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Contraction and Observer Design on Cones

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Abstract— **We consider the problem of positive observer design for positive systems defined on solid cones in Banach spaces. The design is based on the Hilbert metric and convergence properties are analyzed in the light of the Birkhoff theorem. Two main applications are discussed : positive observers for systems defined in the positive orthant, and positive observers on the cone of positive semi-definite matrices with a view on quantum systems.**

I. Introduction

Positive systems arise in several areas, where the state variables represent quantities that do not have a physical meaning when they are negative see, for example, [8], [13], [7]. In the last ten years or so, the design of positive observers for linear positive systems, i.e. observers such that the estimated state respects positivity, has attracted an ever growing attention. Indeed, the requirement that the observer estimates be positive at any time seems desirable since it allows a physical interpretation of the estimation. Positive observers have been studied for classes of linear systems and under specific structural assumptions. In [10] structural properties, including observability, of positive systems have been studied, whereas observers for compartmental systems have been developed [14]. In [1] the positive observer design problem has been dealt with using coordinates transformations and the theory of positive realization [10], [2], thus generalizing the results in [14] and relaxing the conditions under which positive observers exist.

For a class of time-varying non-linear (or linear) systems, we advocate the fact that the use of special "polar" coordinates (a vector is parameterized by its norm, and an element of the unit sphere) as well as the use of a special metric on the unit sphere (more exactly the projective space), the Hilbert projective metric, simplifies the design of observers and the convergence analysis. The key component of our approach is a theorem by Birkhoff, 1957, that characterizes a class of positive mappings that are contractions in the projective space for the Hilbert metric. For those systems, a mere copy of the system provides a simple positive observer which converges exponentially in the projective space (at least). Our approach can thus be related to the one introduced by [9] where the convergence analysis of observers is based on the special choice of a contractive metric. The approach leads to potentially powerful alternatives to the usual methods, which are generally concerned with the conditions under which a Luenberger observer is a positive system itself.

More generally this paper deals with the general question of the design of observers on solid cones in Banach spaces, the positive orthant being a particular case. The estimates are required to remain in the cone at any time. We will prove that the Hilbert projective and the Birkhoff Theorem allow to design meaningful candidate observers with guaranteed convergence properties for a large class of systems. The approach is valid in finite and infinite dimension, and opens the way to the design of simple observers which respect the underlying structure of the problem, on several other cones than the positive orthant. In particular we apply the technique to the cone of positive semi-definite matrices.

The paper is organized as follows: in Section 2, the Hilbert metric and Birkhoff theorem are recalled. In Section 3, a class of observers on solid cones is proposed. In Section 4, the results are applied to the positive orthant. In particular we introduce a system for which it is not possible to build a linear convergent positive observer and for which we present a non-linear convergent observer. In Section 5 we discuss the design of observers on the cone of positive semi-definite matrices.

II. Birkhoff-Bushell Theorem

In this section, we recall basic results presented in [4]. A solid cone K defined on a Banach space $\mathcal{X}$, is a subset of X which is such that 1) $\mathring{K}$, the interior of K, is not empty, 2) $K + K \subset K$, 3) $\lambda K \subset K$ for all $\lambda \geq 0$, and 4) $\{-K\} \cap K = \{0\}$. On such a space, a partial order can be defined by the following relation : $x \leq y$ iff $y - x \in K$. Let us now define two important quantities. Let $x, y \in K \backslash \{0\}$. We let

$$M(x/y) = \inf\{\lambda \in \mathbb{R}_+ | x \leq \lambda y\}$$
$$m(x/y) = \sup\{\mu \in \mathbb{R}_+ | \mu y \leq x\}$$

and $M(x/y) = +\infty$ if the set is empty. The notation is justified by the fact that we always have $m(x/y)\ y \leq x \leq M(x/y)\ y$.

Definition 1: The Hilbert projective metric in $K \backslash \{0\}$ is defined by $d(x, y) = \log(M(x/y)/m(x/y))$.

The metric can be called projective, as we have $d(\lambda x, \mu y) = d(x, y)$ for all $\lambda, \mu > 0$. Let $A : K \to K$ be mapping:

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- A is said positive if $A : \mathring{K} \to \mathring{K}$.
- A is monotone increasing if $x \leq y$ implies $Ax \leq Ay$.
- A is said to be homogeneous of degree p if for all $\lambda > 0$ and $x \in K$ we have $A(\lambda x) = \lambda^p A(x)$.
- The projective diameter of a positive mapping A is defined by $\Delta(A) = \sup \ \{d(Ax, Ay) | x, y \in K\}$.
- The contraction ratio is $k(A) = \inf\{\lambda \in \mathbb{R}_+ | d(Ax, Ay) \leq \lambda d(x, y)\}$.

The Birkhoff theorem allows to characterize a class of systems that are contractions for the Hilbert metric.

Theorem 1: [Birkhoff (1957)] Let A be a monotone increasing mapping which is homogeneous of degree p in $\mathring{K}$. Then for all $x, y \in K$ we have

$$d(Ax, Ay) \leq p \ d(x, y)$$

If A is a positive linear mapping we have for all $x, y \in \mathring{K}$

$$d(Ax, Ay) \leq (\tanh(\frac{\Delta(A)}{4}))d(x, y)$$

III. Projective observer on a cone

Consider the time-varying system on a solid cone K in a banach space $\mathcal{X}$

$$\begin{aligned} x_{k+1} &= A_k(x_k) \\ y_k &= C_k(x_k) \end{aligned} \tag{1}$$

where each A_k is a positive map on K and C is a positive homogenous map of degree q. Then if the initial state x_0 is in the cone, it remains in the cone for all times. We are concerned with the design of observers whose estimated state $\hat{x}$ remains in K for all times.

A. A positive observer

We start from the following decomposition for all $x \in \mathring{K}$:

$$x = rz, \quad (r, z) \in \mathbb{R}_+^* \times (\mathbb{S} \cap \mathring{K})$$

where $\mathbb{S}$ denotes the unit sphere in X and where r is a scaling factor representing the norm of x, i.e. $r_k = \|x_k\|$. We have

$$z_{k+1} = \frac{A_k z_k}{\|A_k z_k\|} \tag{2}$$

which is well defined as A_k is a positive map. As the output map is supposed to be homogeneous of degree q we have

$$\|y_k\| = r_k^q \|C(z_k)\|$$

Thus a simple candidate observer for the complete state is

$$\begin{aligned} \hat{z}_{k+1} &= \frac{A_k \hat{z}_k}{\|A_k \hat{z}_k\|}, \\ \hat{r}_k &= (\frac{\|y_k\|}{\|C(\hat{z}_k)\|})^{1/q} \end{aligned} \tag{3}$$

with $\hat{z}_0 \in \mathring{K}$. The observer is well-defined as $\hat{z}_0 \in \mathring{K}$ and C are positive, and it delivers positive estimates $\hat{x}_k = \hat{r}_k \hat{z}_k$, as $\hat{z}$ remains in $\mathring{K}$.

B. Convergence issues

In this Section we study the asymptotic behavior of the z coordinate. The Birkhoff theorem implies that if A_k is a positive homogeneous map of degree $p \leq 1$, the Hilbert distance between z_k and $\hat{z}_k$ does not increase. As a result the observer does not diverge (at least as long as the estimation of the z coordinate is concerned). Furthermore, when the observer's dynamics is a contraction, exponential convergence can be expected for a large class of systems (see [9] for a general study of observers based on a contraction associated to a specific metric). The Birkhoff theorem allows to highlight some systems that are strict contractions:

Proposition 1: Consider the system (1) and suppose there exists a finite horizon T such that either

1) $\exists p \ (0 < p < 1) \quad \forall k \in \mathbb{N}$ the operator $A_{k+T} \circ \cdots \circ A_{k+1} \circ A_k$ is homogeneous of degree at most p
2) Or $\exists R > 0 \quad \forall k \in \mathbb{N} \ A_{k+T} \circ \cdots \circ A_{k+1} \circ A_k$ is linear with projective diameter $\Delta \leq R$

Then observer (3) is such that $d(\hat{z}_k, z_k)$ converges exponentially to zero.

Proof: The proof is a straightforward application of Birkhoff theorem over a finite horizon. ■

IV. Positive observers for positive linear systems in the positive orthant

In this section we address the particular case of linear systems on the positive orthant $K = \mathbb{R}_+^n = \{x \in \mathbb{R}^n | \forall i \ x_i \geq 0\}$. The Hilbert metric on this cone is defined by:

$$d(x, y) = \max_{i,j} \log(\frac{x_i y_j}{x_j y_i})$$

We consider the linear positive system in $\mathbb{R}_+^n$.

$$x_{k+1} = A_k x_k + B_k u_k, \quad y_k = C_k x_k \tag{4}$$

where A_k is positive, B_k and C_k are non-negative matrices. The positive observer (3) becomes

$$\begin{aligned} \hat{z}_0 \in \mathring{K}, \ \hat{z}_{k+1} &= \frac{A_k \hat{z}_k + B_k u_k}{\|A_k \hat{z}_k + B_k u_k\|}, \\ \hat{r}_k &= \frac{\|y_k\|}{\|C_k(\hat{z}_k)\|} \end{aligned} \tag{5}$$

In continuous time, the linear system becomes

$$\frac{d}{dt}x = A(t)x + B(t)u(t), \quad y = C(t)x \tag{6}$$

with, $A(t)$ Metzler, i.e. $A_{ij}(t) \geq 0$ for $i \neq j$, $B(t)$ and $C(t)$ non-negative, and observer (5) writes

$$\begin{aligned} \frac{d}{dt}\hat{z} &= (1 - \hat{z}\hat{z}^T)[A(t)\hat{z} + B(t)u(t)], \\ \hat{r} &= \frac{\|y(t)\|}{\|C(t)\hat{z}\|} \end{aligned} \tag{7}$$

A. Convergence issues

First of all, theorem 3.5 of [4] proves that the contraction ratio of the map $x \mapsto Ax+z_0$ with $z_0 \geq 0$ is less than the one of A. Thus the additional term $B_k u_k$ helps convergence and needs not be considered in the sequel. We have the following result extending Proposition 1 to the full observer:

Proposition 2: Consider the system (4) and suppose there exists a finite horizon T such that $\exists R > 0 \ \ \forall k \in \mathbb{N}\ A_{k+T} \circ \cdots \circ A_{k+1} \circ A_k$ is linear with projective diameter $\Delta \leq R$, Proposition 1 implies the quantity

$$d(\hat{z}_k, z_k)$$

converges exponentially to zero. Moreover, if there exists $\epsilon, \alpha, \beta > 0$ such that for $k > 0$ the ball center $\hat{z}_k$ and radius ϵ satisfies $S(\hat{z}_k, \epsilon) \subset K$, and $\alpha \leq \|C_k\| \leq \beta$, then observer (5) is such that the quantity

$$|\frac{\hat{r}_k}{r_k} - 1|$$

also converges exponentially to zero.

Proof: Proposition 1 implies that $d(\hat{z}_k, z_k)$ tends exponentially to zero. So $S(z_k, \epsilon/2) \subset K$ for k large enough. So the angle between z_k and any line of C_k is less than, say, $\theta < \pi/2$. It implies that

$$\begin{aligned} |\frac{r_k}{\hat{r}_k} - 1| &\leq \frac{1}{\alpha \cos\theta}\ |\ \|C_k \hat{z}_k\| - \|C_k z_k\|\ | \\ &\leq \frac{1}{\alpha \cos\theta}\ \|C_k(\hat{z}_k - z_k)\| \end{aligned}$$

Moreover, the differential of the map $\mathbb{S}^{n-1} \cap K \ni v \mapsto d(v, w)$ for $w \in \mathbb{S}^{n-1} \cap K$ satisfying $S(w, \epsilon/2) \subset K$ is uniformly bounded from below in all directions on the subset $\{v \in \mathbb{S}^{n-1} \cap K |\ S(v, \epsilon/2) \subset K\}$. So there exists $\gamma > 0$ such that $\|\hat{z}_k - z_k\| \leq \gamma\ d(\hat{z}_k, z_k)$ for k large enough. As a result, $\|\hat{z}_k - z_k\|$ converges exponentially to zero for k sufficiently large and so does $|\frac{r_k}{\hat{r}_k} - 1|$. ■

The previous result provides a convergence result that differs from the usual linear exponential convergence. Indeed the fact that $\hat{r}_k/r_k$ converges exponentially to 1 does not imply the exponential convergence of the linear error $\hat{r}_k - r_k$ and $\hat{x}_k - x_k$. Nevertheless, it could be argued that the exponential convergence of the error $\hat{r}_k/r_k$ is a meaningful alternative that acknowledges the nonlinear nature of the state-space. For instance, this error is invariant to scalings, which is meaningful from a physical point of view as scalings often correspond to a change of units. Such a state error is also naturally found in the theory of symmetry-preserving observers [3].

Moreover, for *bounded* trajectories, exponential convergence of the ratio $\hat{r}_k/r_k$ does imply exponential convergence of the linear error. Finally, it is of interest to observe that the proposition implies (non exponential) convergence for the following metric:

$$d_p(\hat{x}_k, x_k) = \sqrt{d(\hat{z}_k, z_k)^2 + |\log(\hat{r}_k/r_k)|^2} \qquad (8)$$

The metric $|\log(\hat{r}_k/r_k)|$ is called the "natural" metric on $\mathbb{R}_+^*$, see e.g. [6]. The metric (8) is therefore a contractive metric for the observer. This result is reminiscent of the general work of [9].

B. Time-invariant case

Birkhoff is a generalized version of the celebrated Perron theorem on an arbitrary cone by means of Hilbert's geometry. Indeed, in the case where K is the positive orthant, the Perron theorem is a corollary of the Birkhoff theorem. Proposition 2 can thus be formulated directly with the help of Perron-Frobenius theory that deals with applications having a finite projective diameter.

Proposition 3: Consider the time-invariant system

$$x_{k+1} = Ax_k + Bu_k, \quad y_k = Cx_k$$

Suppose that A is primitive, i.e. there exists a natural $T \in \mathbb{N}$ such that all the coordinates A_{ij}^T of the T-th power of A are strictly positive. Then observer (5) has the following properties:

- $d(\hat{z}_k, z_k)$ and $|\frac{\hat{r}_k}{r_k} - 1|$ converge exponentially to zero.
- $d_p(\hat{x}_k, x_k) \to 0$ where d_p is the metric (8).
- if the sequence r_k is bounded, the observer is exponentially convergent in the usual sense, i.e. $\|\hat{x}_k - x_k\| \to 0$ exponentially.

Proof: Theorem 1 and the Banach contraction mapping theorem (or the Perron-Frobenius theorem) imply there exists a vector $v \in \mathring{K}$ (the fixed point) in $\mathring{K} \cap \mathbb{S}$ such that both $d(\hat{z}_k, v)$ and $d(z_k, v)$ tend exponentially to zero (see [4] for more details). For k large enough, it implies that

$$|\frac{r_k}{\hat{r}_k} - 1| \leq \frac{2}{\|Cv\|}\ |\ \|C\hat{z}_k\| - \|Cz_k\|\ | \leq \frac{2}{\|Cv\|}\ \|C(\hat{z}_k - z_k)\|$$

So there exists $\gamma > 0$ such that $|r_k/\hat{r}_k - 1| \leq \gamma \|\hat{z}_k - z_k\|$. On the tangent space to $v/\|v\|$ in $\mathbb{S} \cap \mathring{K}$ the differential of the map $w \mapsto d(v, w)$ is bounded from below in all directions. As a result, $\|\hat{z}_k - z_k\|$ converges exponentially to zero for k sufficiently large and so does $|\frac{r_k}{\hat{r}_k} - 1|$. ■

C. An example

We consider the positive continuous-time system of [5]

$$\dot{x} = \begin{pmatrix} 1 & 3 & 2 \\ 10 & 2 & 4 \\ 3 & 2 & 1 \end{pmatrix} x, \quad y = \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} x$$

As all the off-diagonal coordinates of A are strictly positive, the associated discrete-time map is primitive, and Proposition 3 applies for this system. The present paper allows to derive a convergent positive observer whereas [5] proves that it is *not* possible to build a convergent positive linear observer for this system.

D. Positive observers and measurement noise

Proposition 2 proves that for a whole class of systems, one can build a convergent positive observer with very weak assumptions on the output map. Indeed, for those systems $n-1$ coordinates (the z term) are estimated without the use of the output map. Thus the z estimate is *never* noisy, even when the measurement noise is very large. The remaining

1-dimensional term r is estimated via the following relation: $\hat{r}_k = \|y_k\|/\|C_k(\hat{z}_k)\|$. Note that any non-zero component of the output y_k suffices to estimate the r coordinate, so that we will assume in the sequel the output is a scalar $y_k > 0$.

If the measurement noise is large, the estimation of observer (5) may not be easy to interpret as $\hat{r}_k$ can be as noisy as the output y_k. For a better noise filtering, we propose the following modification:

$$\hat{r}_k = \|A_k(\hat{r}_{k-1}\hat{z}_{k-1}) + B_k u_{k-1}\| + L_r(y_k/C_k(\hat{z}_k) - \hat{r}_k)$$

If L_r is small enough the noise is efficiently filtered. Yet, L_r must be large enough to ensure convergence. For instance if we consider the one-dimensional time-invariant system:

$$\frac{d}{dt} r = ar, \quad y = cr$$

with $a > 0$, we see that convergence is guaranteed as soon as $L_r > a$.

E. Numerical experiments

Consider the continuous-time system [1]:

$$\dot{x} = \begin{pmatrix} -1 & 1 \\ 2 & 0 \end{pmatrix} x, \quad y = \begin{pmatrix} 1 & 0 \end{pmatrix} x \tag{9}$$

for which there exists no convergent linear Luenberger observer. Proposition 3 proves that observer (5) is positive and converges, as the off-diagonal terms of the matrix associated to the continuous time system are strictly positive. In the first numerical experiment we consider the noiseless system (see Fig 1). We see that the estimates of the first and second coordinates of the observer (5), $\hat{x}_1(t)$ and resp. $\hat{x}_2(t)$ are always positive and that the error converges.

In the second experiment, a gaussian white noise with unit standard deviation (more than 10% of the maximum value of the signal) was added. Estimates of the following observer:

$$\begin{aligned} \frac{d}{dt}\hat{z} &= (1 - \hat{z}\hat{z}^T)[A\hat{z} + Bu(t)], \\ \frac{d}{dt}\hat{r} &= \hat{r}(\hat{z}^T A\hat{z}) + L_r(\frac{\|y\|}{\|C(\hat{z})\|} - \hat{r}) \end{aligned} \tag{10}$$

with $L_r = 3\ s^{-1}$ are presented on Figure 2. We see that the noise is efficiently filtered and the observer is still positive and convergent. In both experiments the initial conditions are: $x(0) = (1, 1/10)^T,\ \hat{x}(0) = (1/10, 1)^T$.

As a final remark, note that, looking at the figures we see the error $\hat{r}/r - 1$ seems much more adapted than the usual error $\hat{x} - x$ when the norm of x diverges (i.e. $r \to \infty$). Indeed, the interesting transient behavior of the estimation error is "crushed" by the plot scale as the coordinates of x grow exponentially. For example if we plot $\|\hat{x} - x\|$ over a 6-seconds horizon, the initial errors are barely visible on the plot.

V. Observers in the cone of positive semi-definite matrices

Positive semi-definite matrices appear in various contexts of applied mathematics and engineering. They appear as variables (convex programming, LMI, Lyapunov equation),

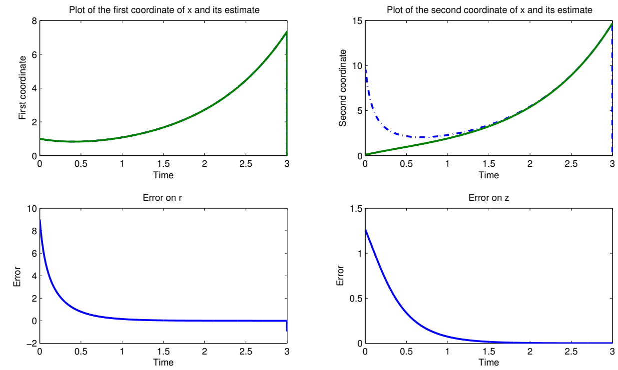

Fig. 1. Results of observer (7) for system (9) with noiseless measurement. Top left: $x_1(t)$ (plain line) and $\hat{x}_1(t)$ (dashed line). They are equal by definition of the observer since the output is x_1. Top right: $x_2(t)$ (plain line) and $\hat{x}_2(t)$ (dashed line). Bottom left: $\hat{r}/r - 1$. Bottom right: $\|\hat{z} - z\|$.

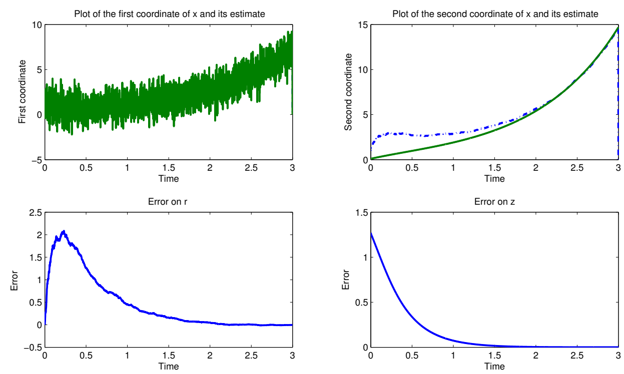

Fig. 2. Results of observer (10) for system (9) with noisy measurement (white noise with unit standard deviation). Top left: Measurement $y(t)$ (plain line) and $\hat{x}_1(t)$ (dashed line). The noise is efficiently filter (see top right graphics) and $\hat{x}_1$ is thus masked by y. Top right: $x_2(t)$ (plain line) and $\hat{x}_2(t)$ (dashed line). Bottom left: $\hat{r}/r - 1$. Bottom right: $\|\hat{z} - z\|$.

and as covariance matrices (statistics, signal processing, Kalman filtering), diffusion tensors (biomedical imaging), and kernels in machine learning. The study of the cone of positive semi-definite hermitian matrices has thus received ever growing attention in the last years, and we propose to apply the theory developped in this paper on this cone. It writes

$$K = \{X \in \mathbb{C}^{n \times n} | X = X^\dagger,\ X \succeq 0\}$$

the Hilbert metric is

$$d(X, Y) = \log(\frac{\lambda_{max}(XY^{-1})}{\lambda_{min}(XY^{-1})})$$

Considering a linear system defined on K

$$X_{k+1} = \mathcal{A}_k(X_k) + \mathcal{B}_k(u), \quad y_k = \mathcal{C}_k(X_k)$$

the whole analysis developped in Section II can be applied. We are now going to discuss a particular possible domain of application : design of quantum filters.

A. Quantum filtering as a positive observer problem

An interesting linear system on K is the evolution of the density matrix characterizing a state in a quantum channel. Indeed consider a quantum channel associated to the Kraus map (which is a positive linear map on K)

$$\mathcal{K}(\rho) = \sum_{\mu=1}^{n} M_\mu \rho M_\mu$$

where ρ is the density matrix, i.e. an hermitian semi-definite positive matrix of trace one, describing the input state, $\mathcal{K}(\rho)$ is the output state, and $\sum_1^m M_\mu M_\mu^\dagger = I$. The evolution is described by the following discrete-time system

$$\rho_{k+1} = \mathcal{M}_{\mu_k}(\rho_k) := \frac{1}{\mathrm{Tr}(M_{\mu_k}\rho_k M_{\mu_k}^\dagger)} M_{\mu_k}\rho_k M_{\mu_k}^\dagger$$

where ρ_k is the quantum state at time t_k and $\mu_k = \{1,..,m\}$ is a random variable such that $\mu_k = j$ with probability $\mathrm{Tr}(M_j \rho_k M_j^\dagger)$. The process preserves the trace and it is a concatenation of a linear map and a renormalization.

The problem of quantum filtering is the following: consider a realization associated to the Kraus map defined above, and assume that at each step the jump $\mu_k \in \{1,..,n\}$ is detected, but the initial state ρ_0 is not known. It is typically an observer problem. The following observer is known as a "quantum filter":

$$\hat\rho_{k+1} = \mathcal{M}_{\mu_k}(\hat\rho_k) \tag{11}$$

and it is a mere copy of the dynamics which takes into account the jump information μ_k. Observer (11) suits in the framework developped in Section 1. Moreover, as the true process ρ_k must be of trace 1, there is no need to estimate the scaling r_k of Section 1. The quantum filters can be analyzed as positive observers in the light of the theory developped in this paper. In particular we have the following sufficient condition for exponential convergence:

Proposition 4: If for any $1 \le \mu \le n$ the map $\mathcal{M}_\mu$ has a finite projective diameter, the quantum filter (11) converges exponentially, i.e. $d(\hat\rho_k, \rho_k) \to 0$ exponentially.

Note that a similar result has already been underlined recently by one of the authors in [12] for applications in consensus. Proposition 4 proves that the Hilbert distance is a good metric to analyze convergence of quantum filters, as the Kraus maps are contractions for the Hilbert metric. Indeed they are linear maps so Theorem 1 proves the the contraction ratio does not exceed 1. The Hilbert metric may thus prove to be a useful alternative to the celebrated trace norm, i.e. $d(\rho_1, \rho_2) = \mathrm{Tr}(|\rho_1 - \rho_2|)$ for the study of quantum filters. See [11] and references therein for more details.

VI. CONCLUSION

In this paper, we proposed a new design method for positive observers on solid cones. The convergence analysis is based on Birkhoff theorem. It allows to build convergent positive observers for a large class of systems, which are homogeneous and not necessarily linear.

The theory was applied to two cones: the positive orthant of the euclidean space and the cone of hermitian positive definite matrices. In the positive orthant, the theory allows to build very simple convergent positive observers for a large class of systems. For instance, we prove exponential convergence of the observer for the linear system $\frac{d}{dt}x = Ax + Bu,\ y = Cx$ as soon as the off-diagonal terms of A are strictly positive, and the construction of the observer is trivial. In the cone of positive definite matrices, the application of Birkhoff theorem provides a framework to design and analyze quantum filters. Another application on the cone of semi-definite positive matrices would be to study the convergence of the distributed Kalman filter with the tools introduced in this article. This is left for future research.

As a concluding remark, note that the method developped in this paper may seem magical as the observer design becomes trivial. However the price to pay is that the convergence speed is not freely chosen as in Luenberger observer design for observable linear systems where the eigenvalues of the closed-loop system can be freely assigned.

REFERENCES

[1] J. Back and A. Astolfi. Design of positive linear observers for positive linear systems via coordinates transformations and positive realizations. *SIAM Journal of Control and Optimization*, 47(1):345–373, 2008.

[2] L. Benvenuti and L. Farina. A tutorial on the positive realization problem. *IEEE Trans. Automat. Control*, 49(5):651–664, 2004.

[3] S. Bonnabel, Ph. Martin, and P. Rouchon. Symmetry-preserving observers. *IEEE Trans. on Automatic Control*, 53(11):2514–2526, 2008.

[4] P.J. Bushell. Hilbert's metric and positive contraction mappings in a banacb space. *Archive for Rational Mechanics and Analysis*, 1973.

[5] N. Dautrebande and G. Bastin. Positive linear observers for positive linear systems. In *Proc. ECC 99*, 1999.

[6] J. Faraut and A. Koranyi. *Analysis on Symmetric Cones*. Oxford Univ. Press, London, U.K., 1994.

[7] L. Farina and S. Rinaldi. *Positive Linear Systems: Theory and Applications*. Wiley, New York, 2000.

[8] J.A. Jacquez and C.P. Simson. Qualitative theory of compartmental systems. *SIAM Reviews*, 35:43–79, 1993.

[9] W. Lohmiler and J.J.E. Slotine. On metric analysis and observers for nonlinear systems. *Automatica*, 34(6):683–696, 1998.

[10] Y. Ohta, H. Maeda, and S. Kodama. Reachability, observability and realizability of continuous-time positive systems. *SIAM Journal of Control and Optimization*, 22(2):171–180, 1984.

[11] P. Rouchon. Fidelity is a sub-martingale for any quantum filter. Technical report, Submitted, Available on Arxiv. 2010.

[12] S. Sepulchre, A. Sarlette, and P. Rouchon. Consensus in non-commutative spaces. In *49th IEEE Conference on Decision and Control*, pages 6596–6601, 2010.

[13] H.L. Smith. *Monotone Dynamical Systems-An introduction to the theory of competitive and cooperative systems*. American mathematical society, Province, RI, 1995.

[14] J.M. van den Hof. Positive linear observers for linear compartmental systems. *SIAM Journal of Control and Optimization*, 36(2):590–608, 1998.