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Well posedness of the model of an extruder in infinite dimension

Mamadou Diagne and Valérie Dos Santos Martins and Françoise Couenne and Bernhard Maschke

***Abstract*— The topic of this paper is to present and analyse a physical model of the extrusion process which is expressed two systems of conservation laws (with source terms) coupled by a moving interface whose relation is derived from the conservation of momentum. After a change of variables on the spatial variables is performed in order to transform the time-varying spatial domain in fixed one, the linearisation of the model around an equilibrium profile is given, the well-posedness in the sense of the existence of a C_0-semigroup of those the coupled systems is proven.**

***Index Terms*— Coupled systems of conservation laws, Well-posedness, infinite dimensional system, Distributed parameter system**

NOMENCLATURE

B	Geometric parameter
c_p $(J\ kg^{-1}\ K^{-1})$	Specific heat capacity
F $(kg\ s^{-1})$	Mass flow rate
F_d $(kg\ s^{-1})$	Net forward mass flow rate
f $(-)$	Filling ratio
K_d $(-)$	Geometric parameter
l (m)	Moving boundary
M $(-)$	Moisture content
N $(rd\ s^{-1})$	Screw speed
P (Pa)	Pressure
$S_{ech}(m^2)$	Exchange area between melt & barrel
$S_{eff}(m^2)$	Effective area
$T(K)$	Melt temperature
$T_F(K)$	Barrel temperature
$V_{eff}(m^3)$	Effective volume
x (m)	Space coordinate
α $(Jm^{-2}s^{-1}K^{-1})$	Heat exchange coefficient
χ $(-)$	Dimensionless space coord.
η $(Pa\ s^{-1})$	Melt viscosity
$\mu(J\ kg^{-1}\ K^{-1})$	Viscous heat generation parameter
$\rho_0(kg\ m^{-3})$	Melt density
ξ (m)	Pitch lenght

INDEX AND SUPERSCRIPT

in	inlet
p	Partially Filled Zone
f	Fully Filled Zone
e	Equilibrium
$-$	Variable in dimensionless coordinate

I. INTRODUCTION

An extruder is made of a barrel containing one or two Archimedean screws rotating inside the barrel. At the output, the extruder is equipped of a die from which the material is extruded from the process (Fig. 1). The process is controlled by the barrel temperature and the screw speed to ensure the desired properties (moisture, density, etc... of the food or the polymer) at the die in presence of perturbations. The physical phenomena involved in the extrusion process,

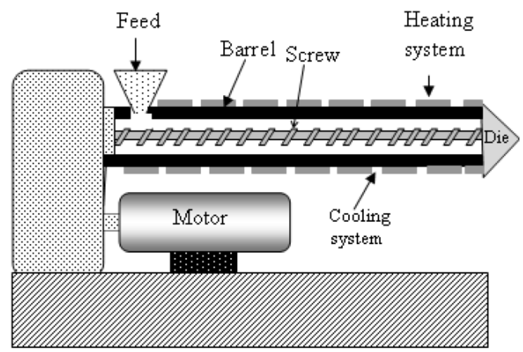

Fig. 1. Description of the mechanism of an extruder

consist in coupled non linear phenomena, such as viscous Newtonian or non Newtonian fluid flows, heat transfer and possibly chemical reactions. The design of an extrusion process involves a complex modular geometry in function of the screw profile, allowing different capacities of mixing along the extruder. The reader is referred to [1] for the steady-state modelling for design purpose and to [2], [3], [4], [5], [6] for dynamical models and for the control to [7] developing a proportional integral PI feedback or [8] developing a multi-variable predictive control.

But the main subject that we shall be interested in, is that the extruder is divided in time-varying spatial zones where the material fills completely or not the volume of the extruder involving a change of causality and order of the system. In this paper we shall follow [9], [10] where an infinite-dimensional model is developed and shall define and analyse a simple 1 dimensional model consisting of two systems of conservation laws (with source terms) coupled by a moving interface.

In the second section the physical model of the extrusion process is recalled in terms of two systems of conservation laws (with source terms) coupled by a moving interface whose relation is derived from the conservation of momentum. In the third section a change of variables on the spatial variables is performed in order to transform the time-varying spatial domain in fixed one, thereby introducing a fictitious convection term in the conservation laws. In the fourth section this model is linearized. In the fifth section, the dynamics of the boundary is integrated to the distributed

Université de Lyon, F-69622, Lyon, France; Université de Lyon1, Villeurbanne, LAGEP, UMR 5007 CNRS, CPE, 43 bd du 11 novembre, 69622 Villeurbanne Cedex, France. e-mail: name@lagep.univ-lyon1.fr

state variables and the obtained linear system is shown to generate a C_0-semigroup.

II. THE PHYSICAL MODEL

Following [9] and [10], the spatial domain of the extruder is split in two parts: the partially and fully filled zones according to the figure 2.

In the partially filled zone (PFZ) (or conveying zone), the pressure is supposed to be constant and equal to the atmospheric pressure P_0. In the fully filled zone FFZ, the filling volume fraction is by definition equal to 1 and the resistance of the die generates a pressure gradient. The difference between the net forward flow at the die and the pumping capacity of the screws causes the displacement of the boundary between the PFZ and FFZ. The dynamic

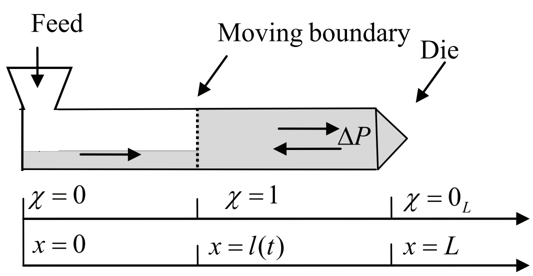

Fig. 2. The 2-zones assumption in the extruder

model is derived from the mass and energy balances on a volume element for each zone under some assumptions:

- the pitch of the screw ξ is uniform;
- the flow is 1D and strictly convective, the melt density ρ_0 and viscosity η are assumed to be constant;
- there exists a boundary between the PFZ and the FFZ corresponding to discontinuity of the filled volume (or filled volume fraction also called filling ratio);
- the extruded melt is composed of some species blended with water

A. Model of the Partially Filled Zone (PFZ):

The mass balance equations in the PFZ, are written on the spatial domain $[0, l(t)]$, in terms of the filling ratio f_p (the filled volume fraction which may be related to the total mass density) and the moisture content M_p [9]. The energy balance is written in terms of the of temperature T_p of the mixture.

The balance equations express the convection through the rotation of the screw at the translational velocity, product on the pitch of the screw ξ and the rotation speed of the screw $N(t)$. The source term Ω_1 groups the heat produced by the viscosity of the material (proportional to $N^2(t)$) and the heat exchange with the barrel (proportional to $(T_{F_p} - T_p)$)[1]:

$$\frac{\partial}{\partial t}\begin{pmatrix} f_p(x,t)\\ M_p(x,t)\\ T_p(x,t)\end{pmatrix} = -\xi N(t) I_3 \frac{\partial}{\partial x}\begin{pmatrix} f_p(x,t)\\ M_p(x,t)\\ T_p(x,t)\end{pmatrix} + \begin{pmatrix} 0\\ 0\\ \Omega_1(f_p, N(t), T_p, T_{F_p})\end{pmatrix} \qquad (1)$$

with $\quad \Omega_1 = \dfrac{\mu_p \eta_p N^2(t)}{f_p(x,t)\rho_0 V_{eff} c_p} + \dfrac{S_{ech}\alpha}{\rho_0 V_{eff} c_p}(T_{F_p} - T_p)$

and $\quad P(x,t) = P_0.$

$\quad V_{eff} = \xi S_{eff}$

B. Model of the Fully Filled Zone FFZ:

In the FFZ zone, the model is reduced to the mass balance of water written in terms of the moisture content M_f and the energy balance written in terms of the temperature T_f. The balances are written on the spatial domain $[l(t), L]$. The speed of convection $\frac{F_d \xi}{\rho_0 V_{eff}}$ is a function of the net flow rate at the die F_d (Eq. 3), F_d being a function of the geometric characteristic of the die K_d, the viscosity η_f and the pressure build-up in this zone $P(x,t)$. The heat transfer with the barrel and viscous dissipation created by the viscosity are defined in the term Ω_2.

$$\frac{\partial}{\partial t}\begin{pmatrix} M_f(x,t)\\ T_f(x,t)\end{pmatrix} = \frac{-F_d \xi}{\rho_0 V_{eff}} I_2 \frac{\partial}{\partial x}\begin{pmatrix} M_f(x,t)\\ T_f(x,t)\end{pmatrix} + \begin{pmatrix} 0\\ \Omega_2(N(t), T_f, T_{F_f})\end{pmatrix} \qquad (2)$$

$$\text{with } \Omega_2 = \frac{\mu_f \eta_f N^2(t)}{\rho_0 V_{eff} c_p} + \frac{S_{ech}\alpha}{\rho_0 V_{eff} c_p}(T_{F_f} - T_f)$$

$$\text{and } F_d = \frac{K_d}{\eta_f}\Delta P \qquad (3)$$

$$\Delta P = (P(L,t) - P_0) \qquad (4)$$

The pressure gradient in this zone is expressed as a function of the difference between the maximum flow and F_d:

$$\frac{\partial P(x,t)}{\partial x} = \eta_f \frac{V_{eff} N(t) \rho_0 - F_d}{B \rho_0} \qquad (5)$$

Let us note that, as a consequence of a constant melt density, the gradient (5) is uniform.

C. Model of the moving interface at $l(t)$:

Following [2], [3], we assume that the two zones are separated by an interface defined by the discontinuity of the filling ratio: in the PFZ the filling ratio satisfies $f_p(x,t) < 1, x \in [0, l(t)[$ with $f_p(l^-, t) < 1$ and in the FFZ $f_p(x,t) = 1$, $x \in]l(t), L]$. The dynamics of the moving boundary is obtained from the global mass balance on the FFZ zone:

$$\frac{dl(t)}{dt} = \frac{F(f_p(l^-,t)) - F_d}{\rho_0 S_{eff}(1 - f_p(l^-,t))} \qquad (6)$$

[1] I_j stands for the identity matrix $j \times j$.

D. Interface relations:

At the interface $x = l(t)$, temperature and moisture content are supposed to be continuous :

$$\begin{aligned} T_p(l^-,t) &= T_f(l^+,t) \\ M_p(l^-,t) &= M_f(l^+,t) \end{aligned}$$

The third coupling relation between the two zones consists in the continuity of the momentum flux (Eq. 7):

$$\begin{aligned} & F(l^-,t)\xi N(t) + P(l^-,t)f(l^-,t)S_{eff} \\ = \quad & F(l^+,t)\frac{F_d\xi}{\rho_0 V_{eff}} + P(l^+,t)S_{eff} \end{aligned} \tag{7}$$

and allows to compute the mass flow F_d at the die (eq. (3)) and the transport velocity in the FFZ by integrating the pressure gradient on $[l^+, L]$ (Eq. 5) and obtaining the pressure:

$$P(L,t) - P_0 = \frac{-[1+\frac{K_d}{B\rho_0}(L-l^+)] + \sqrt{\Delta}}{\frac{2K_d^2}{\eta_f^2\rho_0 S_{eff}^2}} \tag{8}$$

with $\Delta = [1+\frac{K_d}{B\rho_0}(L-l^+)]^2 + \Omega_3\left(f_p(l^-,t), N(t), l^+\right)$

$$\begin{aligned} \text{and } \Omega_3 \quad = \quad & \left(\frac{2K_d}{\eta_f S_{eff}}\right)^2 \left(\frac{\eta_f V_{eff} N(t)}{B\rho_0}(L-l^+)\right. \\ & \left. +\xi^2 N^2(t) f_p(l^-,t) - (1-f_p(l^-,t))\frac{P_0}{\rho_0}\right) \end{aligned}$$

E. Boundary conditions:

The boundary conditions are defined at the inlet of the extruder that is at $x = 0$. It is assumed that the mass flow is continuous and hence equal to the feed rate $F_{in}(t)$ which leads to the boundary condition on the filling ratio :

$$f_p(0,t) = \frac{F_{in}(t)}{\rho_0 N V_{eff}} \tag{9}$$

The mixing phenomena at the inlet are neglected hence the continuity of the temperature and of the moisture content are assumed :

$$\begin{aligned} T_p(0,t) &= T_{in}(t) \\ M_p(0,t) &= M_{in}(t) \end{aligned}$$

where $M_{in}(t)$ and $T_{in}(t)$ are the moisture content and temperature of the matter at the inlet $x = 0$.

III. Model expressed in fixed domains

In order to deal with a system of balance equations in a fixed domain a classical change of spatial variables is performed for the two zones leading to two systems of conservation laws with source terms and in addition a fictitious convection term due to the change of spatial coordinates.

A. Partially filled zone with fixed boundary model

The change of spatial variables from $[0, l(t)]$ onto the interval $[0, 1]$ is defined in this way:

$$\chi(x,t) = \frac{x}{l(t)} \tag{10}$$

And the PDE in (1) becomes:

$$\begin{aligned} & \frac{\partial}{\partial t}\begin{pmatrix} \overline{f}_p(\chi,t) \\ \overline{M}_p(\chi,t) \\ \overline{T}_p(\chi,t) \end{pmatrix} = \alpha_p(\chi,t) I_3 \frac{\partial}{\partial \chi}\begin{pmatrix} \overline{f}_p(\chi,t) \\ \overline{M}_p(\chi,t) \\ \overline{T}_p(\chi,t) \end{pmatrix} \\ & + \begin{pmatrix} 0 \\ 0 \\ \overline{\Omega}_1(\overline{f}_p, N(t), \overline{T}_p, T_{F_p}) \end{pmatrix}, \chi \in (0,1) \end{aligned} \tag{11}$$

with $\alpha_p(\chi,t) = -\frac{1}{l(t)}[\xi N(t) - \chi\frac{dl(t)}{dt}]$

and $\overline{\Omega}_1 = \frac{\mu_p \eta_p N^2(t)}{\overline{f}_p(\chi,t)\rho_0 V_{eff} c_p} + \frac{S_{ech}\alpha}{\rho_0 V_{eff} c_p}(T_{F_p} - \overline{T}_p)$

With those news coordinates, the model equations include one fictive convective term depending on the velocity $\frac{dl(t)}{dt}$ of the boundary .

B. Fully filled zone with fixed boundary model

In this zone, the change of spatial variables from $x \in (l(t), L)$ onto the interval $[0, 1]$ is defined by :

$$\chi(x,t) = \frac{L-x}{L-l(t)} \tag{12}$$

And the PDE in (2) becomes:

$$\begin{aligned} & \frac{\partial}{\partial t}\begin{pmatrix} \overline{M}_f(\chi,t) \\ \overline{T}_f(\chi,t) \end{pmatrix} = \alpha_f(\chi,t) I_2 \frac{\partial}{\partial \chi}\begin{pmatrix} \overline{M}_f(\chi,t) \\ \overline{T}_f(\chi,t) \end{pmatrix} \\ & + \begin{pmatrix} 0 \\ \overline{\Omega}_2(N(t), \overline{T}_f, T_{F_f}) \end{pmatrix}, \ \chi \in (0,1) \end{aligned} \tag{13}$$

with $\alpha_f(\chi,t) = -\frac{1}{L-l(t)}[\frac{F_d\xi}{\rho_0 V_{eff}} + \chi\frac{dl(t)}{dt}]$

with $\overline{\Omega}_2 = \frac{\mu_f C \eta_f N^2(t)}{\rho_0 V_{eff} c_p} + \frac{S_{ech}\alpha}{\rho_0 V_{eff} c_p}(T_{F_f} - \overline{T}_f)$

The net flow at the die F_d is given by those expressions:

$$F_d = \frac{K_d}{\eta_f}\Delta\overline{P} \tag{14}$$

$$\text{with} \qquad \Delta\overline{P} = (\overline{P}(0,t) - P_0) \tag{15}$$

The boundary conditions and the interface relations are easily deduced from their expression in the original spatial variables and are not developed further.

IV. The linearized model in the fixed boundary coordinates

In this section, the linearization of the system around some equilibrium profile is derived.

A. Equilibrium profiles

- The variables f_p and M_p are constant in time and space as it is shown in this equality:

$$\frac{\partial}{\partial t}\begin{pmatrix}\overline{f}_{pe}\\ \overline{M}_{pe}\end{pmatrix} = \frac{\partial}{\partial \chi}\begin{pmatrix}\overline{f}_{pe}\\ \overline{M}_{pe}\end{pmatrix} = 0 \tag{16}$$

- The temperature $\overline{T}_p$ is given by an ODE in χ:

$$\frac{\partial \overline{T}_{pe}}{\partial \chi}(\chi) = \frac{l_e}{\xi N_e}\overline{\Omega}_{1e} \tag{17}$$

- The variable $\overline{M}_f$ which describes the moisture concentration is constant in time and space:

$$\frac{\partial}{\partial t}\overline{M}_{fe} = \frac{\partial}{\partial \chi}\overline{M}_{fe} = 0 \tag{18}$$

- The evolution of the temperature in this zone is driven by a differential equation in χ as in the PFZ:

$$\frac{\partial}{\partial \chi}\overline{T}_{fe}(\chi) = \frac{(L-l_e)\rho_0 V_{eff}}{\xi F_{de}}\overline{\Omega}_{2e} \tag{19}$$

The moving boundary $l(t)$ is fixed at the equilibrium and induces the following relation between the net flow F_{de} at the die and the screw rotational velocity N_e:

$$\frac{dl_e}{dt} \quad = \quad 0 \Leftrightarrow F_{de} = \rho_0 N_e V_{eff} f_e \tag{20}$$

B. Linear model around the equilibrium profile

The linearization of the two systems of PDE's in fixed domain and the dynamics of the moving interface is then obtained as follows.
- The PFZ Linearized model is given by

$$\frac{\partial}{\partial t}\begin{pmatrix}\delta\overline{f}_p\\ \delta\overline{M}_p\\ \delta\overline{T}_p\end{pmatrix} = \begin{pmatrix}0 & 0\\ 0 & 0\\ \beta_{p2,N} & \beta_{p2,T}\end{pmatrix}\begin{pmatrix}\delta N\\ \delta T_{F_p}\end{pmatrix} \tag{21}$$
$$+\begin{pmatrix}0\\0\\ \beta_{p3}\end{pmatrix}\delta l + \begin{pmatrix}0\\0\\ \beta_{p4}\end{pmatrix}\delta\frac{dl}{dt}$$
$$+\left(-\frac{1}{l_e}\xi N_e I_3 \partial_\chi + \begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ \beta_{p1,f} & 0 & \beta_{p1,T}\end{pmatrix}\right)\begin{pmatrix}\delta\overline{f}_p\\ \delta\overline{M}_p\\ \delta\overline{T}_p\end{pmatrix}$$

with $\beta_{p1,f} = -\dfrac{\mu_p C\eta_p N_e^2}{\overline{f}_{pe}^2\rho_0 V_{eff}c_p},\ \beta_{p1,T} - \dfrac{S_{ech}\alpha}{\rho_0 V_{eff}c_p},$

$$\beta_{p2,N} = \frac{\mu_p\eta_p N_e}{\overline{f}_{pe}\rho_0 V_{eff}c_p} - \frac{S_{ech}\alpha(T_{F_{pe}} - \overline{T}_{pe})}{N_e\rho_0 V_{eff}c_p}$$
$$\beta_{p2,T} = \frac{S_{ech}\alpha}{\rho_0 V_{eff}c_p},\ \beta_{p4} = \chi\frac{l_e}{\xi N_e}\beta_{p3}$$
$$\beta_{p3} = \frac{1}{l_e}\left(\frac{\mu_p\eta_p N_e^2(t)}{\overline{f}_{pe}\rho_0 V_{eff}c_p} + \frac{S_{ech}\alpha}{\rho_0 V_{eff}c_p}(T_{F_{pe}} - \overline{T}_{pe})\right)$$

The FFZ Linearized model is given by:

$$\frac{\partial}{\partial t}\begin{pmatrix}\delta\overline{M}_f\\ \delta\overline{T}_f\end{pmatrix} = -\frac{1}{L-l_e}\frac{F_{de}\xi}{\rho_0 V_{eff}}I_2\frac{\partial}{\partial\chi}\begin{pmatrix}\delta\overline{M}_f\\ \delta\overline{T}_f\end{pmatrix} \tag{22}$$
$$+\begin{pmatrix}0 & 0\\ 0 & \beta_{f1}\end{pmatrix}\begin{pmatrix}\delta\overline{M}_f\\ \delta\overline{T}_f\end{pmatrix} + \begin{pmatrix}0 & 0\\ \beta_{f2,N} & \beta_{f2,T}\end{pmatrix}\begin{pmatrix}\delta N\\ \delta T_{F_f}\end{pmatrix}$$
$$+\begin{pmatrix}0\\ \beta_{f3}\end{pmatrix}\delta\overline{P}(0,t) + \begin{pmatrix}0\\ \beta_{f4}\end{pmatrix}\delta l(t) + \begin{pmatrix}0\\ \beta_{f5}\end{pmatrix}\delta\frac{dl}{dt}$$

with $\beta_{f1} = -\dfrac{S_{ech}\alpha}{\rho_0 V_{eff}c_p}$

$$\beta_{f2,N} = \frac{2\mu_f\eta_f N_e}{\rho_0 V_{eff}c_p},\ \beta_{f2,T} = \frac{S_{ech}\alpha}{\rho_0 V_{eff}c_p}$$
$$\beta_{f3} = \frac{-K_d\left(\mu_f\eta_f N_e^2 + S_{ech}\alpha(T_{F_{fe}} - \overline{T}_{fe})\right)}{\rho_0\eta_f F_{de}V_{eff}c_p}$$
$$\beta_{f4} = \frac{\mu_f\eta_f N_e^2 + S_{ech}\alpha(T_{F_{fe}} - \overline{T}_{fe})}{(L-l_e)c_p}$$
$$\beta_{f5} = -\chi\frac{\mu_f\eta_f N_e^2 + S_{ech}\alpha(T_{F_{fe}} - \overline{T}_{fe})}{\xi F_{de}c_p}$$

Linearized dynamics of the moving interface:

$$\frac{d(\delta l)}{dt} = \frac{-K_d}{\rho_0\eta_f S_{eff}(1-\overline{f}_{pe})}\delta P(0,t) + \frac{\xi\overline{f}_{pe}}{(1-\overline{f}_{pe})}\delta N$$
$$+\left[\frac{\xi N_e(1-2\overline{f}_{pe})}{(1-\overline{f}_{pe})^2} + \frac{K_d(\overline{P}_e - P_0)}{\eta_f\rho_0 S_{eff}(1-\overline{f}_{pe})^2}\right]\delta\overline{f}_p(1^-,t) \tag{23}$$

Boundary conditions become:

$$\delta\overline{f}_p(0,t) \quad = \quad \delta\frac{F_{in}(t)}{\rho_0 N V_{eff}}$$
$$\delta\overline{T}_p(0,t) \quad = \quad \delta T_{in}(t),\ \delta\overline{M}_p(0,t) = \delta M_{in}(t)$$

and

$$\delta\overline{P}(0,t) = \frac{\delta N}{\sqrt{\Delta_e}}\left(\frac{\eta_f V_{eff}}{B}(0-l_e) + 2\rho_0\xi^2 N_e\overline{f}_{pe}\right)$$
$$+\frac{\delta\overline{f}_p}{\sqrt{\Delta_e}}(\rho_0\xi^2 N_e^2 + P_0) + \delta l\frac{(\eta_f S_{eff})}{B\sqrt{\Delta_e}}[-\xi N_e$$
$$+\frac{(\eta_f S_{eff})}{2K_d}\left(\sqrt{\Delta_e} - (1 + \frac{K_d}{B\rho_0}(0-l_e))\right)\Bigg] \tag{24}$$

Interface relations are expressed at $\chi = 1$. The continuity of the moisture concentration and the temperature is assumed:

$$\delta\overline{T}_p(1,t) \quad = \quad \delta\overline{T}_f(1,t)$$
$$\delta\overline{M}_p(1,t) \quad = \quad \delta\overline{M}_f(1,t)$$

C. Well posedness of the linearized PFZ & FFZ

The systems associated with each of the zones define control systems. Indeed the operators $\left(-\frac{1}{l_e}\xi N_e I_3\frac{\partial}{\partial\chi} + \tilde{\beta_{p1}}\right)$ and $\left(-\frac{1}{L-l_e}\frac{F_{de}\xi}{\rho_0 V_{eff}}I_2\frac{\partial}{\partial\chi} + \tilde{\beta_{f1}}\right)$ generate each one a $\mathcal{C}_0$-semigroup as it may be proved using the perturbation theory of operators [11] and results developed for hyperbolic systems [12] for the homogeneous systems (21-22), i.e. $U = (\delta N\ \delta T_F)^T = 0$. It may be easily checked that these operators are closed operators and densely defined in $L_2(0,1)$ (and resp. $L_2(0_L,1)$, $\tilde{\beta}$ stands for the matrix associated to β). Indeed:

- $-\frac{1}{l_e}\xi N_e I_3$ and $\tilde{\beta_{p1}}$ are linear and bounded operators, (resp. $-\frac{1}{L-l_e}\frac{F_{de}\xi}{\rho_0 V_{eff}}I_2$ and $\tilde{\beta_{f1}}$)
- the domain of β_{p1} is dense in $L_2(0,1)$ (resp. $\tilde{\beta_{f1}}$ in $L_2(0_L,1)$),
- $-\frac{1}{l_e}\xi N_e I_3$ is invertible, (resp. $-\frac{1}{L-l_e}\frac{F_{de}\xi}{\rho_0 V_{eff}}I_2$).

So as the operator ∂_χ generates a $\mathcal{C}_0$-semigroup, then $\left(-\frac{1}{l_e}\xi N_e I_3\frac{\partial}{\partial\chi}+\tilde{\beta_{p1}}\right)$ (resp. $\left(-\frac{1}{L-l_e}\frac{F_{de}\xi}{\rho_0 V_{eff}}I_2\frac{\partial}{\partial\chi}+\tilde{\beta_{f1}}\right)$) can be viewed as a bounded perturbation (additive and multiplicative one) of the operator ∂_χ. So they still generate a $\mathcal{C}_0$-semigroup [11], [12]. Furthermore the input maps are linear and bounded, hence the systems (21) and (22) define control systems for which the solutions are well-defined.

D. The moving boundary $l(t)$

The linearized dynamics of the moving boundary is defined by replacing (24) into (23) and one obtains the control system:

$$\frac{d(\delta l)}{dt}=\alpha_l\delta l+\alpha_f\delta f_p+\alpha_N\delta N \tag{25}$$

The physically admissible numerical values lead to the positivity of the coefficient α_l hence to an unstable drift system. Such instability is not observed physically and as a conclusion the coupling of the models of the two zones though the interface relations is essential for the well-posedness of the complete system. This will be the topic of the next section.

V. Analysis of the linearized system of the complete 2-zones model

The proof of the existence of solutions for systems of conservation laws through some moving boundary may be analyzed in different ways for instance by considering two systems of PDE's coupled by an ODE and closing the loop after having proved the existence of solutions for the cascaded system [13]. Another approach is to consider a color function, defining the two spatial domain, and augment the state space with this function [14]. In this paper we shall follow some similar route and define a distributed variable associated with the position of the boundary:

$$\delta l(x,t)=\delta l(t).\mathbb{1}_{(0,1)}(\chi) \tag{26}$$

and belongs to the subspace of constant functions $K(0,1)$ (which is isomorphic to $\mathbb{R}$). We shall consider the complete linearized system defined by the state variables $\varphi(\chi,t)$:

$$\varphi^T=\begin{pmatrix}\delta\overline{f}_p & \delta\overline{M}_p & \delta\overline{M}_f & \delta\overline{T}_p & \delta\overline{T}_f & \delta l\end{pmatrix} \tag{27}$$

$\delta\overline{M}_p$, $\delta\overline{M}_f$, $\delta\overline{T}_p$, $\delta\overline{T}_f$ and $\delta\overline{f}_p$ are defined in $H^1(0,1)$, belonging to the state space:

$$X=H^1(0,1)^5\times K(0,1) \tag{28}$$

which is isomorphic to $(H^1(0,1))^5\times\mathbb{R}$. The homogeneous system expression is given then by:

$$\partial_t\varphi(x,t)=A(\chi)\varphi(x,t)=\big(A_1(\chi)+A_2(\chi)\big)\varphi(x,t) \tag{29}$$

The operator $A(\chi)$ can be decomposed in two operators, $A_2(\chi)$ a bounded operator, and $A_1:D(A_1)\subset X\to Y=L^2(0,1)^5\times\mathbb{R}$ composed of the differential operator ∂_χ (for more details, see [15]). Those operators are as follow:

$$A_2=\begin{pmatrix}0&0&0&0&0&0\\0&0&0&0&0&0\\0&0&0&0&0&0\\A_{2,1}&0&0&A_{2,4}&0&A^p_{2,6}\\A_{2,1}&0&0&0&A_{2,5}&A^f_{2,6}\\0&0&0&0&0&0\end{pmatrix} \tag{30}$$

$$\text{with } A_{2,1} = -\frac{\mu_p C\eta_p N_e^2}{\overline{f}_{pe}^2\rho_0 V_{eff}c_p},\ A^p_{2,6}=\beta_{p3} \tag{31}$$

$$A_{2,4}=A_{2,5} = -\frac{S_{ech}\alpha}{\rho_0 V_{eff}c_p} \tag{32}$$

$$A^f_{2,6} = \left(\beta_{f3}(\gamma_l-\gamma_f\frac{\alpha_l}{\alpha_f})+\beta_{f4}\right) \tag{33}$$

The expression of the differential operator A_1 is:

$$A_1 = \begin{pmatrix}\theta^p\partial_\chi&0&0&0&0&0\\0&\theta^p\partial_\chi&0&0&0&0\\0&0&\theta^f\partial_\chi&0&0&0\\\beta_{41}&0&0&\theta^p\partial_\chi&0&\beta_{46}\\\beta_{51}&0&0&0&\theta^f\partial_\chi&\beta_{56}\\\alpha_f\delta_1&0&0&0&0&\alpha_l\end{pmatrix}$$

$$\text{with } \theta^p = -\frac{\xi N_e}{l_e},\theta^f=-\frac{1}{L-l_e}\frac{F_{de}\xi}{\rho_0 V_{eff}} \tag{34}$$

$$\beta_{41}(\chi) = \beta_{p4}(\chi)\alpha_f\delta_1(\chi) \tag{35}$$

$$\beta_{46}(\chi) = \beta_{p4}(\chi)\alpha_l \tag{36}$$

$$\beta_{51}(\chi) = \Big(\beta_{f3}(\chi)\gamma_f+\beta_{f5}(\chi)\alpha_f\Big)\delta_1(\chi) \tag{37}$$

$$\beta_{56}(\chi) = \Big(\beta_{f3}(\chi)\gamma_f\frac{\alpha_l}{\alpha_f}+\beta_{f5}(\chi)\alpha_l\Big) \tag{38}$$

Corollary 1 ([16], Hille-Yosida): Sufficient conditions for a closed, densely defined operator on a Hilbert space to be the infinitesimal generator of $\mathcal{C}_0$-semigroup satisfying $\|T(t)\|\le e^{wt}$, $w\in\mathbb{R},\forall z\in D(A)$, are:

$$Re(\langle Az,z\rangle)\le w\|z\|^2 \quad\text{for}\quad z\in D(A) \tag{39}$$

$$Re(\langle A^*z,z\rangle)\le w\|z\|^2 \quad\text{for}\quad z\in D(A^*) \tag{40}$$

The operators A_1 satisfies the condition $\forall z\in D(A_1)$ (resp. for $D(A_1^*)$):

$$\langle A_1 z,z\rangle\le C\|z\|^2_{H^1}\ ,\langle A_1^*z,z\rangle\le C\|z\|^2_{H^1} \tag{41}$$

using Holder and the triangular inequalities. Indeed, one gets:

$$\langle A_1 z,z\rangle = \int_0^1(A_1z)^Tz \tag{42}$$

$$= \int_0^1\theta^p\partial_\chi z_1z_1+\theta^p\partial_\chi z_2z_2+\theta^f\partial_\chi z_3z_3dx$$

$$+ \int_0^1\theta^p\partial_\chi z_4z_4+\theta^f\partial_\chi z_5z_5+\alpha_l\partial_\chi z_6z_6dx$$

$$+ \int_0^1\beta_{41}z_1z_4+\beta_{46}z_6z_4+\beta_{51}z_1z_5dx$$

$$+ \int_0^1\beta_{56}z_6z_5+\alpha_f\delta_1z_1z_6dx \tag{43}$$

Each terms $\int_0^1 \theta\partial_\chi z_i z_i$ can be bounded by:

$$\int_0^1 \theta\partial_\chi z_i z_i dx \leq |\theta| \|z_i\|^2_{H^1(0,1)} \quad (44)$$

In the same way, each coupled product, like $\int_0^1 \beta_{41} z_1 z_4 dx$, can be bounded using Holder inequalities, e.g.:

$$\int_0^1 \beta_{41} z_1 z_4 dx \leq \sup_{(0,1)} |\beta_{41}| \int_0^1 z_1 z_4 dx$$
$$\leq C_{41} \left(\|z_1\|^2_{H^1(0,1)} + \|z_4\|^2_{H^1(0,1)} \right) \quad (45)$$

The same is done for $\int_0^1 \alpha_f \delta_1 z_1 z_6 dx$ recalling that

$$\delta_1 z_1 = z_1(1) = \int_0^1 z_1' dx$$

and that $\|z_1\|^2_{H^1(0,1)} = \|z_1\|^2_{L_2(0,1)} + \|z_1'\|^2_{L_2(0,1)}$. So there exists a constant $C = \sup(|\theta^p|, |\theta^f|, C_{ij})$ such that

$$\langle A_1 z, z \rangle \leq C \left(\sum_1^6 \|z_i\|^2_{H^1(0,1)} \right) = C\|z\|^2_{H^1(0,1)} \quad (46)$$

The same is done with the adjoint operator $A_1^* : Y^* \to X^*$:

$$A_1^* = \quad (47)$$
$$\begin{pmatrix} -\theta^p\partial_\chi & 0 & 0 & \beta_{41} & \beta_{51} & \alpha_f\delta_1 \\ 0 & -\theta^p\partial_\chi & 0 & 0 & 0 & 0 \\ 0 & 0 & -\theta^f\partial_\chi & 0 & 0 & 0 \\ 0 & 0 & 0 & -\theta^p\partial_\chi & 0 & 0 \\ 0 & 0 & 0 & 0 & -\theta^f\partial_\chi & 0 \\ 0 & 0 & 0 & \beta_{46} & \beta_{56} & \alpha_l \end{pmatrix}$$

and one gets the same constant to bound $\langle A_1^* z, z \rangle$:

$$\langle A_1^* z, z \rangle \leq C \left(\sum_1^6 \|z_i\|^2_{H^1(0,1)} \right) = C\|z\|^2_{H^1(0,1)} \quad (48)$$

and A_1 is the infinitesimal generator of a $\mathcal{C}_0$-semigroup.
All the more, the bounded operator A_2 is a bounded additive perturbation of the operator A_1:

Theorem 1 ([17]): Let X a Banach space and let A the infinitesimal generator of a C_0-semigroup $T(t)$ on X such that $\|T(t)\| \leq Me^{wt}$. If B is a bounded linear operator on X then $A + B$ is infinitesimal generator of a C_0-semigroup $S(t)$ on X such that $\|S(t)\| \leq Me^{(w+M\|B\|)t}$. □

So $A = A_1 + A_2$ still generates a $\mathcal{C}_0$-semigroup $T(t)$ which satisfies $\|T(t)\| \leq e^{(w+\|A_2\|)t}$. The system (29) is well posed [11], [12].
Still using the same results, the system with the control $U(t)$ still generates a $\mathcal{C}_0$-semigroup if bounded inputs are considered and can be viewed as bounded perturbations.

VI. Conclusion

In this paper, a model of an extruder is proposed, which takes into account the moving boundary between the partially and the fully filled zone. The complexity of this system of coupled PDEs and ODE comes from the mobility of the internal interface $l(t)$. A change of space coordinate to define fixed spatial coordinates is developed, and the linearized system is written in those new coordinates. The well posedness of those equations is proved for the coupled systems in the homogeneous case. The system with the control $U(t)$ still generates a $\mathcal{C}_0$-semigroup considering that the variations $(\delta N \ \delta T_F)$ are bounded ones.
The stability problem can then be discussed, noting that if the w of the corollary 1 is negative, then the system is exponentially stable. The majorations realized for the well posedness have to be more precise in order to get $w < 0$.

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