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Stability and Convergence Analysis for a Class of Nonlinear Passive Systems

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***Abstract*—A systematic and general method that proves state boundedness and convergence to nonzero equilibrium for a class of nonlinear passive systems with constant external inputs is developed. First, making use of the method of linear-time-varying approximations, the boundedness of the nonlinear system states is proven. Next, taking advantage of the passivity property, it is proven that a suitable switching storage function can be always obtained to show convergence to the nonzero equilibrium by using LaSalle's Invariance Principle. Numerical and simulation results illustrate the proposed theoretical analysis.**

I. Introduction

THE majority of physical systems found in nature can be modeled as inherently nonlinear systems. In general, nonlinear systems are difficult to be solved while their dynamic performance may sometimes be unexpected. Therefore, stability analysis of nonlinear dynamic systems is of great importance [1].

Stability analysis based on Lyapunov methods is widely used as a standard technique. Unfortunately, this analysis is mainly applied on nonlinear systems without external inputs and it proves stability at the origin [2]. In the cases wherein a nonzero equilibrium exists or an external input appears, Lyapunov techniques need the error dynamics model of the system. However, for nonlinear systems there does not exist a systematic, general method to obtain the error dynamics model since every system has its own nonlinearities.

On the other hand, the most useful tool for the analysis of nonlinear systems with external inputs is the passivity theory [3]. This is directly related to Lyapunov and L_2 stability [4] and can provide a general frame for the analysis, but it fails when the input does not vanish as time passes.

Many Euler-Lagrange [5] or Hamiltonian [3] systems such as electrical, mechanical and electromechanical systems are described by nonlinear dynamic equations which often include a constant input, usually a constant voltage, external force or torque. Even after applying a control law, a constant external input remains on the closed-loop dynamic model [5] and the desired equilibrium is not the origin. Since damping is a common property of all these systems, the passivity can be easily proven with respect to the system output and the constant external input. However, state boundedness is difficult to be proven, while convergence to a specific equilibrium cannot be directly concluded from the general passivity analysis.

In this paper, we connect the passivity analysis with stability and convergence to equilibrium for a class of nonlinear dynamic systems with constant inputs and distinct nonzero equilibriums. In order to prove that the system states remain bounded, the approach that is based on a sequence of linear-time-varying approximations to nonlinear systems is used, as proposed by Tomas-Rodriguez and Banks in [6,7]. This approach avoids using local linearizations in phase space or Lie algebra and replaces the bounded-input-bounded-output (BIBO) and input-to-state (ISS) stability theorems [8,9].

Furthermore, we prove that under some assumptions, often met in real nonlinear systems, passivity is adequate to provide convergence of the states to nonzero equilibriums. To this end, we prove that for this class of nonlinear, passive systems, one can always determine a general, bounded, differentiable, nonincreasing storage function. Particularly, this storage function is constructed as a switching storage function by using as generic function the original storage function easily obtained from the system passivity analysis. The approach of switching storage functions is based on a new efficient idea that has been developed to construct Lyapunov functions for difficult nonlinear systems [10, 11] or hybrid systems [12]. Indeed, this approach effectively overcomes a fundamental problem in nonlinear stability analysis that is the construction of suitable storage functions. Thus, exploiting this possibility, we continue by applying LaSalle's Invariance Principle [1] to prove convergence to the equilibrium.

In Section II, the basic preliminaries used for passivity, stability and convergence analysis are underlined and the basic assumptions needed for our development are provided. In Section III, the state boundedness of the system is proven. In Section IV, an appropriate switching storage function is developed and the convergence analysis is addressed. In Section V, a numerical example is analyzed and simulated to confirm the proposed approach while in Section VI a final conclusion is given.

II. Preliminaries, Definitions And Assumptions

A. *Passivity of nonlinear system*

Let the nonlinear system

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$$\dot{x} = f(x) + B(x)u, \quad x(0) = x_0$$
$$y = h(x) \tag{1}$$

$x \in R^n$, u, $y \in R^m$, f, h, B are smooth and $f(0) = h(0) = 0$.

Theorem 1. [1] Assume that there is a continuous function $V(t) \geq 0$ such that

$$V(t) - V(0) \leq \int_0^t y^T(\tau)u(\tau)d\tau$$

for all functions $u(t)$, for all $t \geq 0$ and all $V(0)$. Then the system with input $u(t)$ and output $y(t)$ is passive.

The storage function $V(x(t))$ at a future time t is equal or less than the sum of the available storage function $V(x(0))$ at an initial time $t_0 = 0$ plus the total energy supplied to the system from the external sources in the interval $[0, t]$, i.e. there exists a nonnegative definite function, $D(x(t)) \geq 0$, such that

$$V(x(t)) - V(x(0)) = -\int_0^t D(x(\tau))d\tau + \int_0^t y^T(\tau)u(\tau)d\tau$$

which can be equivalently written in the derivative form

$$\dot{V}(x(t)) = -D(x(t)) + y^T(t)u(t) . \tag{2}$$

B. Linear approximations

In a variety of nonlinear systems, (1) may put in the following form

$$\dot{x} = A(x)x + B(x)u, \quad x(0) = x_0$$
$$y = C(x)x \tag{3}$$

Theorem 2. [6,7] Consider the nonlinear control system (3) with $x(0) = x_0 \in R^n$. We introduce the sequence of linear, time-varying approximations

$$\dot{x}^{[0]}(t) = A(x_0)x^{[0]}(t) + B(x_0)u^{[0]}(t), \quad x^{[0]}(0) = x_0 \tag{4a}$$

and for $i \geq 1$,

$$\dot{x}^{[i]}(t) = A(x^{[i-1]}(t))x^{[i]}(t) + B(x^{[i-1]}(t))u^{[i]}(t), \quad x^{[i]}(0) = x_0 \tag{4b}$$

If A and B are locally Lipschitz, then system (4) converges to the nonlinear system (3) as $i \to \infty$ and $\lim_{i \to \infty} x^{[i]}(t) \to x(t)$.

C. Stability of linear time-varying systems

Consider the $n-$dimensional linear time-varying dynamic equation

$$\dot{x} = A(t)x + B(t)u, \quad x(0) = x_0 \tag{5}$$

where x is the $n \times 1$ state vector, u is the $m \times 1$ input vector and A, B are $n \times n$ and $n \times m$ matrices respectively. Let $\Phi(t, \tau)$ be the transition matrix of system (5).

Definition 1. [13] A dynamic system of the form of (5) is said to be *totally stable* with respect to the state variable x, or *T-stable* for short, if and only if for any initial state and for any bounded input, the state variables are bounded.

Theorem 3. [13] A system that is described by the linear dynamical equation (5) is totally stable if and only if $\Phi(t, t_0)$ is bounded and

$$\int_{t_0}^{t} \|\Phi(t, \tau)B(\tau)\| d\tau \leq k < \infty$$

for any t_0 and for all $t \geq t_0$.

Consider now a linear time-varying system without external input:

$$\dot{x} = A(t)x, \quad x(t_0) = x_0 \tag{6}$$

and assume that $A(t)$ is continuous and $A(\infty) = \lim_{t \to \infty} A(t)$ exists. Then the following Lemma holds true.

Lemma 1. [6] System (6) is asymptotically stable if the eigenvalues of $A(\infty)$ have negative real parts.

D. Convergence

Definition 2. [1] We say that $x(t)$ approaches a set Q as t approached to infinity, if for each $\varepsilon > 0$ there is a $T > 0$ such that

$$dist(x(t), Q) < \varepsilon, \quad \forall t > T$$

where $dist(p, Q)$ denotes the distance from a point p to the set Q, that is, the smallest distance from p to any point in Q. More precisely,

$$dist(p, Q) = \inf_{x \in Q} \|p - x\| .$$

E. Assumptions

In this paper, we will discuss the stability and convergence of a certain class of nonlinear systems given in the form of (3).

The main assumptions needed for the analysis are the following:

Assumption 1. For nonlinear system (3) it holds:

- For any trajectory $x(t) \in \Omega \subset R^n$, for all $t \geq 0$, matrix $A(x)$ is locally Lipchitz and has eigenvalues with negative real parts.
- Matrix $B(x)$ is constant, i.e. $B(x) = B$.
- Input u is assumed to be constant, i.e. $u(t) = u_c$, $t \geq 0$.

Assumption 2. It holds true that:

- System (3) is passive with respect to the input u and output y, for some storage function $V(x(t)) \geq 0$, in accordance to Theorem 1.
- There exist nonzero equilibrium points for (3): $x_e \in M \subset \Omega$ that are distinct, each satisfying the equation $\dot{V}(x_e) = 0$, with $\dot{V}(x(t))$ given by (2), for

some $u(t) = u_c \neq 0$

- No limit cycles exist in Ω

It is noted that the above assumptions are common for a wide class of dynamical systems while model description (3) represents a majority of electrical, mechanical and electromechanical systems.

Since Assumption 2 holds true, input u of system (1) (or (3) equivalently) is constant and the system actually is a nonlinear autonomous system of the form:

$$\dot{x} = \tilde{f}(x), \quad x(t_0) = x_0 \tag{7}$$

with nonzero equilibrium x_e. Therefore, several stability theorems such as the Invariance Principle (LaSalle's Theorem) or Local Invariant Set Theorem [1] can be applied.

III. Stability Analysis

Since the nonlinear system (3) includes an input u which is constant, i.e. bounded, then the first step is to prove boundedness of the solution $x(t)$ of system (3).

According to the Assumption 1, $A(x)$ and B are locally Lipchitz functions and therefore Theorem 2 can be applied. As a result, one can consider the sequence of linear time-varying approximations given by (4a) and (4b).

Consider first the linear time invariant system (4a):

$$\dot{x}^{[0]}(t) = A(x_0)x^{[0]}(t) + Bu_c, \quad x^{[0]}(0) = x_0 \tag{8}$$

Since $A(x)$ has eigenvalues with negative real parts for any $x \in \Omega$ then the eigenvalues of matrix $A(x_0)$ have negative real parts. As a result it is obvious that state $x^{[0]}(t)$ of system (8) is bounded.

Consider, now, the first system ($i = 1$) of the sequence given by (4b):

$$\dot{x}^{[1]}(t) = A(x^{[0]}(t))x^{[1]}(t) + Bu_c, \quad x^{[1]}(0) = x_0 \tag{9}$$

First, we will examine system (9) without the external input, i.e.

$$\dot{x}^{[1]}(t) = A(x^{[0]}(t))x^{[1]}(t), \quad x^{[1]}(0) = x_0 \tag{10}$$

Since $x^{[0]}(t)$ is bounded, then $A(\infty) = \lim_{t\to\infty} A\left(x^{[0]}(t)\right)$ exists and according to the conditions mentioned, the eigenvalues of this $A(\infty)$ have negative real parts. Then, according to Lemma 1, system (10) is asymptotically stable. One can see, that any initial condition x_0 will result to an asymptotically stable system at the origin independently of t_0. In other words, there is a positive constant c, independent of t_0 and for each $\eta > 0$, there is $T(\eta) > 0$ such that

$$\left\|x^{[1]}(t)\right\| < \eta, \ \forall t \geq t_0 + T(\eta), \ \forall \left\|x^{[1]}(t_0)\right\| < c.$$

As a result, the origin is uniformly asymptotically stable and therefore for the transition matrix $\Phi^{[1]}(t,\tau)$ of (9) or (10), it holds true that

$$\begin{aligned} &\exists\, k_1^{[1]}, k_2^{[1]}, m^{[1]} \geq 0: \ \left\|\Phi^{[1]}(t,t_0)\right\| \leq k_1^{[1]} e^{-k_2^{[1]}(t-t_0)} \\ &\int_{t_0}^{t} \left\|\Phi^{[1]}(t,\tau)\right\| d\tau \leq m^{[1]}, \ \forall t \geq t_0 \geq 0 \end{aligned} \tag{11}$$

Since B is constant, then Theorem 3 can be applied to prove that system (9) is totally stable. Therefore, the solution $x^{[1]}(t)$ is bounded.

Now, proceeding with any $i > 1$, a similar analysis can be sequentially applied on (4b) which results in the boundedness of $\Phi^{[i]}(t,t_0)$ and into the condition

$$\int_{t_0}^{t} \left\|\Phi^{[i]}(t,\tau)\right\| d\tau \leq m^{[i]}, \ \forall t \geq t_0 \geq 0 \ \text{ for some } m^{[i]} \geq 0.$$

Thus, applying Theorem 3, it is proven that any system in the sequence is totally stable producing a bounded state $x^{[i]}(t)$.

Eventually, as $i \to \infty$, then $x(t)$ is bounded since $x(t) = \lim_{i\to\infty} x^{[i]}(t)$.

Thus we have established the following Lemma.

Lemma 2. Under Assumption 1 and for any x_0, the trajectories $x(t)$ of the original nonlinear system (3) are bounded in Ω for all $t \geq 0$.

IV. Convergence To Equilibrium

State boundedness proven in the previous Section is a fundamental property for a nonlinear system with external input. In the analysis that follows, we prove that under the above assumptions, the bounded solution of system (3) converges to any equilibrium $x_e \in M$. Thus, we proceed with the following Lemma.

Lemma 3. Consider nonlinear system (3) satisfying Assumptions 1 and 2. Let $R_V = \{V_1^*, V_2^*, \ldots\}$ be the set of all points of $V\left(x(t)\right)$ on the trajectory $x(t)$ satisfying $\dot{V}\left(x(t)\right) = 0$. Then, there always exists a continuous, differentiable bounded switching storage function $W\left(x(t)\right)$:

$$W\left(x(t)\right) = \begin{cases} W_0\left(x(t)\right) \ \text{when } t \in \left[t_0, t_1\right] \\ W_1\left(x(t)\right) \ \text{when } t \in \left[t_1, t_2\right] \\ \vdots \\ W_{K-1}\left(x(t)\right) \ \text{when } t \in \left[t_{K-1}, t_K\right] \\ W_K\left(x(t)\right) \ \text{when } t \in \left[t_K, t_{K+1}\right] \\ \vdots \end{cases} \tag{12}$$

with

$$W_0\left(x(t)\right) = \frac{1}{2}\left(V\left(x(t)\right) - sV_M - (1-s)V_m\right)^2,$$

$$W_1(x(t))=\frac{1}{2}\left(V(x(t))-sV_m-(1-s)V_M\right)^2\left(\frac{V_1^*-sV_M-(1-s)V_m}{V_1^*-sV_m-(1-s)V_M}\right)^2,$$

$$W_{K-1}(x(t))=\frac{1}{2}\left(V(x(t))-sV_M-(1-s)V_m\right)^2\times$$
$$\prod_{i=1,3,5,...}^{K-2}\left(\frac{\left(V_{i+1}^*-sV_m-(1-s)V_M\right)\left(V_i^*-sV_M-(1-s)V_m\right)}{\left(V_i^*-sV_m-(1-s)V_M\right)\left(V_{i+1}^*-sV_M-(1-s)V_m\right)}\right)^2,$$

$$W_K(x(t))=\frac{1}{2}\left(V(x(t))-sV_m-(1-s)V_M\right)^2\left(\frac{V_K^*-sV_M-(1-s)V_m}{V_K^*-sV_m-(1-s)V_M}\right)^2\times$$
$$\prod_{i=1,3,5,...}^{K-2}\left(\frac{\left(V_{i+1}^*-sV_m-(1-s)V_M\right)\left(V_i^*-sV_M-(1-s)V_m\right)}{\left(V_i^*-sV_m-(1-s)V_M\right)\left(V_{i+1}^*-sV_M-(1-s)V_m\right)}\right)^2,$$

where switching parameter s is used for initialization and is calculated once at $t=t_0$:

$$s=\begin{cases}1, \text{ if } \dot{V}(x(t_0))=-D(x(t_0))+y^T(t_0)u_c\geq 0\\ 0, \text{ if } \dot{V}(x(t_0))=-D(x(t_0))+y^T(t_0)u_c<0\end{cases}$$

and V_M, V_m are constant values such that $V_M>V_{\max}$, $V_m<V_{\min}$ with $V_{\max}$ and $V_{\min}$ be the maximum and minimum values of $V(x(t))$ and t_i represents the time instant on which $V(x(t_i))=V_i^*\in R_V$ wherein $\dot{V}(t_i)$ changes its sign.

Then, the derivative of the storage function $\dot{W}(x(t))$ is nonpositive for any $x(t)$ in Ω.

Proof. System (3) is passive and let $V(x(t))$ be the storage function which proves passivity. Since Assumption 1 holds true, then Lemma 2 implies that the solution $x(t)$ remains bounded in Ω from which it follows that $V(x(t))$ is also bounded, i.e. $V_{\min}\leq V(x(t))\leq V_{\max}$. Without loss of generality assume that initially $\dot{V}(x(t_0))\geq 0$, i.e. $s=1$. Then the following storage function can be used (12):

$$W(x(t))=\begin{cases}\frac{1}{2}\left(V(x(t))-V_M\right)^2, t\in[t_0,t_1]\\ \frac{1}{2}\left(V(x(t))-V_m\right)^2\left(\frac{V_1^*-V_M}{V_1^*-V_m}\right)^2, t\in[t_1,t_2]\\ \vdots\\ \frac{1}{2}\left(V(x(t))-V_M\right)^2\prod_{i=1,3,5,...}^{K-2}\left(\frac{\left(V_{i+1}^*-V_m\right)\left(V_i^*-V_M\right)}{\left(V_i^*-V_m\right)\left(V_{i+1}^*-V_M\right)}\right)^2, t\in[t_{K-1},t_K]\\ \frac{1}{2}\left(V(x(t))-V_m\right)^2\left(\frac{V_K^*-V_M}{V_K^*-V_m}\right)^2\prod_{i=1,3,5,...}^{K-2}\left(\frac{\left(V_{i+1}^*-V_m\right)\left(V_i^*-V_M\right)}{\left(V_i^*-V_m\right)\left(V_{i+1}^*-V_M\right)}\right)^2, t\in[t_K,t_{K+1}]\\ \vdots\end{cases}$$

which obviously is continuous and bounded. The time derivative of $W(x(t))$ in every interval $[t_{K-1},t_K]$ is:

$$\dot{W}(x(t))=\dot{V}(x(t))\left(V(x(t))-V_M\right)\prod_{i=1,3,5,...}^{K-2}\left(\frac{\left(V_{i+1}^*-V_m\right)\left(V_i^*-V_M\right)}{\left(V_i^*-V_m\right)\left(V_{i+1}^*-V_M\right)}\right)^2\leq 0 \quad (13)$$

while in every interval $[t_K,t_{K+1}]$ it is:

$$\dot{W}(x(t))=\dot{V}(x(t))\left(V(x(t))-V_m\right)\left(\frac{V_K^*-V_M}{V_K^*-V_m}\right)^2\prod_{i=1,3,5,...}^{K-2}\left(\frac{\left(V_{i+1}^*-V_m\right)\left(V_i^*-V_M\right)}{\left(V_i^*-V_m\right)\left(V_{i+1}^*-V_M\right)}\right)^2\leq 0 \quad (14)$$

Inequalities (13) and (14) hold true since $V(x(t))$ is monotonic in every interval. Furthermore, one can see that $\dot{W}(x(t_i))=0$ at every switching time instant since $\dot{V}(x(t_i))=\dot{V}(x_i^*)=0$. As a result $\dot{W}(x(t))$ is uniformly continuous and obviously $W(x(t))$ is differentiable in Ω. Therefore the solution $x(t)$ is bounded in a region Ω in which $\dot{W}(x(t))\leq 0$, as shown by (13) and (14).

Note that if initially $\dot{V}(x(t_0))<0$, i.e. $s=0$, a similar analysis shows that the same properties hold true for $W(x(t))$ and it is again true that $\dot{W}(x(t))\leq 0$ in Ω.

Therefore, in any case (for $s=1$ or $s=0$) one can determine a continuous, differentiable, bounded storage function $W(x(t))$ with derivative $\dot{W}(x(t))\leq 0$ in Ω. □

Now, we are ready to apply Lemma 3 to prove convergence to equilibrium.

Theorem 4. The state trajectories $x(t)\in\Omega$, of the passive system (3), satisfying Assumptions 1 and 2, converge to an equilibrium $x_e\in M$.

Proof. Since Assumptions 1 and 2 hold true, Lemma 3 can be applied to provide a $W(x(t))$ which satisfies all the demands required by LaSalle's Invariance Principle [1]. Let $R=\{x_1^*,x_2^*,...\}$ be the set of all points of $x(t)$ within Ω such that $V_i^*,(i=1,2,...):R\to R_V$ and let $\bar{R}=R\cup M$ constitutes the set wherein $\dot{W}(x(t))=0$. Since M is the largest invariant set in $\bar{R}$, according to LaSalle's Invariance Principle, every solution $x(t)$ originating in Ω tends to M as $t\to\infty$. As a result the solution converges to some equilibrium point x_e, i.e. $x(t)\to x_e$ as $t\to\infty$. □

From the analysis of Theorem 4, the following remark can be derived.

Remark 1. Convergence of $x(t)$ to the equilibrium implies that for each $\varepsilon > 0$ there is $T > 0$ such that (Definition 2):

$$dist\left(x(t), M\right) < \varepsilon, \quad \forall t > T. \tag{15}$$

Therefore, one can easily prove that there exists a $\delta(\varepsilon)$ for the storage function $V\left(x(t)\right)$ such that:

$$dist\left(V\left(x(t)\right), V\left(x_e\right)\right) < \delta(\varepsilon), \quad \forall t > T. \tag{16}$$

Taking into account the structure of the switching storage function $W\left(x(t)\right)$, this means that for each $\varepsilon > 0$ there always exists a last time-instant $t_N \leq T$ after which $W\left(x(t)\right)$ remains at $W_N\left(x(t)\right)$, for all $t \geq t_N$, i.e. $W\left(x(t)\right)$ is determined by a finite sequence of switching functions $W_i\left(x(t)\right)$, ($i = 0,1,2,...,N$).

V. Numerical Example

A. Model description and analysis

In order to investigate the theoretical analysis mentioned in the previous Section, we illustrate a numerical example of a nonlinear system given by the following dynamic model:

$$\begin{aligned} \dot{x}_1 &= -3x_1 - 10x_2^2 + u_1 \\ \dot{x}_2 &= 10x_1x_2 - 2x_2 - 5x_2x_3 + u_2 \\ \dot{x}_3 &= 10x_2^2 - 2x_3 \end{aligned} \tag{17}$$

where $x = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^T$ and $u = \begin{bmatrix} u_1 & u_2 \end{bmatrix}^T$ are the state vector and input vector respectively.

Assume that the input vector is constant, i.e.

$$u_1 = 5, \quad u_2 = 3, \quad \forall t \geq 0 \tag{18}$$

Then, one can derive the unique equilibrium point of system (17) given as:

$$x_e = \begin{bmatrix} 0.5345 \\ 0.5828 \\ 1.6983 \end{bmatrix} \tag{19}$$

First, we prove that system (17) is passive. Let the continuous differentiable storage function V as:

$$V\left(x(t)\right) = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 + \frac{1}{4}x_3^2 \tag{20}$$

Taking the time derivative of V it yields:

$$\dot{V} = x_1\dot{x}_1 + x_2\dot{x}_2 + \frac{1}{2}x_3\dot{x}_3 \tag{21}$$

and substituting $\dot{x}$ from the dynamic model (17), equation (21) becomes:

$$\dot{V} = -3x_1^2 - 2x_2^2 - x_3^2 + x_1u_1 + x_2u_2 = -D\left(x\right) + y^T u \tag{22}$$

where we consider as output $y = \begin{bmatrix} x_1 & x_2 \end{bmatrix}^T$, as input $u = \begin{bmatrix} u_1 & u_2 \end{bmatrix}^T$ and function $D(x) = 3x_1^2 + 2x_2^2 + x_3^2$. Then (22) implies that $\dot{V} \leq y^T u$.

Integrating the last expression from zero to t, it yields:

$$V(t) - V(0) \leq \int_0^t y^T(\tau)u(\tau)d\tau \tag{23}$$

According to Theorem 1, inequality (23) proves that system (17) is passive.

Continuing, system (17) can be written in the form (3) as follows:

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{x}_3 \end{bmatrix} = \begin{bmatrix} -3 & -10x_2 & 0 \\ 10x_2 & -2 & -5x_2 \\ 0 & 10x_2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \tag{24}$$

with $A(x)$, $B(x) = B$ being locally Lipschitz functions. Furthermore, it should be noted that matrix $A(x)$ depends only on the state variable x_2 and one can easily prove that the eigenvalues of $A(x)$ have negative real parts for any $x_2 \in \Omega$. Assuming that in Ω no limit cycle exists, then both Assumptions mentioned in Section II.E hold true and in accordance to Theorem 4, system states converge to x_e, given by (19), as $t \to \infty$.

B. Simulation results

System (17) with constant inputs u_1, u_2 given by (18) is simulated by assuming zero initial condition for x_0.

Since the theoretical analysis proves that trajectory $x(t)$ is bounded, then for the storage function $V\left(x(t)\right)$ given by (20), it holds true that $0 \leq V\left(x(t)\right) \leq V_{\max}$. Choosing $V_M = 50 > V_{\max}$ and $V_m = -0.001 < V_{\min}$, an appropriate switching storage function $W\left(x(t)\right)$ in accordance with Lemma 2 can be determined as follows: $W\left(x(t)\right) =$

$$\begin{cases} \frac{1}{2}\left(V\left(x(t)\right) - 50\right)^2, \quad t \in \left[0, 0.353\right] \\ \frac{1}{2}\left(V\left(x(t)\right) + 0.001\right)^2 \left(\frac{0.6251 - 50}{0.6251 + 0.001}\right)^2, \; t \in \left[0.353, 0.4931\right] \\ \frac{1}{2}\left(V\left(x(t)\right) - 50\right)^2 \left(\frac{0.5939 + 0.001}{0.6251 + 0.001}\right)^2 \left(\frac{0.6251 - 50}{0.5939 - 50}\right)^2, \\ \qquad t \in \left[0.4931, \infty\right) \end{cases} \tag{25}$$

Figure 1 shows the time response of the storage function $V\left(x(t)\right)$ which is bounded. Figure 2 illustrates the time response of the storage function $W\left(x(t)\right)$ given by (25) which is continuous, decreasing and bounded. We note that that three time intervals are adequate to prove convergence of the solution to the unique equilibrium.

Figures 3, 4 and 5 illustrate the time responses of the three state variables. It can be observed that all states converge to the unique equilibrium point x_e given by (19) verifying the stability analysis and convergence described in Sections III and IV.

VI. CONCLUSION

In this paper, we have proven that under some mild assumptions, the states of every passive system having distinct equilibrium points converge to one of the equilibriums.

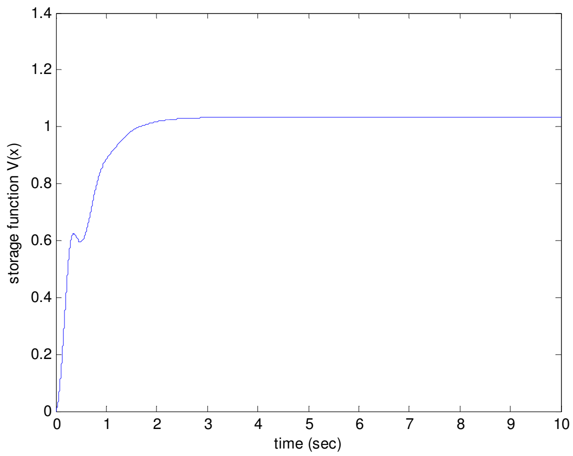

Fig. 1. Time response of the storage function $V\left(x(t)\right)$

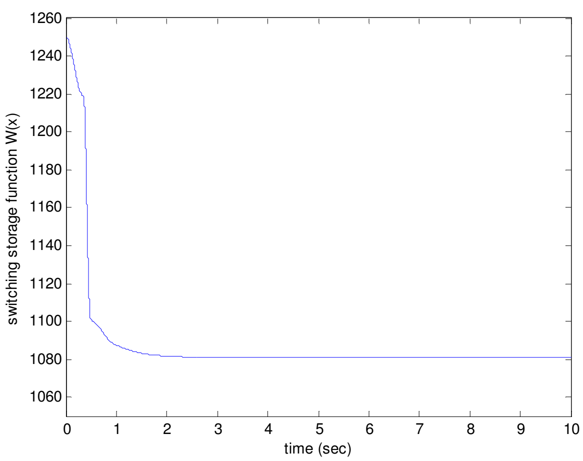

Fig. 2. Time response of the switching storage function $W\left(x(t)\right)$

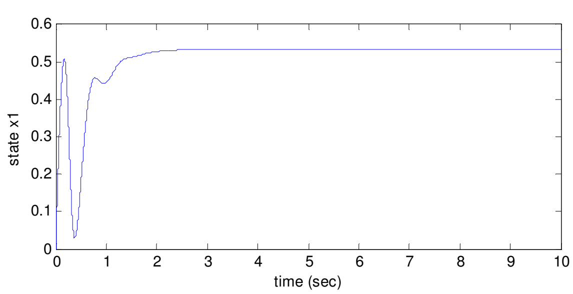

Fig. 3. Time response of state variable x_1

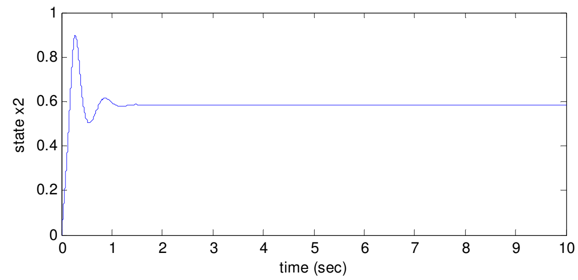

Fig. 4. Time response of state variable x_2

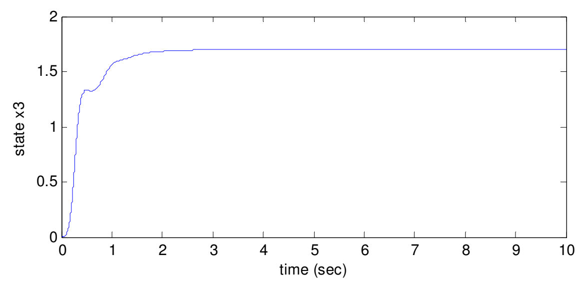

Fig. 5. Time response of state variable x_3

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