minimize input-output gain:

Theorem 5: Let $G(s) = C(sI - A - ELF)^{-1}B + D$ where $B \in \mathbf{R}_+^{n \times 1}$, $C \in \mathbf{R}_+^{1 \times n}$, $D \in \mathbf{R}_+$, $E \in \mathbf{R}_+^{n \times m}$ and $F \in \mathbf{R}^{m \times n}$. Let \mathcal{D} be the set of $m \times m$ diagonal matrices with entries in $[0, 1]$. Suppose that $A + ELF$ is Metzler for all $L \in \mathcal{D}$. Then the following two conditions are equivalent:

- (i) There exists $L \in \mathcal{D}$ such that $A + ELF$ is Hurwitz and $\|G\|_\infty < \gamma$.
- (ii) There exist $p \in \mathbf{R}_+^n$, $q \in \mathbf{R}_+^m$ with

$$\begin{aligned} A^T p + F^T q + C^T &< 0 \\ B^T p + D - \gamma &< 0 \\ q - E^T p &\leq 0 \end{aligned}$$

Moreover, if the conditions of (ii) are satisfied for some p, q , then the conditions of (i) hold for every L such that $q = (EL)^T p$.

Proof. According to Theorem 4, condition (i) holds if and only if the matrix

$$\begin{bmatrix} A + ELF & B \\ C & D - \gamma \end{bmatrix}$$

is Hurwitz, or equivalently, there exists $p \in \mathbf{R}_+^n$ with

$$\begin{bmatrix} A + ELF & B \\ C & D - \gamma \end{bmatrix}^T \begin{bmatrix} p \\ 1 \end{bmatrix} < 0 \quad (2)$$

Given (2), the inequalities of (ii) hold with $q = E^T p$. Conversely, given (ii), the inequalities of (2) follow provided that $q = (EL)^T p$. This proves the desired equivalence between (i) and (ii) in Theorem 5. \square

Example 3 Consider the problem to find $\ell_{ij} \in [0, 1]$ such that the system

$$\begin{aligned} \dot{x}_1 &= \alpha_1 x_1 + \ell_{12}(x_2 - x_1) + u \\ \dot{x}_2 &= \alpha_2 x_2 + \ell_{21}(x_1 - x_2) + \ell_{23}(x_3 - x_2) \\ \dot{x}_3 &= \alpha_3 x_3 + \ell_{32}(x_2 - x_3) + \ell_{34}(x_4 - x_3) \\ \dot{x}_4 &= \alpha_4 x_4 + \ell_{43}(x_3 - x_4) \end{aligned}$$

is stabilized, while the gain from u to x_1 is minimized. The solution is obtained by distributed linear programming solving the inequalities of condition (ii) in Theorem 5 with matrices A , E , F and L specified in Example 2 and

$$\begin{aligned} B^T &= [1 \ 0 \ 0 \ 0] \\ C &= [1 \ 0 \ 0 \ 0] \\ D &= 0 \end{aligned}$$

Once p and q are determined, L can be obtained from the equation $q = (EL)^T p$. \square

VI. ALL INDUCED NORMS ARE THE SAME

In this section we will verify that for positive systems, several notions of system gain are equivalent.

For a sequence $w = \{w(t)\}_{t=0}^{\infty}$ define the norms

$$\|w\|_p = \left(\sum_{t=0}^{\infty} |w(t)|^p \right)^{1/p} \quad \|w\|_{\infty} = \sup_t |w(t)|$$

for $1 \leq p$. Given matrices A, B, C, D , define the transfer function $G(z) = C(zI - A)^{-1}B + D$ with corresponding impulse response $g = \{g(t)\}_{t=0}^{\infty} = (D, CB, CAB, CA^2B, \dots)$. Let $g * w$ denote the convolution of g and w and define

$$\|g\|_{p\text{-ind}} = \sup_w \frac{\|g * w\|_p}{\|w\|_p}$$

for $p = 1, 2$ and ∞ . It is well known that if A is Schur and g scalar, then $\|g\|_{2\text{-ind}} = \max_{\omega} |G(e^{i\omega})|$. Another notation for this norm is $\|G\|_{\infty}$. When $g(t) \geq 0$, the maximum must be attained at $\omega = 0$ since

$$\begin{aligned} |G(e^{i\omega})| &= \left| D + \sum_{t=1}^{\infty} C(e^{-i\omega}A)^{t-1}B \right| \\ &\leq |D| + \sum_t |CA^{t-1}B| = \sum_t g(t) = G(1) \end{aligned}$$

In fact, all p -induced norms are equal for $1 \leq p \leq \infty$:

Theorem 6: Suppose that $g = \{g(t)\}_{t=0}^{\infty}$ is nonnegative with $\sum_{t=0}^{\infty} g(t) < \infty$. Then

$$\|g\|_{p\text{-ind}} = \sum_t g(t) \quad \text{for } 1 \leq p \leq \infty$$

Proof. Let $y = g * w$. Then

$$\begin{aligned} \|y(t)\|_1 &= \sum_{t=0}^{\infty} \left| \sum_{\tau=0}^t g(t-\tau)w(\tau) \right| \\ &\leq \sum_{\tau=0}^{\infty} \left(\sum_{t=\tau}^{\infty} g(t-\tau) \right) |w(\tau)| \\ &\leq \left(\sum_{t=0}^{\infty} g(t) \right) \|w\|_1 \end{aligned}$$

with equality when $w(t) \geq 0$ for all t . Moreover,

$$\begin{aligned} |y(t)| &= \left| \sum_{\tau=0}^{\infty} g(\tau)w(t-\tau) \right| \leq \sum_{\tau=0}^{\infty} g(\tau)|w(t-\tau)| \\ &\leq \sum_{\tau=0}^{\infty} g(\tau)\|w\|_{\infty} \end{aligned}$$

with equality if w is constant. Hence the desired equality

$$\|g\|_{p\text{-ind}} = \sum_t g(t)$$

has been proved for $p = 1$ and $p = \infty$ and it follows for intermediate values of p from the Riesz-Thorin convexity theorem [3, Theorem 7.1.12]. \square

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