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Freshwater-Saltwater Boundary Detection Using Mobile Sensors Part II: Drifter Movement

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Abstract— **In this paper, we study the minimum energy control problem in a flow environment, in which the flow is described by a quadratic function and the objective is to minimize the energy consumption while driving drifters from one point to another. The minimum energy control problem is motivated by the fact that mobile drifters are powered by batteries of limited capacity. We first derive the unconstrained optimal control using the Pontryagin's minimum principle, and then take into account state constraints that arise naturally in our setting. If the dynamics of drifters are restricted to be of Dubin's type, the optimal control is in the form of state feedback, which potentially leads to more robust implementation.**

I. Introduction

In estuaries, saline saltwater can intrude deeply into river channels, due to factors such as tidal forcing, droughts, the diversion of freshwater for agricultural and/or municipal uses, and hurricanes. Saltwater intrusion can have a huge impact on the freshwater supply for agriculture and municipal uses, and the inland biota. The freshwater-saltwater boundary in estuaries acts as an indicator of saltwater intrusion, and has been studied based on simplified 1-D theoretical analysis [1], numerical analysis [2], or static measurements [3]. However, none of these approaches is capable of obtaining a relatively accurate boundary (that might change slowly with time). In a companion paper [4], we study the freshwater-saltwater boundary detection problem by deploying two drifters to explore the boundary, and propose boundary exploration algorithms that search for a point which can reflect the degree of saltwater intrusion.

There are two types of drifter movements in the boundary exploration algorithms [4]: i) a drifter moves along a straight line and stops only when the measured salinity satisfies a certain condition, and ii) a drifter moves from one point to another. In this paper, we focus on the second type of movements, and study control strategies that drive the drifter from one point to another as well as minimizing the energy consumption; the latter consideration is due to the fact that mobile drifters are commonly powered by batteries of limited capacity. To better approximate the river flow in well aligned channels (e.g., irrigation canals), we model the flow using a quadratic function; in contrast, previous work usually assumes that the river flow is approximated as constant, piecewise constant, or affine flows. However, this better approximation makes the minimum energy control problem very challenging. Constraints such as that the optimal trajectory should stay inside the river region impose further challenges.

We first derive the unconstrained optimal control using the Pontryagin's minimum principle [5], and then look into issues such as bounds on the parameters that characterize the optimal control, conditions under which the optimal trajectory stays inside the river region, and the existence of multiple optimal trajectories. For minimum energy control with state constraints that arise naturally in our setting, we first adjoin additional multiplier functions to the Hamiltonian, and then provide a set of equations that need to be solved to obtain the optimal control. If the dynamics of drifters are restricted to be of Dubin's type, the optimal control is in the form of state feedback, which potentially leads to more robust implementation.

Minimum energy control has been considered, e.g. [6]–[8]. In [6]–[8], the goal is to find a path with the minimum energy loss between given source and destination points in piecewise constant regions. Our work differs from [6]–[8] in the cost function. More specifically, in [6], [8], the cost of any path on the surface of a terrain is defined to be the energy loss due to both friction and gravity and could be anisotropic, and in [7], the cost in each region is isotropic. In our work, the cost is defined to be the minimum energy necessary to run against the river flow and is anisotropic. If drifters are operated as Dubin's vehicles, the minimum energy control problem reduces to the minimum time control problem. Minimum time path planning in a flow environment originates from the classic Zermelo's navigation problem. For some recent work, refer to [9]–[11]. This work differs from those references in that the flow is described by a quadratic function and the optimal trajectory is constrained to the river environment.

II. Problem Formulation

For simplicity, we assume that the river banks are parallel and the deployment base is on the freshwater side. Using the Cartesian coordinate system, we define the centerline of the river as the x-axis, and choose the y-axis to be perpendicular to the x-axis and pass through the deployment base. Suppose the distance from the deployment base to the x-axis is $\frac{L}{2}$ (note that $L > 0$ is chosen to be less than the width of the river so that drifters can be operated safely), then the studied river environment can be described by the two dimensional

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region $D := \{(x\ y)^T \in \mathbb{R}^2 \mid -\frac{L}{2} \le y \le \frac{L}{2}\}$, as shown in Fig. 1.

The velocity field in the region D is a mapping $V: D \mapsto \mathbb{R}^2$, assigning $(V_x(x,y)\ V_y(x,y))^T$ to a point $(x\ y)^T \in D$. We assume that $V_y(x,y) = 0$, and[1]

$$V_x(x,y) = v(y) = Ay^2 + B \ , \tag{1}$$

where $A < 0$ and $B > 0$ are constants, and $0 < Ay^2 + B \le B$ for any y satisfying $|y| \le \frac{L}{2}$. A typical velocity profile is plotted in Fig. 1. Note that practical examples of the above flow field could be river flows in irrigation canals and straight channels.

A drifter runs at speed $U = (U_x\ U_y)^T$ relative to the velocity field, and then the dynamics of the drifter can be described by

$$\frac{dx}{dt} = U_x + V_x(x,y), \quad \frac{dy}{dt} = U_y + V_y(x,y) \ .$$

We assume that drifters can run against the river flow to track the freshwater-saltwater boundary, i.e., $\max_{(x\ y)^T \in D} \|V(x,y)\| \le \|U\|_{\max}$; for example, $\|U\|_{\max}$ could be $\sqrt{5}B$ as discussed in Remark 1 of Section III. If a drifter is operated as a Dubin's vehicle, then its velocity magnitude is fixed as $\|U\|$ but the heading angle θ can be controlled, i.e., $U = (\|U\|\cos\theta \quad \|U\|\sin\theta)^T$.

The freshwater-saltwater boundary can be described using the salinity field in the river environment. The salinity field is a mapping $S: D \mapsto \mathbb{R}_0^+$ assigning $S(x,y)$ to a point $(x\ y)^T \in D$, where $\mathbb{R}_0^+$ is the set of nonnegative real numbers and $S(x,y)$ is the salinity at location $(x\ y)^T$. Given a salinity threshold B_{th}, the freshwater-saltwater boundary is defined as a level set $\{(x\ y)^T \in D \mid S(x,y) = B_{th}\}$. A typical boundary is plotted in Fig. 1.

The objective is to deploy a set of drifters from the base and find a point $p_{\min}$ on the boundary with the smallest x coordinate. The drifter deployment problem is studied in detail in a companion paper [4], in which there are two types of drifter movements:

- A drifter moves parallel to the x axis and stops when the salinity measurement satisfies a certain condition (e.g., this type is used in Algorithms 2 and 3 in [4]);
- A drifter moves from one point to another (e.g., this type is used in Algorithms 1, 2, and 3 in [4]).

[1]The dependency of $V_x(x,y)$ on y can be approximated as a quadratic function for well aligned river environments [12], [13].

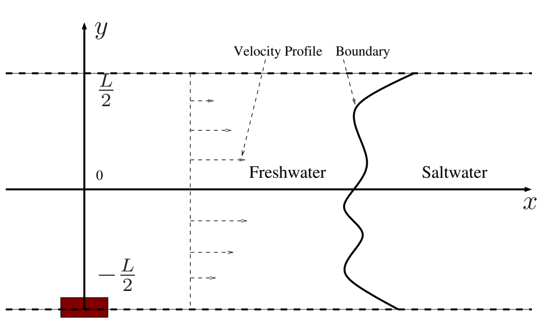

Fig. 1. River environment for drifter deployment.

For the first type of movements, if the drifter moves downstream, no control is necessary; if it moves upstream, a control along the negative x axis can be chosen to achieve the objective since the drifter can run against the current. The second type of movements is the focus of this paper, and we want to find a control strategy that minimizes the energy. More specifically, given p^1 and p^2 inside the region D, the minimum energy control problem is formulated as

$$\begin{aligned} &\min \int_0^{t_f} U^T U dt \ , \\ &\text{s.t. } \frac{dx}{dt} = U_x + Ay^2 + B \ , \ \frac{dy}{dt} = U_y \ , \\ &\quad x(0) = x_{p^1},\ y(0) = y_{p^1}\ , x(t_f) = x_{p^2},\ y(t_f) = y_{p^2} \ . \end{aligned} \tag{2}$$

Note that t_f is free, and $|y_{p^i}| \le \frac{L}{2}$ for $i = 1, 2$ (since the drifter must stay inside the region D).

III. Minimum Energy Control

In this section we study how to obtain the unconstrained minimum energy control using the Pontryagin's minimum principle [5].

Proposition 1 The optimal control to the minimum energy control problem without state constraints is given as $U(t) = -\frac{1}{2}\begin{bmatrix} C_1 \\ P_2(t) \end{bmatrix}$ for $t \in [0, t_f]$, where C_1 is a constant, $P_2(t) = \cosh(t\sqrt{AC_1})P_2(0) - 2\sqrt{AC_1}\sinh(t\sqrt{AC_1})y_{p^1}$, $\cosh(x) = \frac{e^x + e^{-x}}{2}$, and $\sinh(x) = \frac{e^x - e^{-x}}{2}$. If $C_1 \ne 0$, the parameters C_1, $P_2(0)$ and t_f satisfy the following set of equations

$$y_{p^2} = \cosh(t_f\sqrt{AC_1})y_{p^1} - \frac{\sinh(t_f\sqrt{AC_1})P_2(0)}{2\sqrt{AC_1}} \ , \tag{3}$$

$$\begin{aligned} x_{p^2} = &x_{p^1} + \frac{P_2(0)y_{p^1}}{4C_1} - \\ &\frac{(4C_1^2 - 4AC_1 y_{p^1}^2 - 8BC_1 + P_2(0)^2)t_f}{8C_1} + \\ &\frac{(P_2(0)^2 + 4AC_1 y_{p^1}^2)\sinh(2t_f\sqrt{AC_1})}{16C_1\sqrt{AC_1}} - \\ &\frac{P_2(0)y_{p^1}\cosh(2t_f\sqrt{AC_1})}{4C_1} \ , \end{aligned} \tag{4}$$

$$-\frac{C_1^2 + P_2(t_f)^2}{4} + C_1(Ay_{p^2}^2 + B) = 0 \ . \tag{5}$$

If $C_1 = 0$, $P_2(0)$ and t_f satisfy

$$y_{p^2} = y_{p^1} - \frac{P_2(0)t_f}{2} \ , \tag{6}$$

$$x_{p^2} = x_{p^1} + Bt_f + A\left(\frac{P_2(0)^2 t_f^3}{12} - \frac{P_2(0)y_{p^1}t_f^2}{2} + y_{p^1}^2 t_f\right) \ . \tag{7}$$

Proof: The Hamiltonian of the minimum energy control problem is $H = U^T U + P^T(U + N)$, where $P = (P_1\ P_2)^T$

and $N = (Ay^2 + B\ 0)^T$. Using the minimum principle, we obtain the following coupled ODEs besides Eq. (2):

$$\frac{dP_1}{dt} = 0 \ , \tag{8}$$

$$\frac{dP_2}{dt} = -2AP_1 y \ . \tag{9}$$

Since U is chosen to minimize the Hamiltonian, $U = -\frac{1}{2}P$. Plugging U into Eq (2), we obtain the following ODEs:

$$\frac{dx}{dt} = -\frac{1}{2}P_1 + Ay^2 + B \ , \tag{10}$$

$$\frac{dy}{dt} = -\frac{1}{2}P_2 \ . \tag{11}$$

From Eq. (8), we know P_1 is a constant and let $P_1(t) = C_1$ for $t \in [0, t_f]$. Depending on whether C_1 is 0 or not, there are two cases to consider.

Case I If C_1 is nonzero, Eqs. (9) and (11) can be grouped together and rewritten as $\frac{dY}{dt} = MY$, where $Y = (y\ P_2)^T$ and

$$M = \begin{bmatrix} 0 & -\frac{1}{2} \\ -2AC_1 & 0 \end{bmatrix} \ .$$

Therefore, $Y(t) = e^{Mt}Y(0)$. After calculating the matrix exponential e^{Mt}, we obtain

$$y(t) = \cosh(t\sqrt{AC_1})y_{p^1} - \frac{\sinh(t\sqrt{AC_1})P_2(0)}{2\sqrt{AC_1}} \ , \tag{12}$$

$$P_2(t) = \cosh(t\sqrt{AC_1})P_2(0) - 2\sqrt{AC_1}\sinh(t\sqrt{AC_1})y_{p^1} \ , \tag{13}$$

where $P_2(0)$ is the unknown initial condition for $P_2(t)$. Since $y(t_f)$ is given, we obtain the first equation on C_1, $P_2(0)$ and t_f as Eq. (3). Plugging the expression of y in Eq. (12) into Eq. (10) and using $x(0) = x_{p^1}$, we can solve for x as below

$$\begin{aligned} x(t) = \ & x_{p^1} + \frac{P_2(0)y_{p^1}}{4C_1} - \\ & \frac{(4C_1^2 - 4AC_1y_{p^1}^2 - 8BC_1 + P_2(0)^2)t}{8C_1} + \\ & \frac{(P_2(0)^2 + 4AC_1y_{p^1}^2)\sinh(2t\sqrt{AC_1})}{16C_1\sqrt{AC_1}} - \\ & \frac{P_2(0)y_{p^1}\cosh(2t\sqrt{AC_1})}{4C_1} \ . \end{aligned} \tag{14}$$

Since $x(t_f)$ is given, we obtain the second equation on C_1, $P_2(0)$ and t_f as Eq. (4). Since t_f is free and there is no cost imposed on the final state, $H|_{t_f} = 0$. Therefore, we get the third equation on C_1, $P_2(0)$ and t_f as Eq. (5).

By solving Eqs. (3), (4) and (5), we obtain C_1, $P_2(0)$ and t_f, and then the optimal control is $U = -\frac{1}{2}(C_1\ P_2(t))^T$ for $t \in [0, t_f]$.

Case II If C_1 is zero, then $P_1(t) = 0$ for $t \in [0, t_f]$. From Eq. (9), we get $P_2(t) = P_2(0)$ for $t \in [0, t_f]$. From Eq. (11), we obtain

$$y(t) = y_{p^1} - \frac{P_2(0)t}{2} \ . \tag{15}$$

Since we are given $y(t_f)$, we obtain the first equation on t_f and $P_2(0)$ as Eq. (6). Plugging y into Eq. (10), we can solve for x and get

$$x(t) = x_{p^1} + Bt + A\left(\frac{P_2(0)^2t^3}{12} - \frac{P_2(0)y_{p^1}t^2}{2} + y_{p^1}^2 t\right) \ . \tag{16}$$

Based on the given $x(t_f)$, we obtain the second equation on t_f and $P_2(0)$ as Eq. (7). After solving Eqs. (6) and (7), we obtain $P_2(0)$ and t_f, and then the optimal control is given as $U(t) = -\frac{1}{2}(0\ P_2(0))^T$ for $t \in [0, t_f]$. ∎

The following proposition shows that Case II is the limit of Case I when C_1 goes to 0, and can be verified based on Eqs. (12), (13), and (14).

Proposition 2 If the parameter C_1 in Proposition 1 converges to 0, the optimal control (namely, C_1 and the function $P_2(t)$) and the optimal trajectory (namely, the functions $x(t)$ and $y(t)$) in Case I converge pointwise to the optimal control and the optimal trajectory in Case II.

In the following theorem, we propose bounds on C_1, $P_2(0)$ and t_f, which are helpful when guessing an initial solution since Eqs. (3), (4) and (5) are nonlinear.

Theorem 1 In the minimum energy control problem without state constraints, the following results hold:

1) $0 \le C_1 \le \min(4v(y_{p^1}), 4v(y_{p^2}))$;
2) $C_1 = 0$ if and only if $x_{p^1} < x_{p^2}$ and $y_{p^1} = y_{p^2}$;
3) If $C_1 \ge 4v(\frac{L}{2})$, $|y(t)| \le \frac{L}{2}$ for any $t \in [0, t_f]$;
4) If $L \ge 2\sqrt{-\frac{B}{A}}$, $|y(t)| \le \frac{L}{2}$ for any $t \in [0, t_f]$;
5) $|P_2(t)| \le 2B$, $|P_2(0)| \le 2v(y_{p^1})$, and $|P_2(t_f)| \le 2v(y_{p^2})$;
6) $t_f \ge \max\left(\frac{|y_{p^2} - y_{p^1}|}{B}, \frac{|x_{p^2} - x_{p^1}|}{B + \min(2v(y_{p^1}),\ 2v(y_{p^2}))}\right)$;
7) If $y_{p^1} = 0$, then any point p^2 satisfying $x_{p^2} < x_{p^1}$ and $y_{p^2} = 0$ can be reached via two different paths with the same minimum energy cost.

Proof: 1) Since the Hamiltonian does not explicitly depend on t, $H|_{t_f} = 0$ (namely, Eq. (5)) implies $H|_0 = 0$. Therefore, we have $-\frac{C_1^2 + P_2(0)^2}{4} + C_1(Ay_{p^1}^2 + B) = 0$. Note that this is a quadratic equation of C_1 and can be rewritten as

$$C_1^2 - 4(Ay_{p^1}^2 + B)C_1 + P_2(0)^2 = 0 \ . \tag{17}$$

Let C_{11} and C_{12} be the roots of the above equation. Then $C_{11} + C_{12} = 4(Ay_{p^1}^2 + B) = 4v(y_{p^1}) > 0$ because $|y_{p_1}| \le \frac{L}{2}$, and

$$C_{11}C_{12} = P_2(0)^2 \ge 0 \ . \tag{18}$$

Therefore, we must have $C_{11} \ge 0$ and $C_{12} \ge 0$, which implies that $C_1 \ge 0$. Since $C_{11} + C_{12} = 4v(y_{p^1})$, we must have $C_1 \le 4v(y_{p^1})$. Similarly, if we consider Eq. (5), we get $C_1 \le 4v(y_{p^2})$. Thus we have $0 \le C_1 \le \min(4v(y_{p^1}), 4v(y_{p^2}))$.

2) (if part) We prove it by contradiction. Suppose C_1 is nonzero. Since $U = -\frac{P}{2}$, the cost satisfies

$$\int_0^{t_f} U^T U dt = \frac{1}{4}\int_0^{t_f} (C_1^2 + P_2(t)^2)dt \ge \frac{1}{4}C_1^2(t_f - 0) > 0 \ .$$

However, since $x_{p^1} < x_{p^2}$ and $y_{p^1} = y_{p^2}$, $U = (0\ 0)^T$ is feasible with cost 0 (this solution can be obtained from Case II). Therefore, if C_1 is nonzero, the solution is not optimal. A contradiction.

(only if part) If $C_1 = 0$, Eq. (18) implies that $P_2(0)$ must be 0. Therefore, in Case II, Eq. (6) reduces to $y_{p^2} = y_{p^1}$, and Eq. (7) reduces to

$$x_{p^2} = x_{p^1} + (B + Ay_{p^1}^2)t_f = x_{p^1} + v(y_{p^1})t_f > x_{p^1}$$

because $v(y_{p^1}) > 0$ and $t_f > 0$.

3) Based on Eq. (12), we have

$$|y(t)| = |\cosh(t\sqrt{AC_1})y_{p^1} - \frac{\sinh(t\sqrt{AC_1})P_2(0)}{2\sqrt{AC_1}}|$$
$$= |y_{p^1}\cos(t\sqrt{-AC_1}) - \frac{P_2(0)}{2\sqrt{-AC_1}}\sin(t\sqrt{-AC_1})| \quad (19)$$
$$\leq \sqrt{y_{p^1}^2 + \left(\frac{P_2(0)}{2\sqrt{-AC_1}}\right)^2} = \sqrt{y_{p^1}^2 + \frac{P_2(0)^2}{4(-AC_1)}}\ . \quad (20)$$

We have Eq. (19) because $A < 0$ and $C_1 \geq 0$. We obtain $P_2(0)^2$ from Eq. (17), then plug it into Eq. (20) and get

$$|y(t)| \leq \sqrt{\frac{C_1 - 4B}{4A}}\ . \quad (21)$$

Since $A < 0$ and $C_1 - 4B \leq \min(4v(y_{p^1}), 4v(y_{p^2})) - 4B \leq 4v(0) - 4B = 0$, the upper bound on $|y(t)|$ is a nonnegative real number. If $C_1 \geq 4v(\frac{L}{2})$, $C_1 - 4B \geq 4A\frac{L^2}{4}$. Since $A < 0$, we have $|y(t)| \leq \frac{L}{2}$ for any $t \in [0, t_f]$.

4) Using Eq. (21) and the fact that $C_1 \geq 0$, we have

$$|y(t)| \leq \sqrt{\frac{4B}{-4A}} = \sqrt{-\frac{B}{A}}\ . \quad (22)$$

Therefore, if $\sqrt{-\frac{B}{A}} \leq \frac{L}{2}$, i.e., $L \geq 2\sqrt{-\frac{B}{A}}$, $|y(t)| \leq \frac{L}{2}$.

5) Based on Eq. (13), we have

$$|P_2(t)| = |\cosh(t\sqrt{AC_1})P_2(0) - 2\sqrt{AC_1}\sinh(t\sqrt{AC_1})y_{p^1}|$$
$$= |P_2(0)\cos(t\sqrt{-AC_1}) + 2\sqrt{-AC_1}\sin(t\sqrt{-AC_1})y_{p^1}|$$
$$\leq \sqrt{P_2(0)^2 - 4AC_1y_{p^1}^2}\ . \quad (23)$$

We obtain $P_2(0)^2$ from Eq. (17), then plug it into Eq. (23) and get $|P_2(t)| \leq \sqrt{4BC_1 - C_1^2} = \sqrt{4B^2 - (C_1 - 2B)^2} \leq 2B$. To show the upper bound on $|P_2(0)|$, we consider Eq. (17). Since C_1 must be a real number, we have $(4(Ay_{p^1}^2 + B))^2 - 4 \times 1 \times P_2(0)^2 \geq 0$, which implies that $|P_2(0)| \leq 2(Ay_{p^1}^2 + B) = 2v(y_{p^1})$. Note that $|P_2(0)| \leq 2v(y_{p^1}) \leq 2B$ since $A < 0$. Similarly, we can show the upper bound on $P_2(t_f)$ based on Eq. (5).

6) Based on Eq. (11), we have

$$\int_{y(0)}^{y(t_f)} 1dy = \int_0^{t_f} -\frac{P_2(t)}{2}dt$$
$$|y_{p^2} - y_{p^1}| \leq \frac{1}{2}\int_0^{t_f} | - P_2(t)|dt \leq Bt_f\ .$$

Therefore, we have $t_f \geq \frac{|y_{p^2} - y_{p^1}|}{B}$. Based on Eq. (10), we have

$$\int_{x(0)}^{x(t_f)} 1dx = \int_0^{t_f} -\frac{C_1}{2} + Ay^2 + Bdt$$
$$|x_{p^2} - x_{p^1}| = |\int_0^{t_f} -\frac{C_1}{2} + Ay^2 + Bdt| \leq$$
$$\int_0^{t_f} | - \frac{C_1}{2} + Ay^2 + B|dt \leq \int_0^{t_f} |\frac{C_1}{2}| + |Ay^2 + B|dt \leq$$
$$(\min(2v(y_{p^1}),\ 2v(y_{p^2})) + B)t_f\ , \quad (24)$$

where $|Ay^2 + B| \leq B$ in Eq. (24) holds because $0 \leq B + Ay^2$ (due to Eq. (22)) and $A < 0$. Then $t_f \geq \frac{|x_{p^2} - x_{p^1}|}{B + \min(2v(y_{p^1}),\ 2v(y_{p^2}))}$. Putting them together, we get the lower bound.

7) If $y_{p^1} = y_{p^2} = 0$, it can be verified that if $P_2(0)$ is a solution to Eqs. (3), (4) and (5), $-P_2(0)$ is also a solution, which implies that the two controls given as $-\frac{1}{2}\begin{bmatrix} C_1 \\ P_2(t) \end{bmatrix}$ and $-\frac{1}{2}\begin{bmatrix} C_1 \\ -P_2(t) \end{bmatrix}$ both solve the minimum energy control problem and have the same energy cost. ■

Remark 1 If $x_{p^1} > x_{p^2}$, the intuition of 1) in Theorem 1 is quite clear because a drifter has to run against the current in the x direction to reach x_{p^2}. However, it is less intuitive that even if $x_{p^1} \leq x_{p^2}$, the optimal control always makes a drifter run against the current as long as $y_{p^1} \neq y_{p^2}$. 2) shows that the only scenario when a drifter moves following the flow is when $x_{p^1} < x_{p^2}$ and $y_{p^1} = y_{p^2}$. 3) gives a way to check if the optimal trajectory will always stay inside the region D once C_1 is calculated, and the condition only depends on the relative velocity at the boundary of the region D. Note that this condition is only sufficient: for example, as shown in 2), if $x_{p^1} < x_{p^2}$ and $y_{p^1} = y_{p^2}$, C_1 is 0; even though the condition in 3) is violated but the trajectory still stays inside the region D. 4) provides another condition which does not rely on C_1. The condition $L \geq 2\sqrt{-\frac{B}{A}}$ is equivalent to $v(\frac{L}{2}) \leq 0$. However, since we assume that $v(\frac{L}{2}) > 0$, it is possible that the optimal trajectory might leave the region D. Note that 1) and 5) implies that

$$\|U\| = \sqrt{U^TU} = \frac{\sqrt{C_1^2 + P_2(t)^2}}{2} \leq \sqrt{5}B\ ,$$

since $C_1 \leq \min(4v(y_{p^1}), 4v(y_{p^2})) \leq 4B$. The intuition about 7) is that, because the flow is symmetric with respect to the x axis, there could be two different ways but with the same energy cost to move from point p^1 on the x axis to a point p^2 also on the x axis satisfying $x_{p^2} < x_{p^1}$. Such point p^2 is called a conjugate point to p^1 [5]. Note that some solution to Eqs. (3), (4) and (5) could result in a local maximum since the minimum principle only provides necessary conditions for optimal control problems. ■

Example 1 Suppose that the flow is given as $v(y) = -\frac{y^2}{441} + 1$ and p^1 is $(5\ 5)^T$. We consider two scenarios. First, we choose p^2 to be $(9\ 5)^T$. Since $x_{p^1} < x_{p^2}$ and $y_{p^1} = y_{p^2}$,

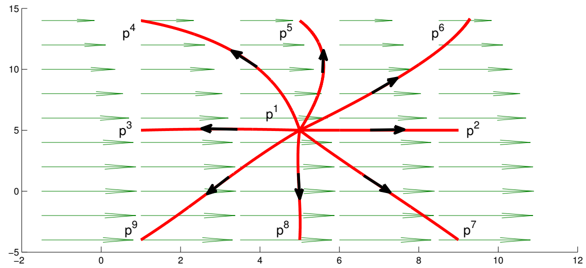

Fig. 2. Optimal trajectories (the red curves) from $p^1 = (5\ 5)^T$ to $p^2 = (9\ 5)^T$, $p^3 = (1\ 5)^T$, $p^4 = (1\ 14)^T$, $p^5 = (5\ 14)^T$, $p^6 = (9\ 14)^T$, $p^7 = (9\ -4)^T$, $p^8 = (5\ -4)^T$ and $p^9 = (1\ -4)^T$.

$C_1 = 0$ and $P_2(0) = 0$ as shown in Theorem 1. In other words, the control U is $(0\ 0)^T$ for $t \in [0, t_f]$, where t_f can be calculated based on Eq. (7) and has the value 4.2404. For the second scenario, we choose the destination p^3 to be $(1\ 5)^T$. Since $x_{p^1} > x_{p^3}$, C_1 must be nonzero. We solve Eqs. (3), (4) and (5) using the nonlinear equation solver *fsolve* in Matlab, and obtain $C_1 = 3.7642$, $P_2(0) = -0.1839$ and $t_f = 4.2539$. The optimal trajectory is shown in Fig. 2, in which the green thin arrows represent flow velocity. Let L be 32, i.e., $|y| \leq 16$ for any point p inside the region D. Since $4v(\frac{L}{2}) = 1.6780 < C_1$, the trajectory must stay inside the region D, and this is also reflected in Fig. 2. Note that $C_1 \leq \min(4v(y_{p^1}), 4v(y_{p^2})) = 3.7732$ and $|P_2(0)| \leq 2v(y_{p^1}) = 1.8866$ hold. Since $\max(\frac{0}{B}, \frac{4}{B+2v(5)}) = 1.3857$, the lower bound of t_f indeed holds. The optimal trajectories for reaching points $p^4, p^5, ..., p^9$ from p^1 are also plotted in Fig. 2. ■

IV. Minimum Energy Control with State Constraints

An optimal trajectory obtained using the method in Section III could leaves the region D (one example is given in Example 2). In this section, we study the minimum energy control problem with the state constraints $-\frac{L}{2} \leq y(t) \leq \frac{L}{2}$.

We first calculate the unconstrained optimal trajectory. If it does leave the region D, we employ the multiplier method in [5] to obtain the optimal control. We assume that $y_{p^1} > 0$ and $y_{p^2} > 0$ so that we only need to consider the constraint $y(t) \leq \frac{L}{2}$. To handle the constraint $S(x, y, t) = y(t) - \frac{L}{2} \leq 0$, we first take its derivative with respect to t to obtain $\frac{dS}{dt} = \frac{dy}{dt} = U_y$, and then we add an additional multiplier $\mu(t)$ to adjoin $\frac{dS}{dt}$ in the Hamiltonian H, i.e., $H = U^T U + P^T(U + N) + \mu U_y$, where $P = (P_1\ P_2)^T$ and $N = (B + Ay^2\ 0)^T$. Since H does not depend on x, we have

$$\frac{dP_1}{dt} = 0, \quad \frac{dP_2}{dt} = -2AP_1 y \ .$$

Since U is chosen to minimize H, we get $U_x = -\frac{P_1}{2}$ and $U_y = -\frac{P_2}{2} - \frac{\mu}{2}$. Note that if $S(x, y, t) = 0$, then $U_y = 0$ and $\mu(t) = -P_2(t) \geq 0$; if $S(x, y, t) < 0$, then $\mu(t) = 0$ and $U_y = -\frac{P_2(t)}{2}$.

In general, the trajectory consists of three pieces: i) from 0 to t_1, the trajectory is an unconstrained arc; ii) from t_1 to t_2, the trajectory is an arc that satisfies $S(x, y, t) = 0$ and $\frac{dS}{dt} = 0$; iii) from t_2 to t_f, the trajectory again is an unconstrained arc. Here t_1 is the time when the trajectory enters the constrained boundary, t_2 is the time when the trajectory leaves the constrained boundary, and $t_1 \leq t_2$.

Based on Theorem 1.2), C_1 must be nonzero; otherwise, the trajectory does not touch $|y| = \frac{L}{2}$. If $t \in [0, t_1)$, $P_1(t) = C_1$, $P_2(t)$, $y(t)$, and $x(t)$ are given in Eqs. (13), (12), and (14) respectively. Since $y(t_1) = \frac{L}{2}$, we get the equation

$$\cosh(t\sqrt{AC_1}) y_{p^1} - \frac{\sinh(t\sqrt{AC_1}) P_2(0)}{2\sqrt{AC_1}} = \frac{L}{2} \ . \tag{25}$$

As $S(x, y, t_1) = 0$ acts as an interior point constraint [5], we have the following equations for P and H

$$P_1(t_1^-) = P_1(t_1^+),\ P_2(t_1^-) = P_2(t_1^+) + \eta,\ H(t_1^-) = H(t_1^+) \ ,$$

where t_1^- (or t_1^+) denotes the time just before (or after) t_1, and η is an unknown constant.

If $t \in (t_1, t_2]$, we have $U_y = 0$. Therefore, $y(t) = \frac{L}{2}$, $P_1(t) = C_1$, $P_2(t) = P_2(t_1^+) - LAC_1(t - t_1^+)$, $\mu(t) = -P_2(t)$, and $x(t) = x(t_1) + (B + A\frac{L^2}{4} - \frac{C_1}{2})(t - t_1)$. Note that $P_2(t_2) = P_2(t_1^+) - LAC_1(t_2 - t_1^+)$.

At t_2, both H and P are continuous. Therefore, if $t \in [t_2, t_f]$, we have $P_1(t) = C_1$, $y(t) = \cosh((t - t_2)\sqrt{AC_1}) y(t_2) - \frac{\sinh((t-t_2)\sqrt{AC_1}) P_2(t_2)}{2\sqrt{AC_1}}$, and $P_2(t) = \cosh((t - t_2)\sqrt{AC_1}) P_2(t_2) - 2\sqrt{AC_1} \sinh((t - t_2)\sqrt{AC_1}) y(t_2)$. Since $y(t_f)$ is given, we obtain the following equation

$$y_{p^2} = \cosh((t_f - t_2)\sqrt{AC_1}) y(t_2) - \frac{\sinh((t_f - t_2)\sqrt{AC_1}) P_2(t_2)}{2\sqrt{AC_1}} \ . \tag{26}$$

Plugging the expression of y into Eq. (10), we can solve for x, and obtain the following equation (since $x(t_f)$ is given)

$$\begin{aligned} x_{p^2} = \ & x(t_2) + \frac{P_2(t_2) y(t_2)}{4C_1} - \\ & \frac{(4C_1^2 - 4AC_1 y(t_2)^2 - 8BC_1 + P_2(t_2)^2)(t_f - t_2)}{8C_1} + \\ & \frac{(P_2(t_2)^2 + 4AC_1 y(t_2)^2) \sinh(2(t_f - t_2)\sqrt{AC_1})}{16C_1\sqrt{AC_1}} - \\ & \frac{P_2(t_2) y(t_2) \cosh(2(t_f - t_2)\sqrt{AC_1})}{4C_1} \ . \end{aligned} \tag{27}$$

Since t_f is free and there is no cost imposed on the final state in the optimization problem, $H|_{t_f} = 0$. Since H does not explicitly depend on t, H is constant and equal to 0 for all t. Therefore, we have $H|_{t_1^-} = H|_{t_2} = H|_{t_f} = 0$, i.e.,

$$0 = -\frac{C_1^2 + P_2(t_1^-)^2}{4} + C_1(B + Ay(t_1^-)^2) \ , \tag{28}$$

$$0 = -\frac{C_1^2}{4} + C_1(B + A(\frac{L}{2})^2) \ , \tag{29}$$

$$0 = -\frac{C_1^2 + P_2(t_f)^2}{4} + C_1(B + Ay_{p^2}^2) \ . \tag{30}$$

By solving Eqs. (25), (26), (27), (28), (29), and (30), we can get C_1, $P_2(0)$, η, t_1, t_2 and t_f, and then the optimal

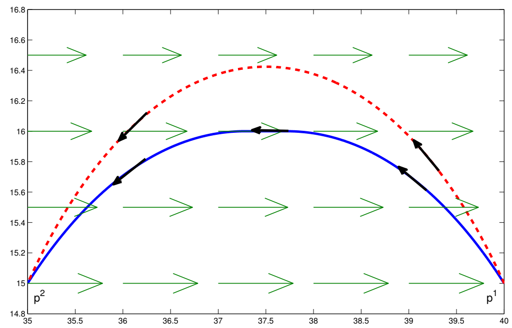

Fig. 3. Constrained (the blue solid curve) and unconstrained (the red dashed curve) optimal trajectories from $p^1 = (40\ 15)^T$ to $p^2 = (35\ 15)^T$.

control is $U = -\frac{1}{2}\begin{bmatrix} C_1 \\ P_2(t) + \mu(t) \end{bmatrix}$ for $t \in [0, t_f]$. Similarly, we can obtain the optimal control for cases in which $y_{p^1} \le 0$ and/or $y_{p^2} \le 0$.

Example 2 We still use the flow parameters in Example 1, but p^1 is at $(40\ 15)^T$ and p^2 is at $(35\ 15)^T$. The unconstrained trajectory is shown as the red dashed curve in Fig. 3, in which the green thin arrows represent flow velocity. The state constraint is given as $y(t) \le \frac{L}{2} = 16$. By solving Eqs. (25), (26), (27), (28), (29), and (30) using the nonlinear equation solver *fsolve* in Matlab, we obtain $C_1 = 1.6780$, $P_2(0) = -0.6869$, $\eta = 0.1278$, $t_1 = 5.7596$, $t_2 = 6.8110$ and $t_f = 12.5736$. The constrained optimal trajectory is shown as the blue solid curve in Fig. 3. ■

V. Minimum Time Control

If we restrict the dynamics of drifters to be of Dubin's type, i.e., $U = (\|U\|\cos\theta \quad \|U\|\sin\theta)^T$, then the minimum energy control problem reduces to a minimum time control problem since the objective function can be rewritten as $\min\ \|U\|^2 \int_0^{t_f} 1dt$. As shown in [5], the heading angle θ satisfies the following differential equation:

$$\frac{d\theta}{dt} = \sin^2\theta\frac{\partial V_y}{\partial x} + \sin\theta\cos\theta(\frac{\partial V_x}{\partial x} - \frac{\partial V_y}{\partial y}) - \cos^2\theta\frac{\partial V_x}{\partial y}$$
$$= -\cos^2\theta\frac{\partial v(y)}{\partial y} = -2Ay\cos^2\theta\ ,$$

since $V_x(x,y) = v(y) = Ay^2 + B$ and $V_y(x,y) = 0$. Then we have

$$\frac{dy}{d\theta} = \frac{dy}{dt}\frac{dt}{d\theta} = -\frac{\|U\|\sin\theta}{2Ay\cos^2\theta}\ . \tag{31}$$

We can rewrite Eq. (31) as $2ydy = -\frac{\|U\|\sin\theta}{A\cos^2\theta}d\theta$, integrate on both sides and we obtain $y^2|_{y_{p^1}}^{y} = -\frac{\|U\|}{A}\sec\theta|_{\theta(0)}^{\theta}$. Now we can get

$$\sec\theta = \sec\theta(0) - \frac{A}{\|U\|}(y^2 - y_{p^1}^2)\ . \tag{32}$$

Essentially, θ is a function of $\theta(0)$, y_{p^1} and y.

Since $\frac{dx}{dy} = \frac{\|U\|\cos\theta + v(y)}{\|U\|\sin\theta}$, we can plug in the expression for θ and the right hand side of $\frac{dx}{dy}$ is a function of y. Now we can integrate both sides and solve for $\theta(0)$ based on $x_{p^1}, y_{p^1}, x_{p^2}, y_{p^2}$. Details are omitted due to lack of space. Once we get $\theta(0)$, we can design a feedback control policy using Eq. (32) since θ only depends on the current y coordinate. If the state constraints $|y| \le \frac{L}{2}$ are considered, we can also solve the minimum time control problem using the additional multiplier approach in [5].

VI. Conclusion

In this paper, we studied the minimum energy control problem in which the river flow is approximated as a quadratic function and the objective is to move a drifter from one point to another with the minimum energy consumption. We derived the optimal control strategy using the minimum principle, and took into account state constraints so that drifters will always stay inside the river region.

In the future, we would like to find approximation algorithms to solve nonlinear Eqs. (3), (4), (5) quickly so that the optimal control can be calculated by less powerful computational devices in real time, and find an upper bound on the final time t_f.

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