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Further results on synergistic Lyapunov functions and hybrid feedback design through backstepping

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Abstract— **We extend results on backstepping hybrid feedbacks by exploiting synergistic Lyapunov function and feedback (SLFF) pairs in a generalized form. Compared to existing results, we delineate SLFF pairs that are "ready-made" and do not require extra dynamic variables for backstepping. From an (weak) SLFF pair for an affine control system, we construct an SLFF pair for an extended system where the control input is produced through an integrator. The resulting hybrid feedback asymptotically stabilizes the extended system when the "synergy gap" for the original system is strictly positive. To highlight the versatility of SLFF pairs, we provide a result on the existence of a SLFF pair whenever a hybrid feedback stabilizer exists. The results are illustrated on the "3D pendulum."**

I. INTRODUCTION

Hybrid feedbacks are commonly used to improve performance and achieve objectives that elude classical feedback designs. Such objectives include global asymptotic stabilization of a point for a system evolving on a manifold that is not topologically equivalent to a Euclidean space, or global asymptotic stabilization of a disconnected set of points.

In a recent series of results, synergistic potential functions are developed and used to achieve robust, global asymptotic stabilization of planar orientation [1], orientation on the 2-sphere [2] (applied to the 3D pendulum in [3]), and rigid-body attitude [4], [5]. Synergistic potential functions are extended to synergistic Lyapunov function and feedback (SLFF) pairs in [6]. For a continuous-time system with embedded logic variables, a Lyapunov function and feedback pair is synergistic when, at places in the state-space where the feedback is ineffective, the logic variable can be switched to decrease the value of the Lyapunov function. The magnitude of the available decrease is called the *synergy gap*. In [6], the synergy gap is defined as an infimum over an appropriate subset of the state space and it is required to be positive for control synthesis and backstepping. In this note, the synergy gap is state dependent. It must be positive away from a desired compact set and everywhere positive for backstepping that achieves global asymptotic stability.

Earlier control algorithms propose a similar scheme exploiting multiple Lyapunov functions. Some have appeared in the context of adaptive control using hysteresis [7], [8] and supervisory control systems [9]. Applications using this feedback scheme have appeared for swing-up and stabilization of an inverted pendulum [9], [10] and for control of a double-tank system [11]. Multiple Lyapunov functions are also proposed for control and analysis in [12].

In Section III we construct a robustly globally asymptotically stabilizing hybrid feedback algorithm using an SLFF pair. In Sections V–VII, we broaden the applicability of (weak) SLFF pairs through backstepping. Starting from a weak SLFF pair for an affine control system, we construct an (non-weak) SLFF pair for an extended system where the control is produced through an integrator. Results of this type for continuous-time systems can be found in [13, Lemma 2.8(ii)] and [14, Theorem 5.3]. Similar results for switched systems appear in [15]; however, the notion of synergism that is crucial for ensuring global asymptotic stability does not appear in [15]. We provide a variety of backstepping results:

- The backstepping algorithm resembles classical backstepping when the assumed weak SLFF pair is *pure* and is *ready-made* relative to a quadratic function. An SLFF pair is *pure* when the Lyapunov function is non-increasing along solutions at every point in the state space when using the feedback. An SLFF pair is *ready-made* when there is an appropriate relationship between the size of the jumps in the feedback law and the synergy gap of the SLFF pair. These definitions are made precise in Section IV and the backstepping algorithm is described in Section V-A.
- If the weak SLFF pair is not pure, a backstepping result can still be obtained when the SLFF is ready-made relative to a linear function. See the algorithm in Section V-B.
- At times, backstepping may not be needed but it still may be desirable to smooth jumps in the control signal. This situation is addressed in Section VI, where the ideal feedback is written in a form that is affine in a function of the logic mode; the latter is then treated as an ideal feedback and implemented dynamically through backstepping.
- For backstepping problems where the SLFF pair is not ready-made, the extra dynamic variable described in the preceding item can be exploited to achieve a backstepping result. See Section VII. This idea also appears in [6].

Notation and terminology: $\mathbb{R}$ ($\mathbb{R}_{\geq 0}$) denotes the (nonnegative) real numbers, and $\mathbb{R}^n$ denotes n-dimensional Euclidean space. Given $x \in \mathbb{R}^n$, $|x|$ denotes its Euclidean norm. The unit n-sphere is $\mathbb{S}^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$. A function is called *smooth* if a sufficient number of its derivatives exist and are continuous so that the derivations make sense. A nonnegative-valued function is said to be positive definite with respect to a set if the function is zero if and only if its argument belongs to the set. For a closed set $\mathbb{X} \subset Q \times \mathbb{R}^n$, where $Q \subset \mathbb{R}$ is a finite set, and a smooth function $V : \mathbb{X} \to \mathbb{R}$, we use $\nabla V(q, z)$ to denote gradient of V relative

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to $z \in \mathbb{R}^n$, with $q \in Q$ considered to be constant. Given a smooth function $\kappa : \mathbb{X} \to \mathbb{R}^m$, we use $\mathcal{D}\kappa(q,z)$ to denote the Jacobian matrix of κ relative to z, i.e., $\mathcal{D}\kappa(q,z)$ is an $\mathbb{R}^{m\times n}$ matrix with ij-th entry given as $\partial\kappa_i(q,z)/\partial z_j$. As in [16], a hybrid system with state $x \in \mathbb{R}^n$ is described by flow and jump sets $C, D \subset \mathbb{R}^n$ and set-valued flow and jump maps $F, G : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$. It satisfies the basic conditions [16] if C and D are closed, F and G are outer semicontinuous and locally bounded, $F(x)$ is nonempty and convex for all $x \in C$, and $G(x)$ is nonempty for all $x \in D$.

II. SLFF PAIRS

We extend the definition of a synergistic Lyapunov function and feedback pair defined in [6]. Consider the system

$$\left.\begin{aligned} \dot{q} &= 0 \\ \dot{z} &= f(q,z,\omega) \end{aligned}\right\} \qquad (q,z) \in \mathbb{X}, \tag{1}$$

where $f : \mathbb{X} \times \mathbb{R}^m \to \mathbb{R}^n$ is continuous, $\omega \in \mathbb{R}^m$ is the control variable, the set $\mathbb{X} \subset Q \times \mathbb{R}^n$ is closed and the set $Q \subset \mathbb{R}$ finite. Let $\mathcal{A} \subset \mathcal{Y} \subset \mathbb{X}$ be such that $\mathcal{A}$ is compact and $\mathcal{Y}$ is closed. We define the set

$$\mathcal{B} := \{(q,z) \in \mathbb{X} : \exists s \in Q,\ (s,z) \in \mathcal{A}\}. \tag{2}$$

A $\mathcal{C}^1$ function $V : \mathbb{X} \to \mathbb{R}_{\geq 0}$ and a continuous function $\kappa : \mathbb{X} \to \mathbb{R}^m$ form a *synergistic Lyapunov function and feedback (SLFF) pair candidate* relative to $(\mathcal{A}, \mathcal{Y})$ if

- $\{(q,z) \in \mathbb{X} : V(q,z) \leq c\}$ is compact for each $c \geq 0$;
- $V(q,z) = 0$ if and only if $(q,z) \in \mathcal{A}$,
- For all $(q,z) \in \mathcal{Y}$,

$$\langle \nabla V(q,z), f(q,z,\kappa(q,z)) \rangle \leq 0. \tag{3}$$

Given an SLFF pair candidate (V,κ), define

$$\mathcal{E} := \{(q,z) \in \mathcal{Y} : \langle \nabla V(q,z), f(q,z,\kappa(q,z)) \rangle = 0\} \tag{4}$$

and let $\Psi \subset \mathcal{E}$ be the largest weakly invariant set [17] for

$$\left.\begin{aligned} \dot{q} &= 0 \\ \dot{z} &= f(q,z,\kappa(q,z)) \end{aligned}\right\} \qquad (q,z) \in \mathcal{E}. \tag{5}$$

For each $(q,z) \in \mathbb{X}$, define

$$\mu_V(q,z) := V(q,z) - \min_{s \in Q} V(s,z). \tag{6}$$

The pair (V,κ) is called a *synergistic Lyapunov function and feedback pair relative to* $(\mathcal{A},\mathcal{Y})$ if

$$\mu_V(q,z) > 0 \qquad \forall (q,z) \in \left(\Psi \cup \overline{\mathbb{X} \setminus \mathcal{Y}}\right) \setminus \mathcal{A}, \tag{7}$$

in which case $\mu_V(q,z)$ is called the *synergy gap* at (q,z). Given a continuous function $\delta : \mathbb{X} \to \mathbb{R}_{\geq 0}$, when

$$\mu_V(q,z) > \delta(q,z) \quad \forall (q,z) \in \left(\Psi \cup \overline{\mathbb{X} \setminus \mathcal{Y}}\right) \setminus \mathcal{A} \tag{8}$$

we say that *the synergy gap exceeds* δ. When δ satisfies

$$\mu_V(q,z) > \delta(q,z) \quad \forall (q,z) \in \left(\Psi \cup \overline{\mathbb{X} \setminus \mathcal{Y}} \cup \mathcal{B}\right) \setminus \mathcal{A}, \tag{9}$$

we say that the synergy gap *totally* exceeds δ. Where the synergy gap is positive, we can change q to reduce V, which is desirable at points in $\Psi \setminus \mathcal{A}$, where the value of V could get stuck during flows, at points in $\left(\overline{\mathbb{X} \setminus \mathcal{Y}}\right) \setminus \mathcal{A}$, where the q-th feedback function is not effective, and possibly at points in $\mathcal{B} \setminus \mathcal{A}$ to ensure that the set $\mathcal{B}$ is stabilized.

Proposition 1: The synergy gap is a continuous function. If (V,κ) *is an SLFF pair, then there exists a continuous function* $\delta : \mathbb{X} \to \mathbb{R}_{\geq 0}$ *that is positive on* $\mathbb{X} \setminus \mathcal{A}$ *such that the synergy gap (totally) exceeds* δ. *If the synergy gap for* (V,κ) *(totally) exceeds the function* δ *then, for each smooth* $\mathcal{K}_\infty$*-function* ρ *having a positive, nondecreasing derivative denoted* ρ', *the pair* $(\rho(V),\kappa)$ *is an SLFF with synergy gap (totally) exceeding the function* $\tilde{\delta}$ *defined as* $\tilde{\delta}(q,z) := \rho'(cV(q,z))(1-c)\delta(q,z)$, *where* c *can be taken arbitrarily in the interval* $(0,1)$.

We show that the existence of an SLFF pair relative to the compact set $\mathcal{A}$ is equivalent to the existence of a feedback

$$\begin{aligned} &\omega = \alpha(q,z) && (q,z) \in C \subset \mathbb{X} \\ &q^+ \in G_c(q,z) \subset Q && (q,z) \in D \subset \mathbb{X} \end{aligned} \tag{10}$$

satisfying the basic conditions and the conditions $\mathcal{A} \subset C$, $C \cup D = \mathbb{X}$, and rendering the compact set $\mathcal{A}$ globally asymptotically stable for the system (1), (10), that is, for

$$\underbrace{\begin{aligned} \dot{q} &= 0 \\ \dot{z} &= f(q,z,\alpha(q,z)) \end{aligned}}_{(q,z) \in C} \qquad \underbrace{\begin{aligned} q^+ &\in G_c(q,z) \\ z^+ &= z \end{aligned}}_{(q,z) \in D.} \tag{11}$$

We start by showing that this asymptotic stabilizability property implies the existence of an SLFF. The opposite implication is established in Theorem 2 given in Section III.

Theorem 1: Suppose the data of (11) satisfies the basic conditions, the compact set $\mathcal{A}$ *is globally asymptotically stable for (11),* $\mathcal{A} \subset C$, *and* $C \cup D = \mathbb{X}$. *Then there exists a smooth function* $V : \mathbb{X} \to \mathbb{R}_{\geq 0}$ *such that* (V,α) *is an SLFF pair relative to* $(\mathcal{A},\mathcal{Y})$ *with* $\mathcal{Y} = C$ *and there exists* $\varepsilon > 0$ *such that the synergy gap (totally) exceeds* δ *where* $\delta(q,z) := \varepsilon V(q,z)$. *If, in addition,* $D \cap \mathcal{A} = \emptyset$ *(and* $\mathcal{B} \setminus \mathcal{A}$ *is closed) then there exist* $\varepsilon_1 > 0, \varepsilon_2 > 0$ *such that the synergy gap (totally) exceeds* δ *with* $\delta(q,z) := \varepsilon_1 V(q,z) + \varepsilon_2$.

III. HYBRID FEEDBACK FROM AN SLFF PAIR

Let (V,κ) denote the SLFF pair and let $\delta : \mathbb{X} \to \mathbb{R}_{\geq 0}$ be continuous. We specify a hybrid controller to globally asymptotically stabilize $\mathcal{A}$ (and $\mathcal{B}$) for (1) as

$$\begin{aligned} C &:= \{(q,z) \in \mathbb{X} : \mu_V(q,z) \leq \delta(q,z)\} \\ \omega &:= \kappa(q,z) \\ D &:= \{(q,z) \in \mathbb{X} : \mu_V(q,z) \geq \delta(q,z)\} \\ G_c(z) &:= \{g_c \in Q : \mu_V(g_c,z) = 0\} \end{aligned} \tag{12}$$

resulting in the closed-loop hybrid system

$$\underbrace{\begin{aligned} \dot{q} &= 0 \\ \dot{z} &= f(q,z,\kappa(q,z)) \end{aligned}}_{(q,z) \in C} \qquad \underbrace{\begin{aligned} q^+ &\in G_c(z) \\ z^+ &= z \end{aligned}}_{(q,z) \in D.} \tag{13}$$

Since δ and μ_V are continuous, C and D are closed. Since μ_V is continuous, G_c is outer semicontinuous. Also, $C \cup D = \mathbb{X}$ and $G_c(z)$ is non-empty for each z such that $(q,z) \in \mathbb{X}$ for some $q \in Q$, in particular, for $(q,z) \in D$.

Theorem 2: Let $\mathcal{Y} \subset \mathbb{X}$, let $\mathcal{A} \subset \mathcal{Y}$ be compact, and let $\delta : \mathbb{X} \to \mathbb{R}_{\geq 0}$ be continuous and positive on $\mathbb{X} \setminus \mathcal{A}$. If (V,κ) is an SLFF pair for (1) relative to $(\mathcal{A},\mathcal{Y})$ with synergy gap (totally) exceeding δ, then $\mathcal{A} \subset C$ and $\mathcal{A}$ ($\mathcal{B}$) is globally asymptotically stable for the closed-loop system (13).

IV. REFINEMENT OF SLFF PAIR PROPERTIES

A. Weak SLFF pairs for affine control systems

We introduce a *weak* synergistic Lyapunov function and feedback pair (weak SLFF) for (1) when $f(q,z,\omega) = \phi(q,z) + \psi(q,z)\omega$ where ϕ and ψ are smooth. Given an SLFF pair candidate (V,κ), with V and κ smooth, define

$$\mathcal{W} := \left\{ (q,z) \in \mathbb{X} : \nabla V(q,z)^\top \psi(q,z) = 0 \right\}. \tag{14}$$

Recall the definition of $\mathcal{E}$ in Section II and let $\Omega \subset \mathcal{E} \cap \mathcal{W}$ denote the largest weakly invariant set for the system

$$\left.\begin{aligned} \dot{q} &= 0 \\ \dot{z} &= \phi(q,z) + \psi(q,z)\kappa(q,z) \end{aligned}\right\} \quad (q,z) \in \mathcal{E} \cap \mathcal{W}. \tag{15}$$

The pair (V,κ) is called a *weak synergistic Lyapunov function and feedback pair relative to* $(\mathcal{A},\mathcal{Y})$ if

$$\mu_V(q,z) > 0 \qquad \forall (q,z) \in \left(\Omega \cup \overline{\mathbb{X} \setminus \mathcal{Y}}\right) \setminus \mathcal{A}. \tag{16}$$

Given a continuous function $\delta : \mathbb{X} \to \mathbb{R}_{\geq 0}$, when

$$\mu_V(q,z) > \delta(q,z) \quad \forall (q,z) \in \left(\Omega \cup \overline{\mathbb{X} \setminus \mathcal{Y}}\right) \setminus \mathcal{A}, \tag{17}$$

we say that *the synergy gap weakly exceeds* δ. If δ satisfies

$$\mu_V(q,z) > \delta(q,z) \quad \forall (q,z) \in \left(\Omega \cup \overline{\mathbb{X} \setminus \mathcal{Y}} \cup \mathcal{B}\right) \setminus \mathcal{A}, \tag{18}$$

we say that the synergy gap weakly *totally* exceeds δ. The next lemma follows immediately from the fact that $\Omega \subset \Psi$ and then comparing (17) to (8).

Lemma 1: If (V,κ) is a smooth SLFF pair with synergy gap (totally) exceeding δ then it is also a weak SLFF pair with synergy gap weakly (totally) exceeding δ.

Example 1 (3-D Pendulum): The reduced dynamics of the 3-D pendulum are given in [18] as

$$\dot{z} = [z]_\times \omega \tag{19a}$$

$$J\dot{\omega} = [J\omega]_\times \omega + mg\,[\nu]_\times z + \tau, \tag{19b}$$

where $z \in \mathbb{S}^2$ is the direction of gravity in the body-fixed frame, $\omega \in \mathbb{R}^3$ is the angular velocity expressed in the body-fixed frame, m is the mass, g is the gravitational constant, ν is the vector from the pivot location to the center of mass, $\tau \in \mathbb{R}^3$ is a vector of input torques, and for any $x, y \in \mathbb{R}^3$, $[x]_\times$ is the 3×3 skew-symmetric matrix that satisfies $[x]_\times y = x \times y$, where $\times$ denotes the vector cross product. We now stabilize the "inverted" point $(z,\omega) = (-\nu/|\nu|, 0)$.

Let Q be a finite set, $\mathbb{X}_0 = Q \times \mathbb{S}^2$, $S \subset Q$, and $\mathcal{A}_0 = S \times \{-\nu/|\nu|\}$. Let $V_0 : \mathbb{X}_0 \to \mathbb{R}$ be positive definite on $\mathbb{X}_0$ relative to $\mathcal{A}_0$ and define $\kappa_0(q,z) = 0$. Clearly, we have that $\left\langle \nabla V_0(q,z), [z]_\times \kappa_0(q,z) \right\rangle = 0$ for all $(q,z) \in \mathbb{X}_0$ so that $\mathcal{Y}_0 = \mathbb{X}_0 = \mathcal{E}_0$ and $\Omega_0 = \mathcal{W}_0 = \{(q,z) \in \mathbb{X}_0 : \nabla V_0(q,z)^\top [z]_\times = 0\}$. The pair (V_0,κ_0) is then a weak SLFF pair for (19a) relative to $(\mathcal{A}_0, \mathbb{X}_0)$ if

$$\inf_{(q,z) \in \Omega_0 \setminus \mathcal{A}_0} \mu_{V_0}(q,z) > 0. \tag{20}$$

To satisfy (20), we may use the synergistic potential functions of [3], [2]. We henceforth assume that the synergy gap weakly totally exceeds a constant $\delta(q,z) = c > 0$. $\square$

B. Pure and Ready-made SLFF pairs

When $\mathcal{Y} = \mathbb{X}$, a (weak) SLFF pair is called a (weak) *pure* SLFF pair. A weak SLFF pair (V,κ) with synergy gap weakly (totally) exceeding $\delta : \mathbb{X} \to \mathbb{R}_{\geq 0}$ is said to be *type I ready-made* with respect to the continuous, positive definite function $\sigma : \mathbb{R}^m \to \mathbb{R}_{\geq 0}$ if there exists a continuous function $\varrho : \mathbb{X} \to \mathbb{R}_{\geq 0}$ such that, $\forall (q,z) \in \mathbb{X}$, $s \in Q$, and $\omega = \kappa(q,z)$,

$$\sigma\big(\omega - \kappa(s,z)\big) - \sigma\big(\omega - \kappa(q,z)\big) \leq \varrho(q,z) \tag{21}$$

and, for all $(q,z) \in (\Omega \setminus \mathcal{A}) \cup \overline{\mathbb{X} \setminus \mathcal{Y}}$,

$$\mu_V(q,z) > \delta(q,z) + \varrho(q,z). \tag{22}$$

Since $\mu_V(q,z) = 0$ for $(q,z) \in \mathcal{A}$, the type I ready-made property implies that

$$\overline{\mathbb{X} \setminus \mathcal{Y}} \cap \mathcal{A} = \emptyset. \tag{23}$$

If κ does not depend on q then, in (21), we can take $\varrho(q,z) = 0$ for all $(q,z) \in \mathbb{X}$. With this choice for ϱ, if (23) holds then (22) follows from (17). According to the last statement of Proposition 1, if δ is positive valued, V is radially unbounded, and the condition (23) holds, then the type I ready-made property is achievable for any σ by modifying the function V as $\rho(V)$ with ρ chosen appropriately.

A weak SLFF pair (V,κ) with synergy gap weakly (totally) exceeding $\delta : \mathbb{X} \to \mathbb{R}_{\geq 0}$ is said to be *type II ready-made* with respect to the continuous, positive definite function $\sigma : \mathbb{R}^m \to \mathbb{R}_{\geq 0}$ if there exists a continuous function $\varrho : \mathbb{X} \to \mathbb{R}_{\geq 0}$ such that, for all $(q,z) \in \mathbb{X}$, $s \in Q$, and $\omega \in \mathbb{R}^m$, (21) holds and, moreover, (22) holds for all $(q,z) \in (\Omega \setminus \mathcal{A}) \cup \overline{\mathbb{X} \setminus \mathcal{Y}}$. In particular, the difference between type I and type II ready-made is in the requirement on ω for which (21) holds: $\omega = \kappa(q,z)$ for type I and $\omega \in \mathbb{R}^m$ for type II. Clearly, if the SLFF pair is type II ready-made then it is type I ready-made. Like for the type I case, if κ is independent of q then, in (21), we can take $\varrho(q,z) = 0$ for all $(q,z) \in \mathbb{X}$.

Example 2 (3-D pendulum): The weak SLFF pair (V_0,κ_0) for (19a) with synergy gap weakly totally exceeding $c > 0$ is type I/II ready made with respect to any positive definite function σ and appropriate function ϱ, since $\kappa_0(q,z) \equiv 0$ does not depend on q. $\square$

V. READY-MADE BACKSTEPPING

The ensuing backstepping results are useful mainly for the case where the SLFF pair for the reduced-order system has a synergy gap (totally) exceeding a positive-valued continuous function δ, i.e., $\delta : \mathbb{X} \to \mathbb{R}_{>0}$. Indeed, the nature of our backstepping results is that the extended system admits an SLFF pair with synergy gap (totally) exceeding the same function δ. If δ is not positive valued then, since it does not depend on the extended state, there is no hope of it being positive valued away from the attractor in the extended state space. In this case, the hybrid control construction based on an SLFF pair given in Theorem 2 would not be applicable.

A. *From a weak, pure, ready-made SLFF pair*

We consider the control system

$$\left.\begin{aligned} \dot{q} &= 0 \\ \dot{\zeta} &= \phi_1(q,\zeta) + \psi_1(q,\zeta)u \end{aligned}\right\} \qquad (q,\zeta) \in \mathbb{X}_1 \tag{24}$$

with $u \in \mathbb{R}^m$, where $\zeta = (z^\top, \omega^\top)^\top$, $\mathbb{X}_1 = \mathbb{X}_0 \times \mathbb{R}^m$ and

$$\phi_1(q,\zeta) = \begin{bmatrix} \phi_0(q,z) + \psi_0(q,z)\omega \\ 0 \end{bmatrix}, \ \psi_1(q,\zeta) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{25}$$

We construct a (non-weak) SLFF pair with synergy gap exceeding a positive-valued function δ by supposing we have a weak, pure, ready-made SLFF pair with synergy gap weakly (totally) exceeding δ for the reduced system

$$\left.\begin{aligned} \dot{q} &= 0 \\ \dot{z} &= \phi_0(q,z) + \psi_0(q,z)\omega \end{aligned}\right\} \qquad (q,z) \in \mathbb{X}_0 \tag{26}$$

with controls $\omega \in \mathbb{R}^m$.

Let $\mathcal{A}_0 \subset \mathbb{X}_0$ be compact. For the system (26), let (V_0, κ_0) be a weak SLFF pair relative to $(\mathcal{A}_0, \mathbb{X}_0)$, with synergy gap weakly (totally) exceeding the continuous function $\delta : \mathbb{X}_0 \to \mathbb{R}_{\geq 0}$. Let $\Gamma \in \mathbb{R}^{m\times m}$ be a symmetric, positive definite matrix and suppose that the SLFF pair is type I ready-made relative to $\sigma(v) := v^\top \Gamma v$. Define

$$\mathcal{A}_1 := \left\{(q,\zeta) \in \mathbb{X}_1 : (q,z) \in \mathcal{A}_0,\ \omega = \kappa_0(q,z)\right\}. \tag{27}$$

For each $(q,\zeta) \in \mathbb{X}_1$, define

$$V_1(q,\zeta) = V_0(q,z) + \sigma(\omega - \kappa_0(q,z)). \tag{28}$$

Let $\theta : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a continuous, positive definite function, and let the smooth function $\Theta : \mathbb{R}^m \to \mathbb{R}^m$ satisfy

$$v^\top \Gamma \Theta(v) + \Theta(v)^\top \Gamma v \leq -\theta(|v|) \quad \forall v \in \mathbb{R}^m. \tag{29}$$

Define

$$\begin{aligned} \kappa_1(q,\zeta) := {} & \Theta(\omega - \kappa_0(q,z)) \\ & - \tfrac{1}{2}\Gamma^{-1}\psi_0(q,z)^\top \nabla V_0(q,z) \\ & + \mathcal{D}\kappa_0(q,z)(\phi_0(q,z) + \psi_0(q,z)\omega). \end{aligned} \tag{30}$$

Theorem 3: Let the compact set $\mathcal{A}_0$ and the smooth functions (V_0, κ_0) be given. Let the compact set $\mathcal{A}_1$ be defined as in (27) and let the pair (V_1, κ_1) be defined by (28)-(30). Suppose, for the system (26), that the pair (V_0, κ_0) is a weak SLFF relative to the pair $(\mathcal{A}_0, \mathbb{X}_0)$ with synergy gap weakly (totally) exceeding the continuous function $\delta : \mathbb{X}_0 \to \mathbb{R}_{\geq 0}$ and the SLFF pair is type I ready-made relative to the function σ defined as $\sigma(v) = v^\top \Gamma v$ where Γ is a symmetric positive definite matrix. Under these conditions, for the system (24)-(25), the pair (V_1, κ_1) is an (non-weak) SLFF pair relative to the pair $(\mathcal{A}_1, \mathbb{X}_1)$ with synergy gap (totally) exceeding δ.

Example 3 (3-D pendulum): Consider the input transformation $\tau = -[J\omega]_\times \omega - mg[\nu]_\times z + Ju$, which renders the angular velocity dynamics (19b) as $\dot{\omega} = u$. We now apply Theorem 3. Let $\sigma(\omega) = \frac{1}{2}\omega^\top J \omega$ (i.e. $\Gamma = J/2$) and let $\Theta(\omega) = J^{-1}\left([J\omega]_\times \omega - \Xi(\omega)\right)$, where $\Xi : \mathbb{R}^3 \to \mathbb{R}^3$ satisfies $\omega^\top \Xi(\omega) \geq \theta(|\omega|)$ and $\theta : \mathbb{R} \to \mathbb{R}$ is a continuous, positive definite function. Applying (30), we arrive at

$$u(q,z) = J^{-1}\left([J\omega]_\times \omega - \Xi(\omega)\right) - J^{-1}[z]_\times^\top \nabla V(q,z),$$

which yields

$$\tau = \kappa_1(q,z) = -mg[\nu]_\times z - \Xi(\omega) - [z]_\times^\top \nabla V_0(q,z). \tag{31}$$

This recovers the feedback of [3]. As a result of Theorem 3, it follows that (V_1, κ_1), with $V_1(q,z,\omega) = V_0(q,z) + \frac{1}{2}\omega^\top J\omega$, is an SLFF pair for (19) relative to $(\mathcal{A}_1, \mathbb{X}_1)$, where $\mathbb{X}_1 = Q \times \mathbb{S}^2 \times \mathbb{R}^3$ and $\mathcal{A}_1 = \{(q,z,\omega) \in \mathbb{X}_1 : q \in S,\ z = -\nu/|\nu|,\ \omega = 0\}$, with gap totally exceeding $\delta(q,z) = c$. □

B. *From a weak, ready-made SLFF pair*

We again consider the control system (24)-(25). Let $\mathcal{A}_0 \subset \mathbb{X}_0$ be compact and let $\mathcal{Y}_0 \subset \mathbb{X}_0$ be closed. The results in this section apply to the case where $\mathcal{Y}_0$ is not necessarily equal to $\mathbb{X}_0$. For the system (26), let (V_0, κ_0) be a weak, SLFF pair relative to $(\mathcal{A}_0, \mathcal{Y}_0)$ with synergy gap weakly (totally) exceeding the continuous function $\delta : \mathbb{X}_0 \to \mathbb{R}_{\geq 0}$. In addition, suppose the SLFF pair is type I ready-made relative to $\sigma(v) := L|v|$ where $L > 0$. Let $\Gamma \in \mathbb{R}^{m\times m}$ be a positive definite, symmetric matrix. Define $\sigma_2(v) := v^\top \Gamma v$ for all $v \in \mathbb{R}^m$. Let $\rho \in \mathcal{K}_\infty$ be smooth, such that $\rho'(s) > 0$ for all $s \geq 0$, and such that $\rho \circ \sigma_2$ is globally Lipschitz with constant less than or equal to L. For example, pick $\rho(s) = c\tilde{\rho}(s)$ where $c > 0$ is sufficiently small and $\tilde{\rho}(s) = s$ for $s \in [0,1]$, $\tilde{\rho}(s) = k\sqrt{s}$ for $s \geq 2$ where $k \geq 1$, and such that $\tilde{\rho}'(s) > 0$ for $s \in [1,2]$ to smoothly connect the value 1 at $s = 1$ to the value $k\sqrt{2}$ at the value $s = 2$. This construction makes the SLFF pair (V_0, κ_0) type II ready-made for the function $v \mapsto \rho(\sigma_2(v))$. Define $\mathcal{Y}_1 := \mathcal{Y}_0 \times \mathbb{R}^m$,

$$V_1(q,\zeta) := V_0(q,z) + \rho(\sigma_2(\omega - \kappa_0(q,z))). \tag{32}$$

and

$$\begin{aligned} \kappa_1(q,\zeta) := {} & \frac{1}{\rho'(\sigma_2(\omega - \kappa_0(q,z)))}\Big[\Theta(\omega - \kappa_0(q,z)) \\ & - \tfrac{1}{2}\Gamma^{-1}\psi_0(q,z)^\top \nabla V_0(q,z) \\ & + \mathcal{D}\kappa_0(q,z)(\phi_0(q,z) + \psi_0(q,z)\omega)\Big]. \end{aligned} \tag{33}$$

Theorem 4: Let the compact set $\mathcal{A}_0 \subset \mathbb{X}_0$ and the closed set $\mathcal{Y}_0 \subset \mathbb{X}_0$ be given. Let ρ and σ_2 be such that $\sigma_2(v) = v^\top \Gamma v$ for all $v \in \mathbb{R}^m$ where $\Gamma \in \mathbb{R}^{m\times m}$ is a symmetric, positive definite matrix, $\rho \in \mathcal{K}_\infty$ is smooth, $\rho'(s) > 0$ for all $s \geq 0$, and $v \mapsto \rho(\sigma_2(v))$ is globally Lipschitz with constant less than or equal to $L > 0$. Let the compact set $\mathcal{A}_1$ be defined as in (27) and let the pair (V_1, κ_1) be defined by (32)-(33). Suppose, for the system (26), that the pair (V_0, κ_0)

is a weak SLFF pair relative to $(\mathcal{A}_0, \mathcal{Y}_0)$ with synergy gap weakly (totally) exceeding the continuous function $\delta : \mathbb{X}_0 \to \mathbb{R}_{\geq 0}$ and the SLFF pair is type I ready-made relative to the function σ given by $\sigma(v) := L|v|$ for all $v \in \mathbb{R}^m$. Under these conditions, for the system (24)-(25), the pair (V_1, κ_1) is a (non-weak) SLFF pair relative to $(\mathcal{A}_1, \mathcal{Y}_1)$ with synergy gap (totally) exceeding δ.

VI. SMOOTHING WITHOUT BACKSTEPPING

Now we consider the situation where the control does not enter through an integrator but we want to remove jumps from the feedback. The ideas described here are also used in Section VII for a backstepping algorithm that does not require the SLFF pair to be ready-made. Henceforth, we work with SLFF pairs having a synergy gap bounded away from a function δ. The synergy gap is said to be (totally) bounded away from a continuous function $\delta : \mathbb{X} \to \mathbb{R}_{\geq 0}$ if there exists a positive real number ε such that the energy gap (totally) exceeds the function $(q, z) \mapsto \tilde{\delta}(q, z) := \delta(q, z) + \varepsilon$. We note that if the synergy gap is (totally) bounded away from a continuous function $\delta : \mathbb{X} \to \mathbb{R}_{\geq 0}$ then, because $\mu_V(q, z) = 0$ for $(q, z) \in \mathcal{A}$, it follows that $\overline{\mathbb{X} \setminus \mathcal{Y}} \cap \mathcal{A} = \emptyset$. We start with the control system

$$\left.\begin{aligned} &\dot{q} = 0 \\ &\dot{z} = \phi_0(q, z) + \psi_0(q, z)\omega \end{aligned}\right\} \qquad (q, z) \in \mathbb{X}_0 \tag{34}$$

with controls $\omega \in \mathbb{R}^m$ for which we suppose we have a (non-weak) SLFF pair relative to the pair $(\mathcal{A}_0, \mathbb{X}_0)$ where $\mathcal{A}_0 \subset \mathbb{X}_0$ is compact with synergy gap (totally) bounded away from the function $\delta : \mathbb{X} \to \mathbb{R}_{\geq 0}$. Let M_0 be the projection of $\mathbb{X}_0$ in the z direction, i.e.,

$$M_0 := \{z \in \mathbb{R}^n : (q, z) \in \mathbb{X}_0 \text{ for some } q \in Q\}.$$

Let N be the cardinality of Q, let $r \leq N$, and let the smooth functions $\beta_0 : M_0 \to \mathbb{R}^m$ and $\vartheta_0 : M_0 \to \mathbb{R}^{m \times r}$ satisfy

$$\kappa_0(q, z) = \beta_0(z) + \vartheta_0(z)\varsigma_0(q) \qquad \forall (q, z) \in \mathbb{X}_0, \tag{35}$$

where $\varsigma_0 : Q \to \mathbb{R}^r$ is some function of the variable q. In turn, we see that, for the system

$$\left.\begin{aligned} &\dot{q} = 0 \\ &\dot{z} = \phi_0(q, z) + \psi_0(q, z)\left(\beta_0(z) + \vartheta_0(z)p\right) \end{aligned}\right\} \quad (q, z) \in \mathbb{X}_0$$

with p as the control variable, the pair (V_0, ς_0) is an SLFF pair with respect to $(\mathcal{A}_0, \mathbb{X}_0)$ with synergy gap (totally) bounded away from δ. Let $\varepsilon > 0$ be such that the synergy gap (totally) exceeds $\tilde{\delta}_1(q, z) := \delta(q, z) + \varepsilon$. Since the set Q is finite, we can easily find a positive definite, symmetric matrix Γ such that, with $\sigma_0(v) = v^\top \Gamma v$, we have

$$\sigma_0(\varsigma_0(q) - \varsigma_0(s)) \leq \frac{\varepsilon}{2} \qquad \forall (q, s) \in Q \times Q. \tag{36}$$

This implies that the SLFF pair (V_0, ς_0), with a synergy gap (totally) exceeding $\tilde{\delta}_2(q, z) := \delta(q, z) + \varepsilon/2$, is type I ready-made for backstepping relative to σ_0. Like in Section V-B, we can also find a function ρ so that, for all $(p, q, s) \in \mathbb{R}^r \times Q \times Q$,

$$\rho(\sigma_0(p - \varsigma_0(s))) - \rho(\sigma_0(p - \varsigma_0(q))) \leq \frac{\varepsilon}{2}. \tag{37}$$

In particular, this implies the SLFF pair (V_0, ς_0), with a synergy gap (totally) exceeding $\tilde{\delta}_2(q, z) := \delta(q, z) + \varepsilon/2$, is type II ready-made for backstepping relative to $\rho \circ \sigma_0$.

Now, using Lemma 1, and depending on whether the original pair (V_0, κ_0) was pure or not, we can apply either Theorem 3 or Theorem 4 to construct a pair (V_1, ς_1) that is an SLFF pair with synergy gap (totally) exceeding $\tilde{\delta}_2$ for

$$\left.\begin{aligned} &\dot{q} = 0 \\ &\dot{z} = \phi_0(q, z) + \psi_0(q, z)\left(\beta_0(z) + \vartheta_0(z)p\right), \ \dot{p} = \alpha \end{aligned}\right\} \\ (q, z, p) \in \mathbb{X}_0 \times \mathbb{R}^r.$$

In particular, from the definition of $\tilde{\delta}_2$, it follows that the synergy gap is (totally) bounded away from δ.

Note that if (V_0, κ_0) was a *weak* SLFF pair for the system (34), this fact would not necessarily guarantee that (V_0, ς_0) is a *weak* SLFF pair for (36), because of the ϑ_0 term that multiplies ψ_0 to generate the input vector field. This observation motivates assuming that (V_0, κ_0) is an (non-weak) SLFF pair for the system (34). In the next section, we will want to allow (V_0, κ_0) to be a weak SLFF pair for the system (34) in anticipation of another backstepping result. We will be able to get away with this weakened assumption because we will come back to the integral of ω, rather than the integral of p, as the control variable.

Example 4 (3-D pendulum): Let $\mathbf{e}_i \in \mathbb{R}^N$ denote the vector with 1 in the ith index and zeros elsewhere. Assuming that (without loss of generality) $Q = \{1, \ldots, N\}$, κ_1—as defined in (31)—can be written as (35). In particular, define

$$\begin{aligned} &\mathcal{V}_0(z) = [V_0(1, z) \ \cdots \ V_0(N, z)]^\top && \vartheta(z) = [z]_\times \mathcal{D}\mathcal{V}_0(z)^\top \\ &\beta(z) = -mg\,[\nu]_\times z - \Xi(\omega) && \varsigma(q) = \mathbf{e}_q, \end{aligned}$$

which yields the closed-loop dynamics of (19) as

$$\dot{z} = [z]_\times \omega \quad J\dot{\omega} = [J\omega]_\times \omega - \Xi(\omega) + \vartheta(z)\mathbf{e}_q.$$

By replacing $\mathbf{e}_q$ with a control variable p, we have that (V_1, ς) is a (non-weak) SLFF pair relative to $(\mathcal{A}_1, \mathbb{X}_1)$ (with V_1, $\mathcal{A}_1$ and $\mathbb{X}_1$ defined in Example 3) with synergy gap totally exceeding $\delta(q, z, \omega) = c$. Suppose also that the synergy gap totally exceeds $c + \epsilon$ and let $\sigma(v) = \frac{\epsilon}{8}|v|^2$ so that for all $(q, s) \in Q \times Q$, $\sigma(\mathbf{e}_q - \mathbf{e}_s) \leq \epsilon/2$ and (V_1, ς) is also type I ready-made with respect to σ.

Now, define $V_2(q, z, \omega, p) = V_1(q, z, \omega) + \sigma(p - \mathbf{e}_q)$, $\mathbb{X}_2 = Q \times \mathbb{S}^2 \times \mathbb{R}^3 \times \mathbb{R}^N$, $\mathcal{A}_2 = \{(q, z, \omega, p) \in \mathbb{X} : q \in S, \ z = -\nu/|\nu|, \ \omega = 0 \ p = \mathbf{e}_q\}$, and

$$\gamma(q, z, \omega, p) = \Theta(p - \mathbf{e}_q) - \mathcal{D}\mathcal{V}_0(z)\,[z]_\times \omega,$$

where $\Theta : \mathbb{R}^N \to \mathbb{R}^N$ satisfies (29) with $\Gamma = I$. It follows from Theorem 3 that (V_2, γ) is an SLFF pair relative to $(\mathcal{A}_2, \mathbb{X}_2)$ with synergy gap totally exceeding $c + \epsilon/2$ for the system

$$\begin{aligned} &\dot{q} = 0 && \dot{z} = [z]_\times \omega \\ &\dot{p} = \alpha && J\dot{\omega} = [J\omega]_\times \omega - \Xi(\omega) + \vartheta(z)p \end{aligned}$$

with α as the control variable.

Having input $(z, \omega) \in \mathbb{S}^2 \times \mathbb{R}^3$, memory states $(q, p) \in Q \times \mathbb{R}^N$, and output τ, the hybrid controller for the 3-D pendulum with smoothing is given as

$$\underbrace{\begin{array}{l}\tau = \beta(z) + \vartheta(z)p, \quad \dot{q} = 0 \\ \dot{p} = \Theta(p - \mathbf{e}_q) - \mathcal{D}\mathcal{V}_0(z)\,[z]_\times \omega\end{array}}_{(q,z,\omega,p) \in C} \qquad \underbrace{\begin{array}{l}q^+ = G(z,\omega,p) \\ p^+ = p\end{array}}_{(q,z,\omega,p) \in D,}$$

where

$$\begin{aligned} C &= \{(q,z,\omega,p) \in \mathbb{X} : \mu_{V_2}(q,z,\omega,p) \le c + \epsilon/2\} \\ D &= \{(q,z,\omega,p) \in \mathbb{X} : \mu_{V_2}(q,z,\omega,p) \ge c + \epsilon/2\} \\ G(z,\omega,p) &= \{g \in Q : \mu_{V_2}(g,z,\omega,p) = 0\}. \end{aligned}$$

If V_0 satisfies (20), this controller globally asymptotically stabilizes $\mathcal{B}_2$, where $\mathcal{B}_2$ is related to $\mathcal{A}_2$ through (2). □

VII. BACKSTEPPING WITHOUT BEING READY-MADE

While the backstepping constructions in this section use extra dynamic states, their advantage is that no preliminary step is needed to make them ready-made for backstepping. Suppose we have a non-weak SLFF pair (V_0, κ_0) with synergy gap (totally) bounded away from δ for

$$\left.\begin{array}{l}\dot{q} = 0 \\ \dot{z} = \phi_0(q,z) + \psi_0(q,z)\omega\end{array}\right\} \quad (q,z) \in \mathbb{X}_0. \tag{38}$$

From the results of Section VI, the pair (V_1, κ_1), of the form

$$\begin{array}{l} V_1(q,z,p) = V_0(q,z) + \sigma(p - \varsigma_0(q)) \\ \kappa_1(q,z,p) = \beta_0(z) + \vartheta_0(z)p, \end{array} \tag{39}$$

is a non-weak SLFF with synergy gap (totally) bounded away from δ for the system

$$\left.\begin{array}{l}\dot{q} = 0 \\ \dot{z} = \phi_0(q,z) + \psi_0(q,z)\omega \\ \dot{p} = \varsigma_1(q,z,p)\end{array}\right\} \quad (q,z,p) \in \mathbb{X}_0 \times \mathbb{R}^r. \tag{40}$$

Moreover, the pair (V_1, κ_1) is both type I and type II ready-made with respect to any function. Indeed, since κ_1 does not depend on q, we can take $\varrho(q,z,p) = 0$ for all $(q,z,p) \in \mathbb{X}_0 \times \mathbb{R}^r$ in (21) and then, since (23) holds because the synergy gap is (totally) bounded away from δ, (22) holds. Now we can apply Theorem 3 or, if the SLFF pair is not pure, Theorem 4 to generate a non-weak SLFF pair (V_2, κ_2) with synergy gap (totally) bounded away from δ for the extended system

$$\left.\begin{array}{l}\dot{q} = 0 \\ \dot{z} = \phi_0(q,z) + \psi_0(q,z)\omega \\ \dot{p} = \varsigma_1(q,z,p) \\ \dot{\omega} = u\end{array}\right\} \quad (q,z,p,\omega) \in \mathbb{X}_0 \times \mathbb{R}^r \times \mathbb{R}^m. \tag{41}$$

Finally, consider the case where (V_0, κ_0) is a weak (rather than non-weak) SLFF pair for (38). In this case it turns out that the SLFF pair (V_1, κ_1) of the form (39) is a weak SLFF pair for the system (40). This fact is explained below. From here, Theorem 3 or 4 can be applied as above to derive a *non-weak* SLFF pair (V_2, κ_2) for the system (41).

Suppose (V_0, κ_0) is a weak SLFF pair for (38). Write the system (40) in the form

$$\left.\begin{array}{l}\dot{q} = 0 \\ \dot{\zeta} = \phi_1(q,\zeta) + \psi_1(q,\zeta)\omega\end{array}\right\} \quad (q,\zeta) \in \mathbb{X}_1 \tag{42}$$

where $\zeta := (z^\top, p^\top)^\top$, $\mathbb{X}_1 := \mathbb{X}_0 \times \mathbb{R}^r$,

$$\phi_1(q,\zeta) := \begin{bmatrix} \phi_0(q,z) \\ \varsigma_1(q,z,p) \end{bmatrix}, \ \psi_1(q,\zeta) := \begin{bmatrix} \psi_0(q,z) \\ 0 \end{bmatrix}. \tag{43}$$

It follows from the definitions that

$$\nabla V_1(q,\zeta)^\top \psi_1(q,\zeta) = \nabla V_0(q,z)^\top \psi_0(q,z).$$

Also, it follows from the proof of Theorems 3 and 4 that

$$\begin{array}{l} \langle \nabla V_1(q,z), \phi_1(q,\zeta) + \psi_1(q,\zeta)\kappa_1(q,\zeta) \rangle = 0 \\ \Longrightarrow \left\{ \begin{array}{l} 0 = \langle \nabla V_0(q,z), \phi_0(q,z) + \psi_0(q,z)\kappa_0(q,z) \rangle \\ p = \varsigma_0(q). \end{array} \right. \end{array}$$

Therefore $\Omega_1 = \{(q,\zeta) \in \mathbb{X}_1 : (q,z) \in \Omega_0, \ p = \varsigma_0(q)\}$. This relationship can be used to arrive at the conclusion that (V_1, κ_1) is a *weak* SLFF for the system (40) with synergy gap (totally) bounded away from δ.

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