2011 50th IEEE Conference on Decision and Control and
European Control Conference (CDC-ECC)
Orlando, FL, USA, December 12-15, 2011

A Lyapunov-based diagnosis signal for fault detection robust tracking problem of a class of sampled-data systems.

M.L. Corradini, A. Cristofaro, R. Giambò and S. Pettinari

Abstract— **This paper addresses a novel Lyapunov-based diagnosis signal design for the Robust Fault Detection of a class of Sampled Data systems whose output vector has to follow an assigned reference. The only signal available for measurements is the output variable. A simulation study on a vehicle suspension system is also reported.**

Index Terms— **Robust Fault Detection Filters, Sampled-Data Systems, Robust Tracking Problem, Lyapunov Function Approach.**

I. Introduction

Over the past two decades, the growing demand for reliability in industrial processes has drawn increasing attention to the problem of Fault Detection (FD). In fact, faults in sensors and actuators are usually associated to increasing operating costs, off-specification production, line shut-down and possible detrimental environment impact. In monitoring and diagnostic of complex dynamical systems when the system under consideration is subject to uncertainties and/or unknown disturbances, robust FDI schemes are usually required. The issue of robustness in FD techniques has been widely studied [2]. In particular, H_∞ optimization [2], [7], [8], [14], [17], [20], aims at reaching an acceptable compromise between disturbance robustness and fault sensitivity, while the adoption of unknown input observer (UIO) [2], [3], [9], [12], [15], [18], [19] is aimed at analytically decoupling the state estimation error from the unknown inputs. Broadly speaking, a diagnosis signal, called residual, is generated within the UIO approach, which should be independent with respect to the system operating state and should be decoupled from disturbances.

Moreover the introduction of computers into signal processing and control occurred in the mid 20th century, brought with increasing persistence, the problem of considering plants with saturating quantized measurements to the attention of the research community. Despite of its large diffusion in industrial digital control systems, at least as far as the author are aware, limited attention has been given to Robust Fault Detection of Sampled Data (SD) systems, where a continuous-time plant is driven by a digital computer (by the aid of analog/digital and digital/analog converters) and may be affected both by faults and uncertain terms, such as unknown disturbances or model uncertainties. In the case of perturbed plants, if matched disturbances affect the continuous-time plant, full decoupling of disturbance terms from faulty signal becomes more difficult in SD systems [22], because uncertainty satisfying the matching condition in the original continous-time plant do loose such property after discretization, therefore unmatched disturbances should be considered. This effect could make even harder the design of robust tools to be used for detecting the eventual occurrence of faults. In the framework of SD systems, the recent papers by Ding et al. [22], [23] should be mentioned, where a direct FD design approach for SD systems is proposed solving an optimization problem based on a well defined operator, but no explicit analytical residual function is provided there. FD analytical approaches for SISO SD systems are studied by the authors in [6], in [4] where some strong conditions of [6] are overcome and in [5] which addresses a robust FD design approach for uncertain MIMO SD systems.

The main contribution of this note is the design of a Lyapunov-based diagnosis signal for the robust Fault Detection of a class of Sampled Data systems whose output vector follows a given reference (tracking problem).

It is considered a completely observable single input single output (SISO), time-invariant continuous-time system which may be affected both by faults and by uncertain terms, such as unknown disturbances or model uncertainties [22], which are assumed bounded by a suitable positive constant as in many other different approaches. Moreover it is supposed that the fault distribution vector does not belong to the same subspace of the disturbance distribution vector and that the system given by the state and the unknown input is not controllable. Setting up suitable assumptions the design of a reduced-order observer and a control input are addressed to guarantee an assigned reference following. Then the presence of a nonzero fault is taken into account a Lyapunov function is disigned to define a fault diagnosis signal.

This paper is organized as follows: in Section II the problem statement is addressed. Assuming initially that no faults affects the continuous time system, Section III deals with the design of a reduced-order filter which causes the estimation error to be bounded by a known constant, and Section IV addresses the robust tracking problem. Section V deals with a novel Lyapunov-based diagnosis signal design problem of a class of uncertain SD system in which the only available signal is the output variable. Finally, in Section VI simulations of a vehicle suspension system are addressed as a worked example of the set-up showed in the previous sections.

In what follows, the symbol $||\cdot||$ refers to the Euclidean norm, while the operator $(\cdot)'$ indicates the transpose operation.

The authors are with the Scuola di Scienze e Tecnologie, Università di Camerino, via Madonna delle Carceri, 62032 Camerino (MC), Italy, Fax: +39 0737 402568, email: {letizia.corradini, andrea.cristofaro, roberto.giambo, silvia.pettinari}@unicam.it

II. PROBLEM STATEMENT

Consider a digital feedback control system consisting of the interconnection of a SISO completely observable continuous-time plant, a digital controller and a A/D converter. The plant is affected by an additive unknown disturbance term and may also undergo possible actuator faults belonging to the classes of abrupt faults (stepwise) or incipient faults (drift-like) [15]. With no loss of generality the continuous-time systems is given in the observability canonical form and it is described as follows

$$\left\{ \begin{array}{l} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}u(t) + \mathbf{r}d(t) + \mathbf{f}\phi(t) \\ y(t) = \mathbf{c}'\,\mathbf{x}(t) \end{array} \right. \quad (1)$$

where $\mathbf{x}(t) \in \mathbb{R}^n$ is the state vector which is not available for measurement, $y(t) \in \mathbb{R}$ is the output, $u(t) \in \mathbb{R}$ is the known input vector, $d(t) \in \mathbb{R}$ is the unknown input (or disturbance), and $\phi(t) \in \mathbb{R}$ is an unknown actuator failures whose distribution matrix $\mathbf{f} \in \mathbb{R}^n$ is supposed known. $\mathbf{A}$, $\mathbf{b}$, $\mathbf{c}'$, $\mathbf{r}$ are known constant matrices. The discretization of plant equations, assuming that u is constant during each sampling interval T_C, provides:

$$\left\{ \begin{array}{l} \bar{\mathbf{x}}(k+1) = \bar{\mathbf{G}}\bar{\mathbf{x}}(k) + \bar{\mathbf{q}}\,u(k) + \bar{\mathbf{\Delta}}(k) + \bar{\mathbf{\Phi}}(k) \\ y(k) = \mathbf{c}'\bar{\mathbf{x}}(k) \end{array} \right. \quad (2)$$

where

$$\bar{\mathbf{G}} = e^{\mathbf{A}\,T_C}, \quad (3)$$

$$\bar{\mathbf{q}} = \left(\int_0^{T_C} e^{\mathbf{A}\tau}\, d\tau \right) \mathbf{b}, \quad (4)$$

$$\bar{\mathbf{\Delta}}(k) = \int_0^{T_C} e^{\mathbf{A}\sigma}\, \mathbf{r}\, d((k+1)T_C - \sigma)\, d\sigma, \quad (5)$$

$$\bar{\mathbf{\Phi}}(k) = \int_{kT_C}^{(k+1)T_C} e^{\mathbf{A}((k+1)T_C - \sigma)} \mathbf{f}\, \phi(\sigma) d\sigma\,. \quad (6)$$

Assuming that observability is preserved by a proper choice of the sampling frequency, with no loss of generality the discretized system (2) can be trasformed in the observability canonical form by a suitable square invertible matrix $\mathbf{M}$ (see for example [1]), obtaining:

$$\left\{ \begin{array}{l} \mathbf{x}(k+1) = \mathbf{G}\mathbf{x}(k) + \mathbf{q}\,u(k) + \mathbf{\Delta}(k) + \mathbf{\Phi}(k) \\ y(k) = \mathbf{c}'\mathbf{x}(k) = x_2(k) \end{array} \right. \quad (7)$$

with $\mathbf{G} = \mathbf{M}^{-1}\,\bar{\mathbf{G}}\,\mathbf{M}$, $\mathbf{q} = \mathbf{M}^{-1}\,\bar{\mathbf{q}}$, $\mathbf{\Delta}(k) = \mathbf{M}^{-1}\,\bar{\mathbf{\Delta}}(k)$, $\mathbf{\Phi}(k) = \mathbf{M}^{-1}\,\bar{\mathbf{\Phi}}(k)$ and $\bar{\mathbf{c}}' = \mathbf{c}'\,\mathbf{M} = \mathbf{c}'$. Partitioning the state vector $\mathbf{x}(t)$ as $\mathbf{x}(t) = (\mathbf{x}_1(t), x_2(t))'$ with $\mathbf{x}_1(t) \in \mathbb{R}^{n-1}$ and $x_2(t) \in \mathbb{R}$, the output signal is exactly the last compotent of the state vector $y(k) = x_2(k)$. Moreover, plant matrices can be partitioned accordingly

$$\mathbf{G} = \begin{pmatrix} \mathbf{G_{11}} & \mathbf{g_{12}} \\ \mathbf{g_{21}}' & g_{22} \end{pmatrix}, \mathbf{q} = \begin{pmatrix} \mathbf{q_1} \\ q_2 \end{pmatrix},$$

and $\mathbf{\Delta}(k) = (\mathbf{\Delta}_1(k), \Delta_2(k))'$, where $\mathbf{G_{11}} \in \mathbb{R}^{(n-1)\times(n-1)}$, $\mathbf{q_1}$, $\mathbf{\Delta}_1(k) \in \mathbb{R}^{n-1}$ and the other matrices have appropriate dimensions.

The addressed problem is defined but some assumptions:

Assumption 1: The state vector is unavailable for measurement except for the output variable $y(k)$.

Assumption 2: The disturbance distribution $\mathbf{r}$ is not multiple of the fault distribution $\mathbf{f}$, that is for any $\alpha \in \mathbb{R}$, $\mathbf{r} \neq \alpha\,\mathbf{f}$.

Assumption 3: The system $(\mathbf{A}, \mathbf{r})$ is not controllable.

The above assumption assures that the subset $\langle \mathbf{A}|Im\mathbf{r}\rangle^{\perp}$ contains a not null vector. Two further hypothesis ensure the existence of a particular control vector u which causes the tracking error to be bounded, and the state vector boundness.

Assumption 4: The scalar q_2 is not null.

Assumption 5: The invariant zeros of the system $(\mathbf{G}, \mathbf{q}, \mathbf{c}')$ are asymptotically stable. (See [10].)

With a slight abuse of notation we have written $\mathbf{x}(k)$ in place of $\mathbf{x}(k\,T_C)$. As in many other different approaches the upper limit of the disturbance is supposed to be known.

Assumption 6: The unknown term $d(t)$ is assumed to be bounded by a known positive constant $\varrho > 0$:

$$|d(t)| \leq \varrho \;\forall\; t \in [0, \infty).$$

III. FILTER DESIGN

Assume no actuator faults affect the continuous-time system (1). Let us design the following reduced-order state observer

$$\hat{\mathbf{x}}_1(k+1) = \mathbf{q_1}u(k) + \mathbf{v}(k) \quad (8)$$

The dynamics of the estimation error, defined as

$$\mathbf{e_1}(k) := \mathbf{x}_1(k) - \hat{\mathbf{x}}_1(k), \quad (9)$$

is given by

$$\mathbf{e_1}(k+1) = \mathbf{G_{11}}\mathbf{x}_1(k) + \mathbf{g_{12}}y(k) + \mathbf{\Delta}_1(k) - \mathbf{v}(k);$$

choosing

$$\mathbf{v}(k) := \mathbf{g_{12}}y(k) + \mathbf{G_{11}}\hat{\mathbf{x}}_1(k), \quad (10)$$

one has

$$\mathbf{e_1}(k+1) = \mathbf{G_{11}}\mathbf{e_1}(k) + \mathbf{\Delta}_1(k). \quad (11)$$

The term $\mathbf{\Delta}_1(k)$ is bounded by $||\mathbf{\Delta}(k)||$ which verifies

$$||\mathbf{\Delta}(k)|| \leq \varrho \int_0^{T_C} \left\| \mathbf{M}^{-1} e^{\mathbf{A}\sigma}\,\mathbf{r} \right\| d\sigma =: \tilde{\varrho}. \quad (12)$$

As a consequence, since the matrix $\mathbf{G_{11}}$ is nilpotent by definition (i.e. ${\mathbf{G_{11}}}^n = 0$), for $k \geq n$ the estimation error verifies

$$||\mathbf{e_1}(k)|| \leq n\tilde{\varrho} \quad (13)$$

IV. ROBUST TRACKING PROBLEM

Assigned a reference signal $y_d(k)$, a control input $u(k)$ is chosen in order to guarantee the asymptotic boundedness of the tracking error

$$\epsilon(k) := y(k) - y_d(k), \quad (14)$$

moreover the asymptotic boundedness of the state vector is addressed.

The dynamics of the error $\epsilon(k)$ is:

$$\begin{aligned} \epsilon(k+1) &= \mathbf{g_{21}}'\hat{\mathbf{x}}_1(k) + g_{22}\epsilon(k) + q_2u(k) + \mathbf{g_{21}}'\mathbf{e_1}(k) \\ &+ \Delta_2(k) - y_d(k+1) + g_{22}y_d(k) \end{aligned}$$

Due to Assumption 4 the control input $u(k)$ can be chosen as

$$\begin{aligned} u(k) &:= -{q_2}^{-1}\mathbf{g_{21}}'\hat{\mathbf{x}}_1(k) - {q_2}^{-1}g_{22}\epsilon(k) + \\ &+ {q_2}^{-1}y_d(k+1) - {q_2}^{-1}g_{22}y_d(k), \end{aligned} \quad (15)$$

so the dynamics of the tracking error reduces to

$$\epsilon(k+1) = \mathbf{g_{21}}'\mathbf{e_1}(k) + \Delta_2(k). \quad (16)$$

Thanks to the error estimation (13) and to the observability canonical form of (7), the tracking error is always bounded by a constant depending on the disturbance bound

$$|\epsilon(k+1)| \leq \tilde{\varrho}\,(1+n), \quad (17)$$

where $\tilde{\varrho}$ is defined in (12).

Since the estimation error is bounded as showed in (13), if the estimated state $\hat{\mathbf{x}}_1$ is bounded, then the asymptotic boundness of the vector state $\mathbf{x}$ is straightforward.

Due to (8), (10) and (15) the dynamics of the estimated state $\hat{\mathbf{x}}_1$ are given by

$$\begin{aligned} \hat{\mathbf{x}}_1(k+1) &= \mathbf{v}(k) + \mathbf{q_1}u(k) = \\ &= \mathbf{H}\,\hat{\mathbf{x}}_1(k) + \bar{\mathbf{h}}\,\epsilon(k) + \\ &+ q_2^{-1}\mathbf{q_1}\,y_d(k+1) + \bar{\mathbf{h}}y_d(k) \end{aligned}$$

where $\mathbf{H} := (\mathbf{G_{11}} - q_2^{-1}\mathbf{q_1}\mathbf{g_{21}}')$ and $\bar{\mathbf{h}} := (\mathbf{g_{12}} - q_2^{-1}\mathbf{q_1}g_{22})$. Since it can be proved the eigenvalues of $\mathbf{H}$ are exactly the invariant zeros of $(\mathbf{G}, \mathbf{q}, \mathbf{c}')$ [4], Assumption 5 implies that $\mathbf{H}$ is Schur stable. Therefore the asymptotic boundness of the estimated state $\hat{\mathbf{x}}_1$ follows straightforwardly observing that

$$||\hat{\mathbf{x}}_1(k+1)|| \leq ||\mathbf{H}^k\hat{\mathbf{x}}_1(0)|| + \sum_{i=0}^{k-1}||\mathbf{H}^i||C_0,$$

where

$$C_0 := \tilde{\varrho}(1 + n||\mathbf{g_{21}}'||)||\bar{\mathbf{h}}|| + (||\bar{\mathbf{h}}|| + ||q_2^{-1}\mathbf{q_1}||)\sup_{j\in\mathbb{N}}|y_d(j)|$$

and, as a consequence

$$\limsup_{k\to\infty}||\hat{\mathbf{x}}_1(k)|| \leq \sum_{i=0}^{\infty}||\mathbf{H}^i||C_0 < \infty.$$

V. Fault Detection Problem

Let us introduce now a nonzero fault $\phi(t) \in \mathbb{R}$. Using Assumption 3 a sufficient condition for faults detectability based on a Lyapunov function is provided.

The estimation error dynamics (9) for $k \geq n$ modifies as

$$\mathbf{e_1}(k+1) = \sum_{i=0}^{n-1}{\mathbf{G_{11}}}^i(\boldsymbol{\Delta}_1(k-i) + \boldsymbol{\Phi}_1(k-i)).$$

Now, by definition, one has

$$\boldsymbol{\Delta}(k) \in Im([\mathbf{r}\ \bar{\mathbf{A}}\mathbf{r}\ \ldots\ \bar{\mathbf{A}}^{n-1}\mathbf{r}]) = \langle\bar{\mathbf{A}}|Im\mathbf{r}\rangle$$

where $\bar{\mathbf{A}} := \mathbf{M}^{-1}\mathbf{A}$ and $\langle\bar{\mathbf{A}}|Im\mathbf{r}\rangle$ is the controllable subspace of $(\bar{\mathbf{A}}, \mathbf{r})$ defined by Wonham in [21].

Due to Assumption 3 there exists $\mathbf{p} \in \langle\bar{\mathbf{A}}, \mathbf{r}\rangle^\perp$ such that $\mathbf{p}'\bar{\mathbf{A}}^i\mathbf{r} = 0$ for all $i = 0, 1, \ldots, n-1$, hence such that $\mathbf{p}'\boldsymbol{\Delta}(k\,T_C) = 0$ for every instant $k\,T_C$. The vector $\mathbf{p}$ can be partitioned as $(\mathbf{p}_1, p_2)'$ with $\mathbf{p}_1 \in \mathbb{R}^{n-1}$ and $p_2 \in \mathbb{R}$ according to the previous partitions.

Regarding the tracking error one has

$$\epsilon(k+1) = \mathbf{g_{21}}'\mathbf{e_1}(k) + \Delta_2(k) + \Phi_2(k).$$

Setting $\mathbf{w}'(k) := [\mathbf{e_1}(k)\ \epsilon(k)]'$ one has

$$\mathbf{w}(k+1) = \overline{\mathbf{G}}\,\mathbf{w}(k) + \begin{bmatrix} \boldsymbol{\Delta}_1(k) \\ \Delta_2(k) \end{bmatrix} + \begin{bmatrix} \boldsymbol{\Phi}_1(k) \\ \Phi_2(k) \end{bmatrix}$$

with $\overline{\mathbf{G}} := \begin{bmatrix} \mathbf{G_{11}} & 0 \\ \mathbf{g_{21}}' & 0 \end{bmatrix}$.

By means of a change of coordinates $\bar{\mathbf{w}}(k) := \mathbf{T}^{-1}\,\mathbf{w}(k) = (\mathbf{e_1}(k), \mathbf{p}'\mathbf{w}(k))'$ through the matrix

$$\mathbf{T}^{-1} = \begin{bmatrix} \mathbf{I}_{n-1} & 0 \\ \mathbf{p}_1' & p_2 \end{bmatrix}, \quad \mathbf{T} = \begin{bmatrix} \mathbf{I}_{n-1} & 0 \\ -p_2^{-1}\mathbf{p}_1' & p_2^{-1} \end{bmatrix} \quad (18)$$

where $\mathbf{I}_{n-1}$ is the identity matrix of dimension $n-1$, the dynamics of $\mathbf{w}(k)$ are transformed as follows

$$\begin{aligned} \bar{\mathbf{w}}(k+1) &= \begin{bmatrix} \mathbf{G_{11}} & 0 \\ \mathbf{p}_1'\mathbf{G_{11}} + p_2\mathbf{g_{21}}' & 0 \end{bmatrix}\bar{\mathbf{w}}(k) + \\ &+ \begin{bmatrix} \boldsymbol{\Delta}_1(k) \\ 0 \end{bmatrix} + \begin{bmatrix} \boldsymbol{\Phi}_1(k) \\ \mathbf{p}_1'\boldsymbol{\Phi}_1(k) + p_2\Phi_2(k) \end{bmatrix}. \end{aligned}$$

The state matrix $\boldsymbol{\Omega} := \mathbf{T}^{-1}\overline{\mathbf{G}}\mathbf{T}$ is Schur stable, as all its eigenvalues are placed in the origin of the complex plane. The fault-free system can be rewritten as

$$\bar{\mathbf{w}}(k+1) = \boldsymbol{\Omega}\,\bar{\mathbf{w}}(k) + \mathbf{s}(k)$$

where $\mathbf{s}(k) = [\boldsymbol{\Delta}_1(k)\ 0]'$ and due to (12)

$$||\mathbf{s}(k)|| \leq \tilde{\varrho}. \quad (19)$$

Let us define a Lyapunov function

$$V(k) := \bar{\mathbf{w}}(k)'\mathbf{L}\bar{\mathbf{w}}(k) \quad (20)$$

where $\mathbf{L}$ is the solution of the algebraic equation

$$\boldsymbol{\Omega}'\mathbf{L}\boldsymbol{\Omega} - \mathbf{L} = -\boldsymbol{\Psi}, \quad (21)$$

with a symmetric positive-definite matrix $\boldsymbol{\Psi} > 0$. It is worth to note that, since $\boldsymbol{\Omega}$ is a stable matrix the existence of the solution $\mathbf{L}$ is ensured. Setting the increment of the Lyapunov function

$$\mathcal{L}_V(k) := V(k+1) - V(k),$$

we have

$$\mathcal{L}_V(k) = -\bar{\mathbf{w}}(k)'\boldsymbol{\Psi}\bar{\mathbf{w}}(k) + 2\,\mathbf{s}(k)'\,\mathbf{L}\,\boldsymbol{\Omega}\,\bar{\mathbf{w}}(k) + \mathbf{s}(k)'\mathbf{L}\,\mathbf{s}(k)$$

and due to (19)

$$\begin{aligned} \mathcal{L}_V(k) &\leq -\bar{\mathbf{w}}(k)'\boldsymbol{\Psi}\bar{\mathbf{w}}(k) + 2\tilde{\varrho}||\mathbf{L}\,\boldsymbol{\Omega}\bar{\mathbf{w}}(k)|| + ||\mathbf{L}||\tilde{\varrho}^2 \leq \\ &\leq -\psi||\mathbf{w}(k)||^2 + ||\mathbf{L}||\tilde{\varrho}^2 + \\ &+ 2\tilde{\varrho}||\mathbf{L}\boldsymbol{\Omega}\mathbf{T}^{-1}||\,||\mathbf{w}(k)|| \end{aligned}$$

with

$$\psi := ||(\mathbf{T}^{-1})'\boldsymbol{\Psi}\mathbf{T}^{-1}||.$$

The inequality $\mathcal{L}_V(k) \leq 0$ is satisfied for

$$||\mathbf{w}(k)|| \geq w_0,$$

where

$$w_0 := \frac{\tilde{\varrho}\left(||\mathbf{L}\mathbf{\Omega}\mathbf{T}^{-1}|| + \sqrt{||\mathbf{L}||\psi + ||\mathbf{L}\mathbf{\Omega}\mathbf{T}^{-1}||^2}\right)}{\psi}. \quad (22)$$

This means that, outside such interval, the norm of $\mathbf{w}(k)$ is decreasing at any time step and this fact can be employed for diagnosis purposes. On the other hand, not the whole error vector $\mathbf{w}(k)$ but only the tracking error $\epsilon(k)$ is available for measurement. We can give a sufficient condition for the fault occurence considering the worst case, as explained in the next statement.

Lemma 1: Let us suppose that for a time step $k > 0$ the variable $\epsilon(k)$ is out of the sector, i.e. $|\epsilon(k)| > w_0$. If the condition $|\epsilon(k+1)| > |\epsilon(k)| + n\tilde{\varrho}$ holds for a given k, then a fault has occurred at some time step $k_1 < k$.

Proof: Let us suppose that $|\epsilon(k)| > w_0$. Since

$$||\mathbf{w}(k)|| \geq |\epsilon(k)| > w_0$$

we have also $||\mathbf{w}(k)|| > w_0$. Out of the sector, in the absence of faults, one must have $||\bar{\mathbf{w}}(k+1)|| < ||\bar{\mathbf{w}}(k)||$ and also $||\mathbf{w}(k+1)|| < ||\mathbf{w}(k)||$, as the Lyapunov function can be rewritten as $V(k) = \mathbf{w}(k)'(\mathbf{T}^{-1})'\mathbf{L}\mathbf{T}^{-1}\mathbf{w}(k)$. Since $||\mathbf{e_1}(k)|| \leq n\tilde{\varrho}$, the observation error cannot contribute to any eventual growth of $||\mathbf{w}(k)||$ for more than $n\tilde{\varrho}$. Therefore:

$$\begin{aligned} ||\mathbf{w}(k+1)|| \geq |\epsilon(k+1)| \quad &> \quad |\epsilon(k)| + n\tilde{\varrho} \geq \\ &\geq \quad |\epsilon(k)| + ||\mathbf{e_1}(k)|| > ||\mathbf{w}(k)|| \end{aligned}$$

hence whenever $|\epsilon(k+1)| > |\epsilon(k)| + n\tilde{\varrho}$, then $||\mathbf{w}(k+1)|| > ||\mathbf{w}(k)||$, this meaning that a fault has necessarily occurred.

The previous development requires that $w_0 < \tilde{\varrho}(1+n)$ in view of (17).

Setting $\mathbf{L}$ and $\mathbf{\Psi}$ such that

$$\psi(1+n) - 2||\mathbf{L}\mathbf{\Omega}\mathbf{T}^{-1}|| - ||\mathbf{L}|| > 0, \quad (23)$$

the inequality $w_0 < \tilde{\varrho}(1+n)$ holds true. ∎

VI. Application to a Vehicle Suspension System

In this section, a physical application is presented as a worked example of the set-up showed in the previous sections. A vehicle suspension system can be reduced to the so-called quarter-car model, shown in Fig. 1, where an additional force ΔF_u resulting from semi-active components has been added [11], and the Coulomb friction F_C has been neglected for simplicity. The tire is typically modeled by a single linear spring.

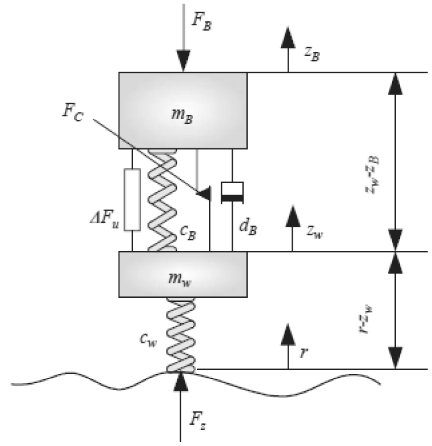

Fig.1 - The quarter-car model.

The classical quarter-car model can be derived (see [16])

$$\begin{aligned} \ddot{z}_B(t) = &-\frac{d_B}{m_B}\dot{z}_B(t) + \frac{d_B}{m_B}\dot{z}_W(t) - \frac{c_B}{m_B}z_B(t) + \\ &+ \frac{c_B}{m_B}z_W(t) - \frac{F_C}{m_B} + \frac{1}{m_B}\Delta F_u - \frac{1}{m_B}F_B \\ \ddot{z}_W(t) = &\frac{d_B}{m_W}\dot{z}_B(t) - \frac{d_B}{m_B}\dot{z}_W(t) + \frac{c_B}{m_W}z_B(t) + \\ &- \frac{c_B + c_W}{m_W}z_W(t) + \frac{c_W}{m_W}r(t) - \frac{F_C}{m_W} - \frac{1}{m_W}\Delta F_u \end{aligned}$$

where c_B and c_W stand for stiffness of body spring and of tire respectively, d_B is the body damping coefficient supposed to be constant. m_B and m_W are the body and wheel mass, z_B, z_W and r stand for the vertical body, wheel, and road displacement, F_Z is the dynamic car load and F_B is the gravity force which is negligible because z_B and z_W are the distance from the equilibrium. The road has a displacement $|r(t)| \leq 0.01$ m. The state vector $\mathbf{x} = [x_1, x_2, x_3, x_4]'$ has been built as follows: $x_1 = \dot{z}_B$, $x_2 = z_B$, $x_3 = \dot{z}_W$, and $x_4 = z_W$. Finally, the input function $u(t) = \Delta F_u$ has been taken into account. The continuous time system is

$$\dot{\mathbf{x}}(t) = \begin{bmatrix} -\frac{d_B}{m_B} & -\frac{c_B}{m_B} & \frac{d_B}{m_B} & \frac{c_B}{m_B} \\ 1 & 0 & 0 & 0 \\ \frac{d_B}{m_W} & \frac{c_B}{m_W} & -\frac{d_B}{m_W} & -\frac{c_B+c_W}{m_W} \\ 0 & 0 & 1 & 0 \end{bmatrix} \mathbf{x}(t) +$$

$$\begin{bmatrix} \frac{1}{m_B} \\ 0 \\ -\frac{1}{m_W} \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 0 \\ \frac{1}{m_W} \\ 0 \end{bmatrix} c_W\, r(t) + \begin{bmatrix} \frac{1}{m_B} \\ 0 \\ -\frac{1}{m_W} \\ 0 \end{bmatrix} \phi(t)$$

$$y(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix} \mathbf{x}(t)$$

where $\phi(t)$ is an eventual actuator fault. According to [13], the following coefficient values have been used: $m_B = 375\,kg$, $m_W = 20\,kg$, $c_B = 130000\,N/m$, $c_W = 10^5\,N/m$, $d_B = 9800\,N \cdot sec/m$. Discretizing the system with a sampling interval $T_C = 0.02$ sec, the matrices of the discretized plant are

$$\bar{\mathbf{G}} = \begin{bmatrix} 0.9997 & -0.111 & -25.4 & -2866 \\ 0.009975 & 0.9913 & -2.026 & -241.4 \\ 4.771\,10^{-5} & 0.009216 & 0.8177 & -21.61 \\ 6.404\,10^{-8} & 1.465\,10^{-5} & 0.001653 & -0.03565 \end{bmatrix},$$

$$\bar{\mathbf{q}} = \begin{bmatrix} 0.133 & 0.0006427 & 2.673\,10^{-5} & 4.159\,10^{-8} \end{bmatrix}',$$

and it can be verified that Assumptions 4 and 5 hold true, in fact the plant zeros are $-0.2118,\ 0.7633 \pm 0.6466i$.

A control input $u(t) = u(k\,T_C)$ for $t \in [k\,T_C, (k+1)\,T_C[$ verifies (15). A road displacement of the form $r(t) = 0.01 * sin(t)$ has been considered, therefore the disturbance term turns out to be bounded by a constant $\rho = c_W * 0.01 = 1000$. It can be verified that the controllability matrix of $(\mathbf{A}, \mathbf{r})$ has a small determinant ($\simeq 0.75$) with respect to matrix coefficients, therefore Assumption 2 is fulfilled too. The vector $\mathbf{p}$ can be determined as $\mathbf{p}' = \left[0\ 0\ 0\ 10^{-5}\right]$, and setting

$$\mathbf{\Psi} = \begin{bmatrix} 0.001 & 0 & 0 & 0 \\ 0 & 0.001 & 0 & 0 \\ 0 & 0 & 0.001 & 0 \\ 0 & 0 & 0 & 0.00001 \end{bmatrix}$$

one gets a solution

$$\mathbf{L} = \begin{bmatrix} 0 & -2.28 \cdot 10^{-5} & 1.86 \cdot 10^{-9} & 0 \\ -2.28 \cdot 10^{-5} & 0.002 & 3.55 \cdot 10^{-15} & 0 \\ 1.86 \cdot 10^{-9} & 3.55 \cdot 10^{-15} & 0.001 & 0 \\ 0 & 0 & 0 & 0.001 \end{bmatrix}$$

with $||\mathbf{L}|| = 0.0020003$ which gives a bound $w_0 = 0.028882$ greater than $(n+1)\tilde{\rho} = 0.0030879$. Plant initial conditions have been chosen as $\mathbf{x}(0)' = [0\ 0.3\ 0\ 0.3]$.

A reference signal $y_d(t) = 0$ has been chosen, and Fig. 2 shows that the output signal follows the given signal when no faults affect the continuous time system. Fig. 3 shows that the estimation error $\mathbf{e_1}(k)$ verifies (13) in the fault free case. Then an abrupt fault $\phi(t)$ of intensity equal to $30\,\rho$ has considered to occur in (1) for $t \geq 50\ s$. Fig. 4 displays the dynamics of the control input $u(k)$. Following the procedure of Lemma 1, detection is performed at time $t = 51\ s$. In particular, the tracking error is reported in Fig. 5, where a dotted line shows the tracking bound $(n+1)\tilde{\varrho}$ (17) and a dash-dot line represents w_0 (22). As proved in Lemma 1, since $|e(k)| \geq w_0$ and $g(k) = |e(k+1)| - |e(k)| - n\,\tilde{\varrho} > 0$ as showed in Fig. 6, we are certain a fault has occured. In Fig. 7 and Fig. 8 an incipient fault of amplitude equal to ρ has considered to occur in (1) for $t \geq 50\ s$.

VII. Conclusions

These notes addressed the design of a Lyapunov-based diagnosis signal for the Robust Fault Detection of a class of Sampled-Data systems whose output vector has to follow an assigned reference. Under some suitable assumptions, a reduced-order observer and a control input have been designed to solve the robust tracking problem. The only signal available for measurements is the output. A physical application of a vehicle suspension system has been studied as a worked example.

References

[1] P. J. Antsaklis and A. N. Michel, *Linear Systems*. Boston: Birkhauser, 2006.

[2] J. Chen and R. J. Patton, *Robust model-based fault diagnosis for dynamic systems*. Norwell, MS: Kluwer Academic Publishers, 1999.

[3] J. Chen, R. J. Patton, and H. Y. Zhang, "Design of unknown input observers and robust fault detection filters," *Int. J. Control.*, vol. 63, no. 1, pp. 85–105, 1996.

[4] M. L. Corradini, A. Cristofaro, R. Giambò, and S. Pettinari, "Design of robust fault detection filters for plants with quantized information," in *Proc. 8th Workshop on Advanced Control and Diagnosis*, Ferrara, IT, 2010.

[5] ——, "Robust fault detection filters for a class of mimo uncertain sampled-data systems," in *Proc. of Conference on Control and Fault-Tolerant Systems (SysTol2010)*, Nice, France, 2010.

[6] M. L. Corradini, R. Giambò, and S. Pettinari, "Design of robust fault detection filters for plants with quantized information," in *Proc. 7th Workshop on Advanced Control and Diagnosis*, Zielona Gòra, PL, 2009.

[7] S. X. Ding, *Model-based fault diagnosis tecniques: Design schemes, algorithms, and tools*. Berlin Heidelberg: Spring-Verlag, 2008.

[8] A. Edelmayer and J. Bokor, "Scaled h_∞ filtering for sensitivity optimization of detection filters," *Budapest, Hungary. SAFEPROCESS*, 2000.

[9] A. Edelmayer, J. Bokor, and Z. Szabo, "Robust detection and estimation of faults by exact fault decoupling and h disturbance attenuation in linear dynamical systems," in *Proc. Amer. Cont. Conf., ACC'06*, 2006, pp. 5716–5721.

[10] C. Edwards and S. K. Spurgeon, *Sliding Mode Control: Theory and Applications*. London: Taylor and Francis, 1998.

[11] D. Fischer and R. Isermann, "Mechatronic semi-active and active vehicle suspensions," *Control Eng. Pract.*, vol. 12, pp. 1353–1367, 2004.

[12] P. M. Frank, "Fault diagnosis in dynamic system using analytical and knoledge based redendancy - a survey and some new results," *Automatica*, vol. 26, pp. 459–474, 1990.

[13] G. F. Franklin, J. D. Powell, and A. Emami-Naeini, *Feedback control of Dynamic Systems*. Englewood Cliffs, New Jersey: Prentice Hall, 2002.

[14] D. Henry and A. Zolghadri, "Design and analysis of robust residual generators for systems under feedback control," *Automatica*, vol. 41, no. 2, pp. 251–264, 2005.

[15] R. Isermann, "Supervision, fault-detection and fault diagnosis methods - an introduction," *Control Eng. Pract.*, vol. 5, no. 5, pp. 639–657, 1997.

[16] ——, *Fault-diagnosis systems. An introduction from Fault Detection to Fault Tolerance*. Berlin Heidelberg: Springer-Verlag, 2006.

[17] R. S. Mangoubi and R. S. Mangoubi, *Robust estimation and failure detection: A concise treatment*. Springer Berlin, 1998.

[18] R. J. Patton and J. Chen, "Parity space approach to model-based fault diagnosis," in *Proceedings SAFEPROCESS*, 1994, pp. 256–261.

[19] ——, "Observer-based fault detection and isolation: robustness and applications," *Control Eng. Pract.*, vol. 5, no. 5, pp. 671–682, 1997.

[20] M. L. Rank and H. Niemann, "Norm based design of fault detectors," *International Journal of Control*, vol. 72, no. 9, pp. 773–783, 1999.

[21] W. M. Wonham, *Linear multivariable control: a geometric approach. Second Ediction*. Springer-Verlag, 1979.

[22] P. Zhang and S. X. Ding, "On fault detection in linear discrete-time, periodic, and sampled-data systems," *Journal of Control Science and Engineering*, 2008.

[23] P. Zhang, S. X. Ding, G. Z. Wang, and D. H. Zhou, "An fdi approach for sampled-data systems," in *Proc. American Control Conference*, Arlington, USA, 2001, pp. 2702–2707.

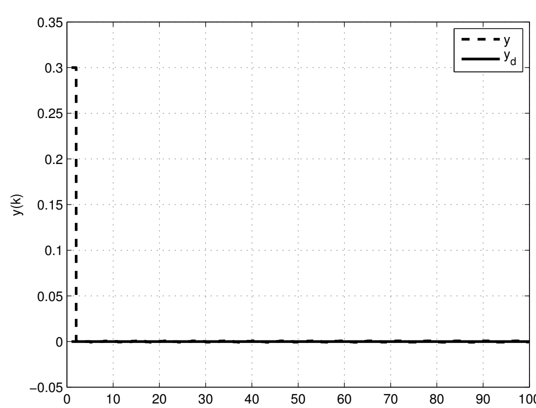

Fig. 2 - Robust tracking in the fault free case.

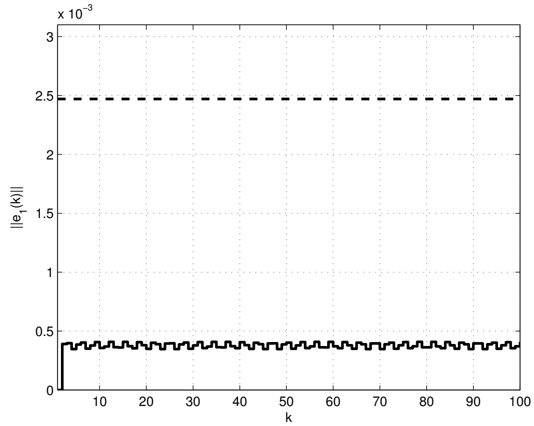

Fig. 3 - Estimation error in the fault free case. The dashed lines is $n\tilde{\rho}$.

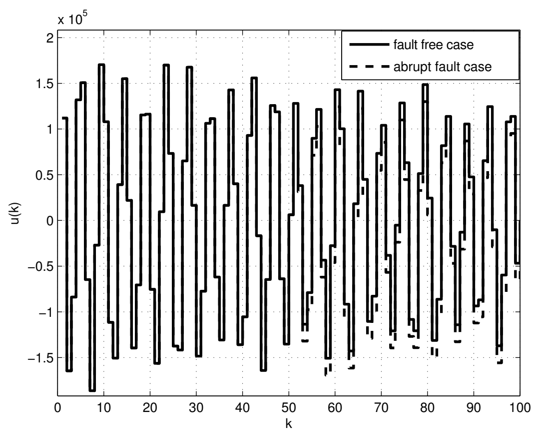

Fig.4 - Control input.

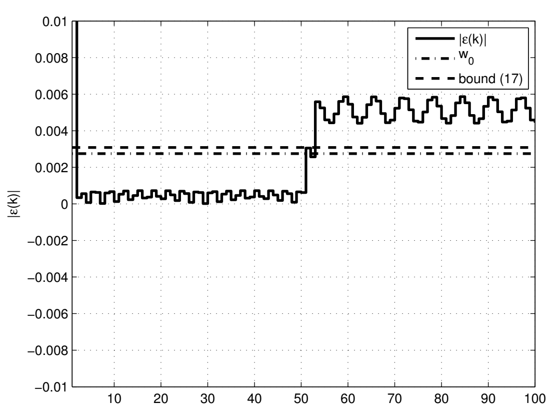

Fig. 5 - Tracking error when an abrupt fault $\phi(t) = 30\rho$ occurs at $t = 51\,s$.

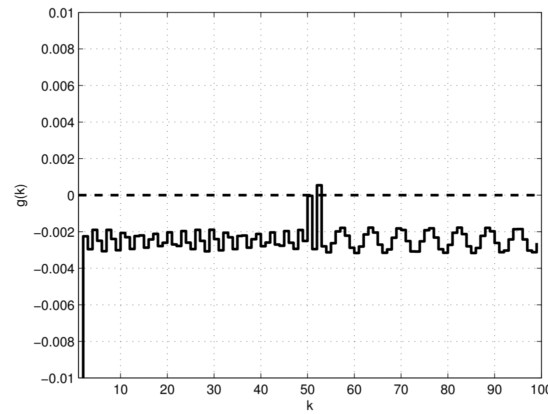

Fig. 6 - Function $g(k) = |\epsilon(k+1)| - |\epsilon(k)| - n\,\tilde{\varrho}$ when an abrupt fault $\phi(t) = 30\rho$ occurs at $t = 51\,s$.

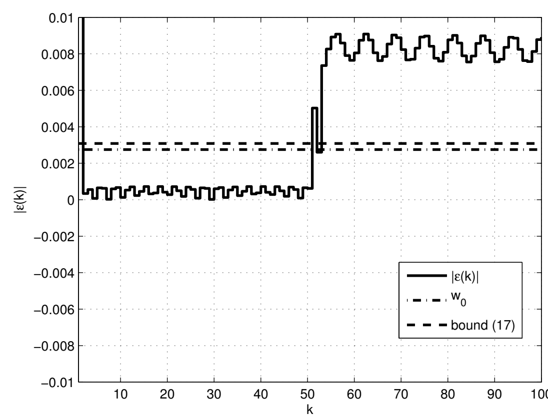

Fig. 7 - Tracking error when an incipient fault of amplitude ρ occurs at $t = 51\,s$.

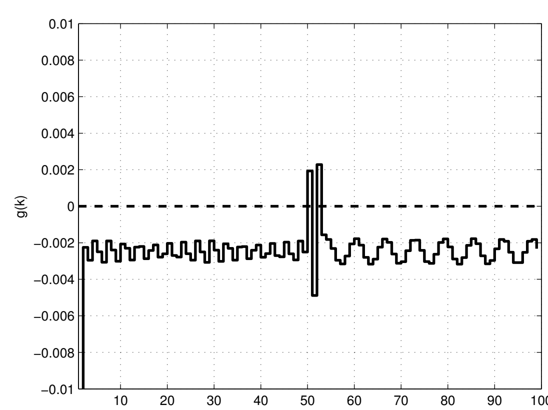

Fig. 8 - Function $g(k) = |\epsilon(k+1)| - |\epsilon(k)| - n\,\tilde{\varrho}$ when an incipient fault of amplitude ρ occurs at $t = 51\,s$.