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Observability and reachability of grid graphs via reduction and symmetries

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Abstract— **In this paper we investigate the observability and reachability properties of a network system, running a Laplacian based average consensus algorithm, when the communication graph is a grid. More in detail, we characterize the structure of the grid eigenvectors by means of suitable decompositions of the graph. For each eigenvalue, based on its multiplicity and on suitable symmetries of the corresponding eigenvectors, we provide necessary and sufficient conditions to characterize all and only the nodes from which the network system is observable (reachable). We discuss the proposed criteria and show, through suitable examples, how such criteria reduce the complexity of the observability (respectively reachability) analysis of the grid.**

I. INTRODUCTION

Distributed computation in network control systems has received great attention in the last years. One of the most studied problems is *average consensus*. Given a network of processors, the objective is to compute the average of the initial states by performing local computation and exchanging local information. A survey on these algorithms and their performance may be found e.g. in [1] and references therein. We are interested in studying reachability and observability of a network system running average consensus, when only a subset of nodes is controlled by an external input or measured by an external sensor.

In this paper we will concentrate on a network system with fixed undirected communication graph topology running a Laplacian based average consensus algorithm. The dynamical system arising from a consensus network with fixed topology is a linear time-invariant system and the problem of understanding if the network state may be reconstructed is an observability problem. Observability and reachability are dual problems in linear systems theory and can be studied using the same tools. However, in the literature the reachability (controllability) point of view is the one that has received more attention.

The reachability (controllability) problem for a leader-follower network was introduced in [2] for a single control node. Intensive simulations were provided showing that it is "unlikely" for a Laplacian based consensus network to be completely controllable. In [3], see also [4], "necessary and sufficient" conditions were provided to characterize the reachability and observability of path and cycle graphs in terms of simple rules from number theory. In [5] and [6], see also [7], necessary conditions for controllability, based on suitable properties of the graph, have been provided. Other contributions on the controllability of network systems can be found in [8], [9], [10]. Observability has been studied for the first time in [11], where necessary conditions for observability, as in the dual reachability setting investigated in [5] and [6], were provided. A parallel research line investigates a slightly different property called *structural observability* [12]. Here, the objective is to choose the nonzero entries of the consensus matrix (i.e. the state matrix of the resulting network system) in order to obtain observability from a given set of nodes. It is worth noting that observability and reachability of a network system are necessary structural properties in many network problems as estimation, intrusion detection and formation problems, [12], [13], [14], [15].

In this paper we extend partial results provided in [16] on the observability and reachability of grid graphs. In [16] we analyzed simple grids, i.e. grids whose eigenvalues have multiplicity one. Here we study arbitrary grids. The contribution of the paper is twofold. First, we characterize the structure of the Laplacian eigenvectors of a grid. Namely, we show that, on the basis of a prime number factorization of the grid dimensions, the eigenvector components present symmetries related to suitable subgrid partitions of the main grid. Also, in each subgrid, the eigenvector components show different symmetries depending on the symmetries of the path eigenvectors that generate the eigenspace.

Second, we provide necessary and sufficient conditions that completely characterize the observability (reachability) of grid graphs. More in detail, on the basis of the node labels, suitable polynomial evaluations and eigenvector symmetries, we are able to: (i) identify all and only the observable (reachable) nodes of the graph, (ii) say if the graph is observable (reachable) from a given set of nodes and (iii) construct a set of observation (leader) nodes from which the graph is observable (reachable).

The paper is organized as follows. In Section II we introduce some preliminary definitions and properties of undirected graphs, describe the network model used in the paper and set up the observability and reachability problems. In Section III we investigate suitable symmetries of the path eigenvectors that are at the basis of the new results on grid graphs. In Section IV we analyze the symmetries in the structure of the grid graph eigenvectors and, on this basis, we provide necessary and sufficient conditions for the observability (reachability) of the graph. For space constrains all proofs are omitted in this paper and will be provided in a forthcoming document.

Notation: We let $\mathbb{N}$, $\mathbb{N}_0$, the $\mathbb{R}_{>0}$ and $\mathbb{R}_{\geq 0}$ denote the natural numbers, the non-negative integer numbers, positive real numbers and the non-negative real numbers, respectively. We denote 0_d, $d \in \mathbb{N}$, the vector of dimension d with zero components and $0_{d_1 \times d_2}$, $d_1, d_2 \in \mathbb{N}$, the matrix with

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d_1 rows and d_2 columns with zero entries. For $i \in \mathbb{N}$ we let e_i be the i-th element of the canonical basis, e.g. $e_1 = [1 \ 0 \ \dots \ 0]^T$. For a matrix $A \in \mathbb{R}^{d_1 \times d_2}$ we denote $[A]_{ij}$ the $(i,j)th$ element and $[A]_i$ the ith column of A. For a vector $v \in \mathbb{R}^d$ we denote $(v)_i$ the ith component of v so that $v = [(v)_1 \dots (v)_d]^T$. Also, we denote $\Pi \in \mathbb{R}^{d \times d}$ the permutation matrix reversing all the components of v so that $\Pi v = [(v)_d \dots (v)_1]^T$ (the j-th column of Π is $[\Pi]_j = e_{n-j+1}$).

II. Preliminaries and problem set-up

In this section we present some preliminary terminology on graph theory, introduce the network model, set up the observability problem and provide some standard results on observability of linear systems that will be useful to prove the main results of the paper.

A. Preliminaries on graph theory

Let $G = (I, E)$ be a static undirected graph with set of nodes $I = \{1, \dots, n\}$ and set of edges $E \subset I \times I$. We denote $\mathcal{N}_i$ the set of neighbors of agent i, that is, $\mathcal{N}_i = \{j \in I \mid (i,j) \in E\}$, and $\mathrm{d}_i = \sum_{j \in \mathcal{N}_i} 1$ the degree of node i. The maximum degree of the graph is defined as $\Delta = \max_{i \in I} \mathrm{d}_i$. The degree matrix D of the graph G is the diagonal matrix defined as $[D]_{ii} = \mathrm{d}_i$. The adjacency matrix $A \in \mathbb{R}^{n \times n}$ associated to the graph G is defined as

$$[A]_{ij} = \begin{cases} 1 & \text{if } (i,j) \in E \\ 0 & \text{otherwise.} \end{cases}$$

The Laplacian L of G is defined as $L = D - A$. The Laplacian is a symmetric positive semidefinite matrix with k eigenvalues in 0, where k is the number of connected components of G. If the graph is connected the eigenvector associated to the eigenvalue 0 is the vector $\mathbf{1} = [1 \ \dots \ 1]^T$.

Next, we introduce the notion of cartesian product of graphs. Let $G = (I, E)$ and $G' = (I', E')$ be two undirected graphs. The cartesian product $G \,\square\, G'$ is a graph with vertex set $I \times I'$ (i.e. the cartesian product of the two vertex sets) and edge set defined as follows. Nodes $[i, i'] \in I \times I'$ and $[k, k'] \in I \times I'$ are adjacent in $G \,\square\, G'$ if either $i = k$ and $(i', k') \in E'$ or $i' = k'$ and $(i, k) \in E$. The cartesian product is commutative and associative. Thus, a $d \in \mathbb{N}$ dimensional product graph, $\prod_{\ell=1}^{d} G_\ell$, is constructed by combining the above definition with the associative property.

We introduce the special graphs that will be of interest in the rest of the paper. A *path graph* is a graph in which there are only nodes of degree two except for two nodes of degree one. The nodes of degree one are called external nodes, while the other are called internal nodes. From now on, without loss of generality, we will label the external nodes with 1 and n, and the internal nodes so that the edge set is $E = \{(i, i+1) \mid i \in \{1, \dots, n-1\}\}$. Since it will be extensively used in the rest of the paper, we provide the explicit expression of the path Laplacian, L_n,

$$L_n = \begin{bmatrix} 1 & -1 & 0 & \dots & 0 \\ -1 & 2 & -1 & \dots & 0 \\ \vdots & & \ddots & & \\ 0 & & -1 & 2 & -1 \\ 0 & & 0 & -1 & 1 \end{bmatrix} \qquad (1)$$

A d-dimensional *grid graph* is the cartesian product of d paths (of possibly different length). In a grid graphs the nodes have degree from d up to $2d$. We call the nodes with degree d *corner nodes*. Corner nodes are obtained from the product of external nodes in the paths.

B. Observability and reachability in a network of agents running average consensus

We consider a collection of agents labeled by a set of identifiers $I = \{1, \dots, n\}$, where $n \in \mathbb{N}$ is the number of agents. We assume that the agents communicate according to a *time-invariant undirected* communication graph $G = (I, E)$, where $E = \{(i,j) \in I \times I \mid i \text{ and } j \text{ communicate}\}$. The agents run a consensus algorithm based on a Laplacian control law (see e.g. [1] for a survey). The dynamics of the agents evolve in continuous time ($t \in \mathbb{R}_{\geq 0}$) and are given by

$$\dot{x}_i(t) = -\sum_{j \in \mathcal{N}_i} (x_i(t) - x_j(t)), \ i \in \{1, \dots, n\}.$$

Using a compact notation the dynamics may be written as

$$\dot{x}(t) = -Lx(t), \ t \in \mathbb{R}_{\geq 0},$$

where $x = [(x)_1 \dots (x)_n]^T = [x_1 \dots x_n]^T$ is the vector of the agents' states and L is the graph Laplacian.

Remark 2.1 (Discrete time system): The observability analysis performed in the paper can be easily extended to suitable discrete time versions of the above continuous time model, see, e.g., [4].

Next, we describe the scenario that motivates our observability analysis. We imagine that an external processor (*not* running the consensus algorithm) collects information from some nodes in the network. We call these nodes *observation nodes*. In particular, we assume that the external processor may read the state of each observation node. Equivalently, we can think of one or more observation nodes, running the consensus algorithm, that have to reconstruct the state of the network by processing only their own state. We can model these two scenarios with the following mathematical framework. For each observation node $i \in I$, we have the following output

$$y_i(t) = x_i(t).$$

Therefore the output matrix is $C_i = \begin{bmatrix} e_i^T \end{bmatrix}$.

If the set of observation nodes I_o in the network has cardinality greater than one, say $I_o = \{i_1, \dots, i_p\} \subset \{1, \dots, n\}$, then the output is $y_{I_o}(t) = \begin{bmatrix} x_{i_1}(t) & x_{i_2}(t) & \dots & x_{i_p}(t) \end{bmatrix}^T$. Therefore, the output matrix is $C_{I_o} = \begin{bmatrix} e_{i_1} \mid & \dots & \mid e_{i_p} \end{bmatrix}^T$. It is a well known result in linear systems theory that the observability properties of the pair (L, C_{I_o}) correspond to the controllability properties of the pair $(L^T, C_{I_o}^T) = (L, C_{I_o}^T)$. The associated dual network system is

$$\dot{x}(t) = -Lx(t) + C_{I_o}^T u(t), \qquad (2)$$

where $u \in \mathbb{R}^p$ is the input vector. It follows easily that each component $(u)_\nu$ fully controls the dynamics of the i_ν-th node, so that this turns to be the model of a leader-follower network. Thus, our results apply also to the controllability problem in a leader-follower network, where the observation nodes correspond to the leader nodes. For the sake of space, from now on we will concentrate on the observability.

Remark 2.2: Straightforward results from linear system theory can be also used to prove that the controllability problem studied in [5] and [6] and the dual observability problem studied in [11] can be equivalently formulated in our set up. □

C. Standard results on observability of linear systems

The observability problem consists of looking for nonzero values of $x(0)$ that produce an identically zero output $y(t)$.

An important result on the reachability (observability) of time-invariant linear systems is the Popov-Belevich-Hautus (PBH) lemma, e.g. [18].

Combining the PBH lemma with the fact that the state matrix is symmetric (and therefore diagonalizable) the following corollary may be proven.

Corollary 2.3: Let X_{no} be the unobservable subspace associated to the pair (L, C), where L is a symmetric matrix. Then X_{no} is spanned by vectors v_l satisfying, for $\lambda \in \mathbb{R}$,

$$\begin{aligned} Cv_l &= 0_p \\ Lv_l &= \lambda v_l. \end{aligned}$$

In the rest of the paper we will call *unobservable eigenvalues and eigenvectors* the eigenvalues and eigenvectors for which (2.3) is satisfied.

III. Symmetries of the path Laplacian eigenvectors

In this section we provide results on the structure and symmetries of the Laplacian eigenvectors of a path graph that will be used to study the observability of grid graphs via suitable, symmetry based, subgrid partition. Next lemma characterizes the symmetry of the path Laplacian eigenvectors.

Lemma 3.1 (Symmetry of the path Laplacian eigenvectors): Any eigenvector v of the Laplacian of a path graph satisfies either $v = \Pi v$ or $v = -\Pi v$, where Π is the usual permutation matrix.

In the rest of the paper we will denote S^+ (respectively S^-) the set of vectors satisfying $v = \Pi v$ (respectively $v = -\Pi v$). An important property of S^+ and S^- is that each one is the orthogonal complement of the other, i.e. $(S^+)^\perp = S^-$.

Next lemma relates the eigenstructure of a given path P to the eigenstructure of any path with length multiple of the length of P.

Lemma 3.2 (Laplacian eigenstructure of P_n and P_{kn}): Let $\lambda_1, \ldots, \lambda_n$ be the eigenvalues of the Laplacian L_n of length n and $v_1, \ldots, v_n$ the corresponding eigenvectors. Then any path of length kn, for some $k \in \mathbb{N}$, with Laplacian matrix L_{kn} satisfies:

(i) $\lambda_1, \ldots, \lambda_n$ are eigenvalues of L_{kn};

(ii) each eigenvector $w_i \in \mathbb{R}^{kn}$ of L_{kn} associated to λ_i, $i \in \{1, \ldots, n\}$, has the form

$$w_i = \begin{bmatrix} v_i \\ \Pi v_i \\ v_i \\ \vdots \end{bmatrix}.$$

Exploiting the result in the above lemma by using the result in Lemma 3.1, it follows easily that $w_i = \begin{bmatrix} v^T & v^T & v^T & \ldots \end{bmatrix}^T$ for $v = \Pi v$ (and thus $w_i = \Pi w_i$) and $w_i = \begin{bmatrix} v^T & -v^T & v^T & \ldots \end{bmatrix}^T$ for $v = -\Pi v$ (and thus $w_i = -\Pi w_i$).

IV. Observability of grid graphs

In this section we give the main results of the paper on the observability (reachability) of grid graphs.

First, we introduce some useful notation. Given a d-dimensional grid graph $G = P_1 \square \ldots P_d$, we denote $i = [(i)_1, \ldots, (i)_d]$ a node of G, where the component $(i)_\kappa$ identifies the position of the node on the κth path. Also, given a Laplacian eigenvector of G, $w \in \mathbb{R}^{n_1 \cdots n_d}$, we say "the component $[(i)_1, \ldots, (i)_d]$ of w" meaning "the component $(i)_1 \cdot (n_1 \cdot n_2 \cdot \ldots \cdot n_d) + (i)_2 \cdot (n_2 \cdot \ldots \cdot n_d) \ldots + (i)_d$ of w".

For the sake of clarity we provide the analysis and results for two dimensional grids ($d = 2$). The results for higher dimensions are based on similar arguments and are omitted for the sake of space.

A. Laplacian eigenstructure of cartesian-product graphs

An important property of graphs obtained as the cartesian product of other graphs is that the Laplacian can be obtained from the Laplacian of their constitutive graphs by using the Kronecker sum of two matrices, see [17]. Given two matrices $A \in \mathbb{R}^{d \times d}$ and $B \in \mathbb{R}^{l \times l}$, with $[A]_{ij} := a_{ij}$, their Kronecker product $A \otimes B \in \mathbb{R}^{dl \times dl}$ is defined as

$$A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \ldots & a_{1d}B \\ a_{21}B & a_{22}B & \ldots & a_{2d}B \\ \vdots & \ldots & & \vdots \\ a_{d1}B & a_{12}B & \ldots & a_{dd}B \end{bmatrix},$$

and their Kronecker sum as

$$A \oplus B = A \otimes I_l + I_d \otimes B.$$

Given the cartesian product of the graphs $G_1, \ldots, G_d$ with Laplacian matrices $L_1, \ldots, L_d$, the Laplacian $L_\square$ of $G_1 \square \ldots \square G_d$ is given by $L_\square = L_1 \oplus \ldots \oplus L_d$. This structure on the Laplacian induces a structure also on its eigenvalues and eigenvectors. We state it in the next lemma, see [17].

Lemma 4.1: (Laplacian eigenstructure of cartesian product graphs) Let $G_1, \ldots, G_d$ be $d \in \mathbb{N}$ undirected graphs and $G = G_1 \square \ldots \square G_d$ their cartesian product. Let $\lambda_1^\kappa, \ldots, \lambda_{n_\kappa}^\kappa$ be the Laplacian eigenvalues of the graphs G_κ and $v_1^\kappa, \ldots, v_{n_\kappa}^\kappa$ the corresponding eigenvectors for $\kappa \in \{1, \ldots, d\}$. The Laplacian eigenvalues and their corresponding eigenvectors of G are

$$\lambda_{i_1} + \lambda_{i_2} + \ldots + \lambda_{i_d} \quad \text{and} \quad v_{i_1} \otimes v_{i_2} \otimes \ldots \otimes v_{i_d},$$

for $i_1 \in \{1, \ldots, n_1\}, \ldots, i_d \in \{1, \ldots, n_d\}$. □

Next, we define a simple cartesian product graph.

Definition 4.2 (Simple cartesian-product graphs): Let G and G' be two undirected graphs and let $\{\lambda_1, \ldots, \lambda_k\}$ and $\{\lambda'_1, \ldots, \lambda'_\kappa\}$ be the sets of distinct eigenvalues among all the Laplacian eigenvalues of respectively G and G'. We say that the graph $G_\square = G \square G'$ is *simple* if the set $\{\lambda_i + \lambda'_\alpha \mid i \in \{1, \ldots, k\}, \alpha \in \{1, \ldots, \kappa\}\}$ contains only distinct eigenvalues. □

B. Symmetries of the grid eigenvectors

Next, we provide tools to recognize symmetries in the grid eigenvectors, based on the graph structure, which will play a key role in the observability analysis.

Without loss of generality, let $\lambda = \lambda_{1,1} + \lambda_{1,2} = \ldots = \lambda_{\mu,1} + \lambda_{\mu,2}$ be an eigenvalue of geometric multiplicity $\mu \in \mathbb{N}$, with $\lambda_{1,1}, \ldots, \lambda_{\mu,1}$ (respectively $\lambda_{1,2}, \ldots, \lambda_{\mu,2}$) eigenvalues of P_1 (respectively P_2) and corresponding eigenvectors $v_1, \ldots v_\mu$ (respectively $w_1, \ldots, w_\mu$). The corresponding eigenspace V_λ is given by

$$V_\lambda = \{v \in \mathbb{R}^{n_1 \cdot n_2} | v = \sum_{i=1}^{\mu} \alpha_i (v_i \otimes w_i), \alpha_i \in \mathbb{R}\}. \qquad (3)$$

Before stating the main results of this section, we need to introduce some useful notation. Given a path P_n of length $n \in \mathbb{N}$, for $\{i, j\} \subset \{1, \ldots, n\}$, $i < j$, we denote $P_{i:j}$ the sub-path of P_n with node set $\{i, \ldots, j\}$ (e.g., $P_{2:4}$ is the sub-path with node set {2,3,4}). Let $G = P_{l \cdot n_1} \Box P_{m \cdot n_2}$ with $P_{l \cdot n_1}$ of dimension $l \cdot n_1$ and $P_{m \cdot n_2}$ of dimension $m \cdot n_2$. We call $G_{ij} = P_{((i-1)n_1+1):(in_1)} \Box P_{((j-1)n_2+1):(jn_2)}$, for $i \in \{1, \ldots, l\}$ and $j \in \{1, \ldots, m\}$, an $n_1 \times n_2$ subgrid of G, see Figure 1.

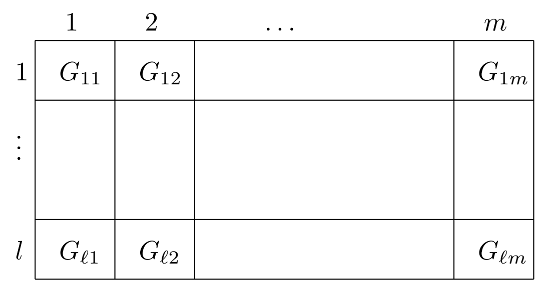


Fig. 1: Partition of a grid into subgrids

Let $v \in \mathbb{R}^{l \cdot n_1 \cdot m \cdot n_2}$ be a vector of G, we call the *subvector of* v *associated to* G_{ij} the vector $v_{ij} \in \mathbb{R}^{n_1 \cdot n_2}$ with components $(v_{ij})_{[\nu, \ell]}$, $\nu \in \{1, \ldots, n_1\}$ and $\ell \in \{1, \ldots, n_2\}$, given by $(v_{ij})_{[\nu, \ell]} = (v)_{[(i\text{-}1)n_1+\nu,\, (j\text{-}1)n_2+\ell]}$.

Informally, the subvector v_{ij} of v is constructed by selecting the components of v that fall into the subgrid G_{ij}.

Next, given a grid $G = P_{n_1} \Box P_{n_2}$, with P_{n_1} and P_{n_2} paths of length n_1 and n_2 respectively, we introduce two useful operators that flip the components of a vector v associated to a grid G. Formally, consider the matrices

$$(\Pi_{n_1} \otimes I_{n_2}) = \begin{bmatrix} 0_{n_2 \times n_2} & 0_{n_2 \times n_2} & \ldots & 0_{n_2 \times n_2} & I_{n_2} \\ 0_{n_2 \times n_2} & 0_{n_2 \times n_2} & \ldots & I_{n_2} & 0_{n_2 \times n_2} \\ & & \iddots & & \\ 0_{n_2 \times n_2} & I_{n_2} & \ldots & 0_{n_2 \times n_2} & 0_{n_2 \times n_2} \\ I_{n_2} & 0_{n_2 \times n_2} & \ldots & 0_{n_2 \times n_2} & 0_{n_2 \times n_2} \end{bmatrix}$$

and

$$(I_{n_1} \otimes \Pi_{n_2}) = \begin{bmatrix} \Pi_{n_2} & 0_{n_2 \times n_2} & \ldots & 0_{n_2 \times n_2} & 0_{n_2 \times n_2} \\ 0_{n_2 \times n_2} & \Pi_{n_2} & \ldots & 0_{n_2 \times n_2} & 0_{n_2 \times n_2} \\ & & \ddots & & \\ 0_{n_2 \times n_2} & 0_{n_2 \times n_2} & \ldots & \Pi_{n_2} & 0_{n_2 \times n_2} \\ 0_{n_2 \times n_2} & 0_{n_2 \times n_2} & \ldots & 0_{n_2 \times n_2} & \Pi_{n_2} \end{bmatrix}.$$

Given a vector $v \in \mathbb{R}^{n_1 \cdot n_2}$ associated to the grid G, with components $(v)_{[\nu, \ell]}$, $\nu \in \{1, \ldots, n_1\}$ and $\ell \in \{1, \ldots, n_2\}$, let $v_1 = (\Pi_{n_1} \otimes I_{n_2})v$ and $v_2 = (I_{n_1} \otimes \Pi_{n_2})v$. The vectors v_1 and v_2 are related to v by

$$(v_1)_{[\nu,\, \ell]} = (v)_{[n_1 - \nu + 1,\, \ell]},$$

and

$$(v_2)_{[\nu,\, \ell]} = (v)_{[\nu,\, n_2 - \ell + 1]},$$

for $\nu \in \{1, \ldots, n_1\}$ and $\ell \in \{1, \ldots, n_2\}$. Finally, the composition of the two operators satisfies $(\Pi_{n_1} \otimes I_{n_2})(I_{n_1} \otimes \Pi_{n_2}) = (\Pi_{n_1} \otimes \Pi_{n_2})$. Thus, when applied to a vector v, the composed operator flips both components. That is, denoting $v_3 = (\Pi_{n_1} \otimes \Pi_{n_2})v$, we have

$$(v_3)_{[\nu,\, \ell]} = (v)_{[n_1 - \nu + 1,\, n_2 - \ell + 1]},$$

for $\nu \in \{1, \ldots, n_1\}$ and $\ell \in \{1, \ldots, n_2\}$.

Lemma 4.3: Let $G_0 = P_{n_1} \Box P_{n_2}$ with P_{n_1} and P_{n_2} paths of length respectively n_1 and n_2. Any eigenvalue λ of the Laplacian L_0 of G_0 is an eigenvalue of the Laplacian L of $G = P_{l \cdot n_1} \Box P_{m \cdot n_2}$ for any $l \in \mathbb{N}$ and $m \in \mathbb{N}$.

We are now ready to characterize the eigenvector symmetries by suitable subgrid partitions.

Theorem 4.4: Let $G_0 = P_{n_1} \Box P_{n_2}$ be a grid of dimension $n_1 \times n_2$ with P_{n_1} and P_{n_2} paths of dimension respectively n_1 and n_2. Take any grid $G = P_{l \cdot n_1} \Box P_{m \cdot n_2}$ of dimension $ln_1 \times mn_2$ and let G_{ij}, $i \in \{1, \ldots, l\}$ and $j \in \{1, \ldots, m\}$, be a partition into subgrids of dimension $n_1 \times n_2$.

Then for each eigenvalue (possibly non-simple) common to L and L_0, any associated eigevector v of L can be decomposed into subvectors v_{ij} relative to the subgrids G_{ij} where

$$v_{ij} = (\Pi_{n_1} \otimes I_{n_2})^{(i-1)} (I_{n_1} \otimes \Pi_{n_2})^{(j-1)} v_0$$

for $i \in \{1, \ldots, n_1\}$ and $j \in \{1, \ldots, n_2\}$, where v_0 is an eigenvector of L_0 associated to λ.

The above theorem has a nice and intuitive graphical interpretation, as shown in Figure 2. Given a grid G and an eigenvector v associated to an eigenvalue λ, we can associate a symbol to each node depending on the value of the eigenvector component. Next, we partition the grid G into subgrids of dimension $n_1 \times n_2$. Given the symbols in the subgrid G_{11}, the symbols in a subgrid $G_{i,j}$, for $i \in \{1, \ldots, l\}$ and $j \in \{2, \ldots, m\}$, are obtained by a reflection of the subgrid $G_{i,j-1}$ with respect to the horizontal axis, while the symbols in a subgrid $G_{i,j}$, for $i \in \{2, \ldots, l\}$ and $j \in \{1, \ldots, m\}$, are obtained by a reflection of the subrgid $G_{i-1,j}$ with respect of the vertical axis.

Next, we analyze the eigenvector components of a subgrid whose dimensions are prime, or equivalently the components

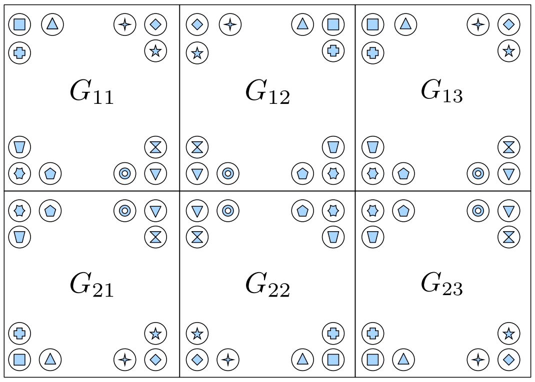


Fig. 2: Graphical interpretation of Theorem 4.4

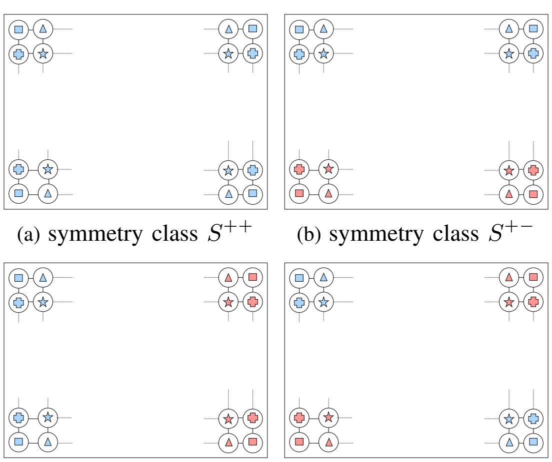
of eigenvectors associated to eigenvalues that are not eigenvalues of smaller subgrids.

Proposition 4.5: Let $G_0 = P_{n_1} \square P_{n_2}$ be a grid of dimension $n_1 \times n_2$. For any eigenvalue λ, let V_λ be the associated eigenspace, with structure as in equation (3). Then, each eigenvector of the basis, $(v_i \otimes w_i)$, satisfies one of the four relations:

$$\begin{aligned}(v_i \otimes w_i) &= (\Pi \otimes I)(v_i \otimes w_i) = (I \otimes \Pi)(v_i \otimes w_i)\\ (v_i \otimes w_i) &= (\Pi \otimes I)(v_i \otimes w_i) = -(I \otimes \Pi)(v_i \otimes w_i)\\ (v_i \otimes w_i) &= -(\Pi \otimes I)(v_i \otimes w_i) = (I \otimes \Pi)(v_i \otimes w_i)\\ (v_i \otimes w_i) &= -(\Pi \otimes I)(v_i \otimes w_i) = -(I \otimes \Pi)(v_i \otimes w_i).\end{aligned}$$

In the following we denote the set of vectors satisfying each one of the four relations in the proposition respectively as S^{++}, S^{+-}, S^{-+} and S^{--}.

The result in Proposition 4.5 can be easily explained by using a graphical interpretation. We associate a symbol to each node depending on the value of the eigenvector component. Also, we denote with the same symbol but different colors, nodes that have components of opposite sign. Each of the four cases in the proposition correspond to a scheme in Figure 3.



(a) symmetry class S^{++} (b) symmetry class S^{+-}

(c) symmetry class S^{-+} (d) symmetry class S^{--}

Fig. 3: Graphical interpretation of Proposition 4.5

This proposition has an important impact on the symmetries of general eigenvectors belonging to the same eigenspace, when the dimensions n_1 and n_2 are prime numbers (and thus for each subgrid of a general grid). Clearly, any eigenvector of V_λ can be written as a linear combination of the basis vectors, and thus, using the proposition, by the sum of at most four vectors each one having one of the four symmetries. Thus, in order to identify the symmetries of a general vector, we just need to identify nodes with the same symbol and color in different classes. If basis vectors of at least three different classes are present, by inspection in Figure 3, no symmetries are present. On the contrary, if all basis vectors belong to the same class, then also the linear combination does. Interesting symmetries arise from the linear combination of basis vectors belonging to two of the four classes. Namely, a general eigenvector v satisfies:

- $v_{(\nu,\ell)} = v_{(n_1-\nu+1,\ell)}$ if the two classes share the first symbol (e.g., S^{++} and S^{+-});
- $v_{(\nu,\ell)} = v_{(\nu,n_2-\ell+1)}$ if the two classes share the second symbol (e.g., S^{++} and S^{-+});
- $v_{(\nu,\ell)} = v_{(n_1-\nu+1,n_2-\ell+1)}$ if the two classes do not share any symbol (e.g., S^{++} and S^{--});

C. Observability analysis

In this section we provide necessary and sufficient condition to characterize all and only the nodes from which the network system is observable. First, we need a well known result in linear systems theory, see, e.g., [18].

Lemma 4.6: If a state matrix $A \in \mathbb{R}^{n\times n}$, $n \in \mathbb{N}$, has an eigenvalue with geometric multiplicity $\mu > p$, then for any $C \in \mathbb{R}^{p\times n}$ the pair (A, C) is unobservable. □

The previous lemma applied to the grid Laplacian says that, in case the grid is non simple with maximum eigenvalue multiplicity μ, then the grid is not observable from a set of observation nodes of cardinality less than μ.

Using Corollary 2.3, it follows straight that we can study the observability properties of the grid separately for each eigenvalue. Namely, to guarantee observability, we need to show that for each eigenvalue of the grid Laplacian L, there does not exists any eigenvector satisfying the condition in (2.3), i.e. having zero in some components.

If λ is simple, the corresponding eigenspace V_λ in (3) is given by $V_\lambda = \{v \in \mathbb{R}^{n_1\cdot n_2} | v = \alpha_1(v_1 \otimes w_1), \alpha_1 \in \mathbb{R}\}$. Thus, finding the zeros of any eigenvector in V_λ is equivalent to finding the zeros of the eigenvectors v_1 and w_1 and replicate them according to the Kronecker product structure. Clearly, with this observation in hand, the analysis of any simple eigenvalue can be performed by using the tools for simple grid graphs developed in [16].

For eigenvalues with multiplicity greater than one, next two considerations are important. First, not all the eigenvectors of λ have the structure of a Kronecker product. Second, consistently with Lemma 4.6, it is always possible to find an eigenvector $v \in V_\lambda$ with an arbitrary component equal to zero, for a suitable choice of the coefficients α_i in (3). Thus, the observability analysis does not depend only on the zero components of the path eigenvectors, but also on the symmetries in the grid eigenvector components. That is, for the eigenvalue under investigation, we want to answer to the following question. If we find an eigenvector with zero in an arbitrary component ℓ, what are the other components that are zero in the chosen eigenvector? We provide the analysis for non-simple eigenvalues of multiplicity two, leaving the generalization to a discussion.

On the basis of the eigenvector symmetries identified in Theorem 4.4, we can study the observability of a subgrid with prime dimensions. In particular, we concentrate our analysis on the sector of components $[\nu, \ell]$ with $\nu \in \{1, \dots, \frac{n_1-1}{2}\}$ and $\ell \in \{1, \dots, \frac{n_2-1}{2}\}$.

Next lemma provides useful properties of the eigenvector components in a subgrid with prime length dimensions.

Lemma 4.7: Let $G_0 = P_{n_1} \square P_{n_2}$ be a grid of dimension $n_1 \times n_2$. Then, any Laplacian eigenvector $u = v \otimes w$ of the grid, with v and w respectively eigenvectors of P_{n_1} and P_{n_2} associated to eigenvalues λ_v and λ_w, has components $u_{[\nu,\ell]}$, $\nu \in \{1, \dots, n_1\}$ and $\ell \in \{1, \dots, n_2\}$ satisfying

(i) $u_{[\nu,\ell]} = p_\nu(\lambda_v) \cdot p_\ell(\lambda_w) \cdot (v)_1 \cdot (w)_1$, where $p_r(s)$ is the polynomial of degree $(r-1)$ defined as $p_2(s) = 1 - s$ for $r = 2$ and, denoting $p_1(s) = 1$, by the recursion

$$p_r(s) = (2 - s) p_{r-1}(s) - p_{r-2}(s) \tag{4}$$

for $r \geq 3$;

(ii) if n_1 and n_2 prime, then $p_\nu(\lambda_v) \neq 0$ and $p_\ell(\lambda_w) \neq 0$ for any $\nu \in \{1, \dots, \frac{n_1-1}{2}\}$ and $\ell \in \{1, \dots, \frac{n_2-1}{2}\}$.

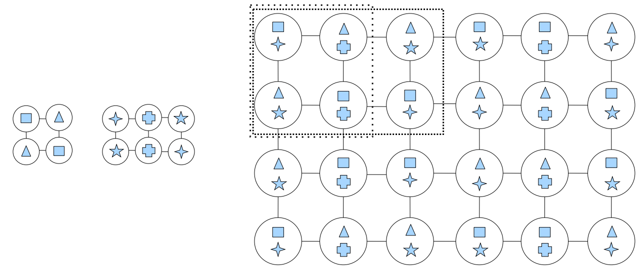
Next theorem gives necessary and sufficient conditions for two eigenvector components to be both zero in a subgrid with prime dimensions.

Theorem 4.8: Let $G_0 = P_{n_1} \square P_{n_2}$ be a grid of dimension $n_1 \times n_2$ with n_1 and n_2 prime. Let $\lambda = \lambda_{1,1} + \lambda_{1,2} = \lambda_{2,1} + \lambda_{2,2}$ be an eigenvalue of multiplicity two, with $\lambda_{1,1}$ and $\lambda_{2,1}$ ($\lambda_{1,2}$ and $\lambda_{2,2}$) eigenvalues of P_{n_1} (P_{n_2}). Let V_λ be the associated eigenspace. Then there exists an eigenvector $v \in V_\lambda$ with zero components $[\nu_1, \ell_1]$ and $[\nu_2, \ell_2]$, $\nu_1, \nu_2 \in \{1, \dots, \frac{n_1-1}{2}\}$ and $\ell_1, \ell_2 \in \{1, \dots, \frac{n_2-1}{2}\}$, if and only if

$$\frac{p_{\nu_2}(s)}{p_{\nu_1}(s)}\Big|_{s=\lambda_{1,1}} \cdot \frac{p_{\ell_2}(s)}{p_{\ell_1}(s)}\Big|_{s=\lambda_{1,2}} = \frac{p_{\nu_2}(s)}{p_{\nu_1}(s)}\Big|_{s=\lambda_{2,1}} \cdot \frac{p_{\ell_2}(s)}{p_{\ell_1}(s)}\Big|_{s=\lambda_{2,2}}, \tag{5}$$

where $p_r(s)$ is the polynomial of degree $r - 1$ defined by the recursion in equation (4).

Next, we show a graphical interpretation of the of the observability results obtained by combining the results of Theorem 4.4, Proposition 4.5 and Theorem 4.8. We present it through an example. In Figure 4 we show a two dimensional grid of length 4×6. It can be easily tested that this grid has two non-simple eigenvalues of multiplicity two, namely $\lambda_1 = 2$ and $\lambda_2 = 3$. We partition the grid into subgrids of dimensions 2×2 and 2×3. The eigenvalue $\lambda_1 = 2$ (respectively $\lambda_2 = 3$) is an eigenvalue of multiplicity two in the subgrid 2×2 (2×3). The eigenvectors generating V_{λ_1} (V_{λ_2}) belong to S^{+-} and S^{-+} (S^{++} and S^{--}), which gives the symmetries in Figure 4 (a) according to Proposition 4.5 and subsequent discussion. Replicating the subgrid symbols according to Theorem 4.4 we get the structure in Figure 4 (b). Given a set of observation nodes, the grid is observable if and only if the nodes do not have any symbol in common. If, for example, the observation nodes share the top symbol, then the eigenvalue $\lambda = 2$ (of the subgrid 2×2) is unobservable.



(a) subgrids 2x2 & 2x3 (b) subgrid partition

Fig. 4: Graphical interpretation of the observability analysis.

V. Conclusions

In this paper we have characterized the observability (by duality the reachability) of grid graphs in terms of suitable graph decompositions, symmetries in the structure of the grid eigenvectors and simple rules from number theory. In particular, we have shown what are all and only the unobservable set of nodes and provided simple routines to choose a set of observation nodes that guarantee observability.

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