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Learning equilibria in constrained Nash-Cournot games with misspecified demand functions

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***Abstract*— We consider a constrained Nash-Cournot oligopoly where the demand function is linear. While cost functions and capacities are public information, firms only have partial information regarding the demand function. Specifically, firms either know the intercept or the slope of the demand function and cannot observe aggregate output. We consider a learning process in which firms update their profit-maximizing quantities and their beliefs regarding the unknown demand function parameters, based on disparities between observed and estimated prices. A characterization of the mappings, corresponding to the fixed point of the learning process, is provided. This result paves the way for developing a Tikhonov regularization scheme that is shown to learn the correct equilibrium, in spite of the multiplicity of equilibria. Despite the absence of monotonicity of the gradient maps, we prove the convergence of constant and diminishing steplength distributed gradient schemes under a suitable caveat on the starting points. Notably, precise rate of convergence estimates are provided for the constant steplength schemes.**

I. Introduction

The Nash solution concept [7] has been extensively analyzed and applied in economics, engineering and applied sciences and finds relevance in the examination of strategic behavior in noncooperative games. In such settings, the Nash equilibrium is a tuple of strategies from which no player can profit from unilaterally deviating. In this paper, we consider a deterministic Nash-Cournot game, in which a common homogeneous commodity is being produced by several firms and its price is specified completely by a function of the aggregate output. In such a game, the ith player solves Opt(x_{-i}), defined as

$$\begin{aligned} \min \quad & f_i(x;\theta) \triangleq \Big(c_i(x_i) - p(X;\theta)x_i\Big) \\ \text{subject to} \quad & x_i \in K_i, \end{aligned}$$

where $x \triangleq (x_1,\ldots,x_N)^T$, x_i denotes the output of firm i, $c_i(\cdot)$ denotes firm i's cost function, and $K_i \triangleq [0, Cap_i]$ with Cap_i being the capacity of firm i. The price function of the commodity, denoted by $p(X;\theta)$, is defined as

$$p(X;\theta) \triangleq a^* - b^* X,$$

where $X = \sum_{i=1}^N x_i$ and $\theta = (a^*, b^*)$. The associated Nash-Cournot equilibrium is given by a tuple $x^* = (x_i^*)_{i=1}^N$ where $x_i^* \in \text{SOL}(\text{Opt}(x_{-i}^*))$ for $i = 1,\ldots,N$, $\text{SOL}(\text{Opt}(x_{-i}^*))$ denotes the solution of Opt(x_{-i}) and $x_{-i} = (x_j)_{j\neq i}$.

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Cournot models predate the Nash solution concept and a host of variants have been analyzed [8], [9]. An oft-used assumption in game-theoretic models is one which requires that player payoffs are public knowledge and every player is able to forecast the choices of his adversaries. As noted by Kirman [5], a firm's information sets may be incomplete as manifested by a regime where firms have imperfect information of the payoffs of their adversaries. In a Cournot setting, firms may have an incorrect specification of the demand function. Naturally, firms can ascertain that their estimates differ from observations, leading to an adjustment process. In effect, firms *learn* the parameters of the game while participating in the game.

Our work is inspired by a series of papers by Szidarovszky, Bischi and their coauthors [2], [3], [10] where firms competing in a Nash-Cournot attempt to learn a parameter of the demand function while playing the game. In [1], in an unconstrained regime with linear costs, the authors examine the stability of learning the equilibrium and one of the unknown parameters of θ (either a^* or b^*). In particular, they consider two cases:

(Case 1): The slope b^* is known, but a^* is unknown.
(Case 2): a^* is known, but the slope b^* is unknown.

It is shown that this process is globally stable for case 1 and unstable when considering case 2.

In this paper, we consider the learning of equilibria when one component of θ is unknown and the aggregate output X is unobservable by the firms. In particular, if b^* and X are unknown, then our goal lies in developing algorithms that construct a sequence $z^k = (x^k, b^k)$ such that

$$\lim_{k\to\infty} z^k = z^*,$$

where $z^* = (x^*, b^*)$.

Broadly speaking, our focus is on *constrained* Nash-Cournot problems; such an extension is not a trivial one in that gradient-based learning now involves introduces the use of a projection operator. In such a regime, we prove that the mappings associated with the variational problems are $\mathbf{P}$ and $\mathbf{P}_0$ maps for cases 1 and 2, respectively. Notably, while such a variational problem has a unique solution in the context of Case 1 while such uniqueness cannot be claimed when learning b^* (Case 2). Despite this lack of uniqueness, we develop a Tikhonov regularization scheme that is guaranteed to converge to the correct equilibrium, under suitable conditions. The convergence of standard gradient-based distributed schemes cannot be immediately claimed since the mappings are not monotone (but admit a weaker

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$\mathbf{P}_0$ property). However, by imposing a condition on the starting points, we demonstrate that constant and diminishing steplength schemes are convergent. Of particular importance is the rate estimate provided in a constant steplength regime.

The rest of the paper is organized as follows. In Section II, we provide an equivalent variational inequality condition for the Nash-Cournot equilibrium. In Section III, we propose the fixed-point problems for learning both equilibria and unknown parameters for both case 1 and case 2. The properties of the mappings associated with the fixed-point problems are also analyzed. Section IV focuses on a centralized Tikhonov regularization scheme, while Section V deals with a distributed scheme under some initial conditions. Some numerical results are provided in Section VI.

Throughout the paper, we use $\|x\|$ to denote the Euclidean norm of a vector x, i.e., $\|x\| = \sqrt{x^T x}$. We use Π_K to denote the Euclidean projection operator onto a set K, i.e., $\Pi_K(x) \triangleq \mathrm{argmin}_{y \in K} \|x - y\|$. A square matrix H is said to be a $\mathbf{P}$-matrix if every principal minor of H is positive. Similarly, H is a $\mathbf{P_0}$-matrix if every principal minor of H is nonnegative.

II. Problem description

Under the convexity of the cost function, the equilibrium conditions of the game are sufficient and can be compactly stated as a variational inequality VI(K,F), given a closed and convex set $K \subseteq \mathbb{R}^n$ and a mapping $F : \mathbb{R}^n \to \mathbb{R}^n$. Recall that VI$(K,F)$ requires an $x \in K$ satisfying

$$(y-x)^T F(x) \geq 0, \quad \text{for all } y \in K. \tag{1}$$

In the current setting, the set K is given by $K \triangleq \prod_{i=1}^N K_i$, and the mapping $F(x)$ is defined as

$$F(x) \triangleq \begin{pmatrix} c_1'(x_1) + b^*(X + x_1) - a^* \\ \vdots \\ c_N'(x_N) + b^*(X + x_N) - a^* \end{pmatrix}, \tag{2}$$

where a^* and b^* are assumed to be available and X is observable by every agent. Furthermore, $F(x)$ can be easily shown to be Lipschitz continuous (with constant L) and strongly monotone where the latter implies that there exists an $\eta > 0$ such that for all $x, y \in K$, we have that $(F(x) - F(y))^T (x-y) \geq \eta \|x-y\|^2$. The strong monotonicity and Lipschitz continuity allow for developing a simple distributed scheme of the form:

$$x^{k+1} = \Pi_K \left(x^k - \gamma F(x^k) \right). \tag{3}$$

In fact, if $\gamma < 2\eta/L^2$, then the sequence $\{x^k\}$ produced by (3) converges to the unique solution of the VI(K,F).

However, in this paper, our interest lies in *learning* the equilibrium when either a^* or b^* is unknown. Instead, the prices can be observed but the aggregate output is not available to any agent. We denote a_i and b_i as the estimates of a^* and b^*, respectively, by the ith player. Since, the aggregate output is not observable by any player, every player builds an estimate of adversarial decisions. We denote x_{ij} as the estimate of the jth player's output by the ith player and x_{ii} as the ith player's true output. Given the price p, every player i adjusts his belief regarding a^* or b^* as well as his belief regarding adversarial outputs.

In accordance with standard assumptions in a noncooperative game, we assume that the costs and capacities are common knowledge across all players. It is worth noting that a host of distributed algorithms for computing equilibria impose a different informational assumption in that players only know their payoffs and strategy sets and can observe aggregate output X; In contrast, in this setting, players do not know their payoffs perfectly (since either a^* or b^* are unknown) and X *cannot* be observed.

III. Characterization of learning problems

In this section, we propose a learning scheme in which firms compute profit-maximizing production levels for themselves and their adversaries, given their current beliefs regarding the unknown parameter. Simultaneously, they also update their belief based on differences between observed and estimated prices. In this section, we characterize the mapping associated with the fixed point of the adjustment process. In fact, this characterization proves crucial in the development of a Tikhonov scheme in Section IV.

Assumption 1 (A1): Suppose the cost functions c_i are convex and twice continuously differentiable functions for $i = 1, \dots, N$ and $K_i = [0, Cap_i]$ for $i = 1, \dots, N$.

Assumption 2 (A2): Suppose the demand function is defined as $p(X;\theta) = a^* - b^* X$ where $\theta = (a^*, b^*) > 0$.

We consider the case where a^* is known, but b^* is unknown (case 2). The analysis of the case when b^* is known but a^* is unknown (case 1) is similar and simpler. Therefore, we only consider case 2 in the present and following sections. Via a projected gradient step, the ith firm updates its strategy x_{ii} and its belief of the jth firm's strategy, namely x_{ij}, based on b_i^k. Furthermore, b_i^k is modified as per the difference between estimated and observed prices. Our goal lies in the development of single timescale schemes where both updates occur simultaneously. Therefore, the associated fixed-point problem is

Next, we consider the case where a^* is known, but b^* is unknown. An analogous learning process has a fixed point given by the following:

$$\begin{aligned} x_{ij} &= \Pi_{K_j} \left(x_{ij} - \gamma \nabla_{x_{ij}} f_i(x; b_i) \right), \\ b_i &= \Pi_{\mathbb{R}_+} \left(b_i - \gamma \left(p(\bar{X}; b^*) - p(X_i; b_i) \right) \right), \end{aligned} \tag{4}$$

for $i, j = 1, \dots, N$, succinctly stated as $z = \Pi_K(z - \gamma F(z))$, where $z = (x, b)$,

$$K \triangleq \underbrace{\hat{K} \times \cdots \times \hat{K}}_{N \text{ terms}} \text{ with } \hat{K} \triangleq \prod_{j=1}^N K_j \times \mathbb{R}_+, \tag{5}$$

and $F(z)$ is defined as

$$F(z) = \begin{pmatrix} (\nabla_{x_{ij}} f_i(x_i; b_i))_{i,j} \\ (p(\bar{X}; b^*) - p(X_i; b_i))_i \end{pmatrix}. \tag{6}$$

The equivalent variational inequality VI(K,F) is given by

$$\text{Find } z^* \in K: \quad (z-z^*)^T F(z^*) \geq 0, \quad z \in K. \tag{7}$$

Next, we show that the mapping F associated with VI (7) is a $\mathbf{P_0}$-mapping on K (Proposition 3), i.e., the Jacobian matrix $\nabla F(z)$ of the mapping F is a $\mathbf{P_0}$-matrix for all $z \in K$. Let $A_i = b_i(I+ee^T)+E_i$, $B_i = (x_i + (e^T x_i)e)e_i^T$, $C_i = b_i e_i e^T - b^* e e_i^T$, and $D = \sum_{i=1}^N (e^T x_i) e_i e_i^T$, where $x_i = (x_{i1},\ldots,x_{iN})^T$, e and e_i are the column of ones and the ith unit vector in $\mathbb{R}^N$, respectively, and E_i is an $N \times N$ diagonal matrix with $c_j''(x_{ij})$ as its jth diagonal entry. Let $H(z) = \nabla F(z)$ for all $z \in K$. Then,

$$H(z) = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \triangleq \begin{pmatrix} A_1 & & & B_1 \\ & \ddots & & \vdots \\ & & A_N & B_N \\ C_1 & \cdots & C_N & D \end{pmatrix}, \tag{8}$$

and thus $\det(H(z)) = \det(A)\det(D - CA^{-1}B)$. For showing $H(z)$ is a $\mathbf{P_0}$-matrix for all $z \in K$, we consider two cases: (1) Every component of b is positive (Lemma 1) (2) Some component of b is zero (Lemma 2) to claim the final result.

Lemma 1: Suppose (A1) and (A2) hold and $F(z)$ and K are defined by (6) and (5), respectively. Additionally, suppose $b > 0$. Then the mapping F is a $\mathbf{P}_0$-mapping on K.

Proof: It suffices to show that $H(z)$ defined in (8) is a $\mathbf{P_0}$-matrix for all $z \in K$ when $b > 0$. Given $z = (x,b)$, let $H = H(z)$. Since $z \in K$, it follows that $x \geq 0$. Note that the matrix E_i is a diagonal matrix with nonnegative diagonal entries for all i as a result of convex cost functions. Also recall that the sum of a diagonal positive semidefinite matrix and a $\mathbf{P_0}$-matrix is a $\mathbf{P_0}$-matrix. Therefore, we only need to show that H is a $\mathbf{P_0}$-matrix when $E_i = 0$ for all i.

We consider any principal submatrix of H denoted by $H_{\alpha\alpha}$, where $\alpha = \cup_{i=1}^{N+1}\alpha_i$ and $\alpha_i \subseteq \{(i-1)N+1,\ldots,iN\}$ are non-overlapping index sets. Then, $H_{\alpha\alpha}$ is given by

$$\begin{pmatrix} \tilde{A} & \tilde{B} \\ \tilde{C} & \tilde{D} \end{pmatrix} \triangleq \begin{pmatrix} A_{\alpha_1\alpha_1} & & & B_{\alpha_1\alpha_{N+1}} \\ & \ddots & & \vdots \\ & & A_{\alpha_N\alpha_N} & B_{\alpha_N\alpha_{N+1}} \\ C_{\alpha_{N+1}\alpha_1} & \cdots & C_{\alpha_{N+1}\alpha_N} & D_{\alpha_{N+1}\alpha_{N+1}} \end{pmatrix}.$$

Thus, $\det(H_{\alpha\alpha}) = \det(\tilde{A})\det(\tilde{D} - \tilde{C}\tilde{A}^{-1}\tilde{B})$. Let I_{n_i} and e^{n_i} denote the identity matrix and the column of ones in $\mathbb{R}^{n_i \times n_i}$ and $\mathbb{R}^{n_i}$, respectively, where $n_i = |\alpha_i|$, $i = 1,\ldots,N+1$. Also, for $i = 1,\ldots,N$ and $j \in \{i, N+1\}$, let $e_i^{n_j}$ denote the $i^{(j)}$-th unit vector in $\mathbb{R}^{n_j}$, where $i^{(j)}$ is the position of $(j-1)N+i$ in α_j. Then, assuming $E_i = 0$, we have for $i = 1,\ldots,N$,

$$\begin{aligned} A_{\alpha_i\alpha_i} &= b_i(I_{n_i} + e^{n_i}(e^{n_i})^T), \\ B_{\alpha_i\alpha_{N+1}} &= (x_{\alpha_i} + (e^T x_i)e^{n_i})(e_i^{n_{N+1}})^T, \\ C_{\alpha_{N+1}\alpha_i} &= b_i e_i^{n_{N+1}}(e^{n_i})^T - b^* e^{n_{N+1}}(e_i^{n_i})^T, \\ D_{\alpha_{N+1}\alpha_{N+1}} &= \sum_{i=1}^N (e^T x_i) e_i^{n_{N+1}}(e_i^{n_{N+1}})^T. \end{aligned}$$

Since $A_{\alpha_i\alpha_i}^{-1} = \frac{1}{b_i}(I_{n_i} - \frac{1}{n_i+1}e^{n_i}(e^{n_i})^T)$, $\tilde{C}\tilde{A}^{-1}\tilde{B}$ is given by

$$\begin{aligned} &\sum_{i=1}^N C_{\alpha_{N+1}\alpha_i} A_{\alpha_i\alpha_i}^{-1} B_{\alpha_i\alpha_{N+1}} \\ &= \sum_{i=1}^N \frac{1}{b_i}\Big[(b_i e_i^{n_{N+1}}(e^{n_i})^T - b^* e^{n_{N+1}}(e_i^{n_i})^T) \\ &\times (I_{n_i} - \frac{1}{n_i+1}e^{n_i}(e^{n_i})^T)(x_{\alpha_i} + (e^T x_i)e^{n_i})(e_i^{n_{N+1}})^T\Big] \\ &= \sum_{i=1}^N \frac{1}{b_i}\Big[(b_i e_i^{n_{N+1}}(e^{n_i})^T - b^* e^{n_{N+1}}(e_i^{n_i})^T) \\ &\quad \times (x_{\alpha_i} - \frac{1}{n_i+1}e^{n_i}(x_{\alpha_i}^T e^{n_i}) + (e^T x_i)e^{n_i} \\ &\quad - \frac{n_i}{n_i+1}e^{n_i}(e^T x_i))(e_i^{n_{N+1}})^T\Big]. \end{aligned}$$

Further simplification of this expression leads to

$$\begin{aligned} &\sum_{i=1}^N \Big[(e_i^{n_{N+1}}(e^{n_i})^T - \frac{b^*}{b_i}e^{n_{N+1}}(e_i^{n_i})^T) \\ &\quad \times (x_{\alpha_i} + \frac{1}{n_i+1}e^{n_i}(e^T x_i - x_{\alpha_i}^T e^{n_i}))(e_i^{n_{N+1}})^T\Big] \\ &= \sum_{i=1}^N \Big[e_i^{n_{N+1}}(e_i^{n_{N+1}})^T(x_{\alpha_i}^T e^{n_i}) \\ &\quad + \frac{n_i}{n_i+1}e_i^{n_{N+1}}(e_i^{n_{N+1}})^T(e^T x_i - x_{\alpha_i}^T e^{n_i}) \\ &\quad - \frac{b^*}{b_i}e^{n_{N+1}}(e_i^{n_{N+1}})^T(x_{\alpha_i}^T e_i^{n_i}) \\ &\quad - \frac{b^*}{(n_i+1)b_i}e^{n_{N+1}}(e_i^{n_{N+1}})^T(e^T x_i - x_{\alpha_i}^T e^{n_i})\Big]. \end{aligned}$$

This implies that $\tilde{D} - \tilde{C}\tilde{A}^{-1}\tilde{B}$ is

$$\begin{aligned} &= \sum_{i=1}^N \frac{1}{n_i+1}e_i^{n_{N+1}}(e_i^{n_{N+1}})^T(e^T x_i - x_{\alpha_i}^T e^{n_i}) \\ &+ \sum_{i=1}^N \frac{b^*}{b_i}e^{n_{N+1}}(e_i^{n_{N+1}})^T(x_{\alpha_i}^T e_i^{n_i} + \frac{1}{n_i+1}(e^T x_i - x_{\alpha_i}^T e^{n_i})). \end{aligned} \tag{9}$$

Since $e^T x_i \geq x_{\alpha_i}^T e^{n_i}$ for all $i = 1,\ldots,N$, the first item on the right hand side of equation (9) is a diagonal matrix with nonnegative diagonal entries. The second item is a matrix of rank 1 with all elements being nonnegative, and thus is a $\mathbf{P_0}$-matrix. Recall that the sum of a diagonal positive semidefinite matrix and a $\mathbf{P_0}$-matrix is a $\mathbf{P_0}$-matrix. Therefore, $\tilde{D} - \tilde{C}\tilde{A}^{-1}\tilde{B}$ is also a $\mathbf{P_0}$-matrix and has nonnegative determinant. Since $\det(\tilde{A}) = \prod_{i=1}^N \det(A_{\alpha_i\alpha_i}) \geq 0$, we have $\det(H_{\alpha\alpha}) = \det(\tilde{A})\det(\tilde{D} - \tilde{C}\tilde{A}^{-1}\tilde{B}) \geq 0$ for all $\alpha = \cup_{i=1}^{N+1}\alpha_i$ with $\alpha_i \subseteq \{(i-1)N+1,\ldots,iN\}$, i.e., H is a $\mathbf{P_0}$-matrix. ■

We now show that the mapping is also $\mathbf{P}_0$ when at least one component of b is allowed to be zero.

Lemma 2: Suppose (A1) and (A2) hold and $F(z)$ and K are defined by (6) and (5), respectively. Additionally, suppose $b \geq 0$ with $b_i = 0$ for $i \in \mathcal{I}$ where $\mathcal{I} \triangleq \{i : b_i = 0\}$ and $|\mathcal{I}| > 0$. Then the mapping F is a $\mathbf{P}_0$-mapping on K.

Proof: It suffices to show that $H(z)$ defined in (8) is a $\mathbf{P_0}$-matrix for all $z \in K$ when $b \geq 0$ with $b_i = 0$ for $i \in \mathcal{I}$. We use the notation in Lemma 1. Given $z = (x, b)$, let $H = H(z)$. Consider the structure of the principal submatrix $H_{\alpha\alpha}$ of H when $E_i = 0$ for all i. It is obvious that $\det(H_{\alpha\alpha}) = 0$ if at least two components of b are zero, or if $n_i \geq 2$ for some i with $b_i = 0$. Therefore, we only need to consider the case when $b_l = 0$, $b_i > 0$ for $i \neq l$ and $n_l = 1$ for some $l \in \{1, \ldots, N\}$.

By doing a column transformation, we can get the new $C_{\alpha_{N+1}\alpha_i}$ denoted by $\hat{C}_{\alpha_{N+1}\alpha_i}$, where $\hat{C}_{\alpha_{N+1}\alpha_i} = b_i e_i^{n_{N+1}} (e^{n_i})^T$ for $i \neq l$, and $\hat{C}_{\alpha_{N+1}\alpha_l} = -b^* e^{n_{N+1}} (e_l^{n_l})^T$. By doing a row transformation, we obtain the new $A_{\alpha_l\alpha_i}$ denoted by $\hat{A}_{\alpha_l\alpha_i}$, where

$$\begin{aligned}\hat{A}_{\alpha_l\alpha_i} &= -B_{\alpha_l\alpha_{N+1}} D^{-1}_{\alpha_{N+1}\alpha_{N+1}} \hat{C}_{\alpha_{N+1}\alpha_i} \\ &= \begin{cases} 0, & \text{for } i \neq l, \\ v, & \text{for } i = l, \end{cases}\end{aligned}$$

for some scalar $v > 0$. Furthermore, we get the new $B_{\alpha_l\alpha_{N+1}}$ denoted by $\hat{B}_{\alpha_l\alpha_{N+1}}$, where $\hat{B}_{\alpha_l\alpha_{N+1}} = 0$, and $D_{\alpha_{N+1}\alpha_{N+1}}$ remains the same. Let $\hat{A}, \hat{B}, \hat{C}$ and $\hat{D}$ denote the associated matrices for $\tilde{A}, \tilde{B}, \tilde{C}$ and $\tilde{D}$ after transformations. Then,

$$\hat{D} - \hat{C}\hat{A}^{-1}\hat{B} = \sum_{i \neq l} \frac{1}{n_i + 1} e_i^{n_{N+1}} (e_i^{n_{N+1}})^T (e^T x_i - x_{\alpha_i}^T e^{n_i}).$$

Since $e^T x_i \geq x_{\alpha_i}^T e^{n_i}$ for all i, we have $\det(\hat{D} - \hat{C}\hat{A}^{-1}\hat{B}) \geq 0$. Also, $\det(\hat{A}) = \prod_{i=1}^{N} \det(\hat{A}_{\alpha_i\alpha_i}) > 0$. Therefore, $\det(H_{\alpha\alpha}) = \det(\hat{A}) \det(\hat{D} - \hat{C}\hat{A}^{-1}\hat{B}) \geq 0$ for all α, i.e., H is a $\mathbf{P_0}$-matrix. ■

Proposition 3: Suppose (A1) and (A2) hold and $F(z)$ and K are defined by (6) and (5), respectively. Then the mapping F is a $\mathbf{P}_0$-mapping on K.

Proof: It follows from Lemma 1 and Lemma 2. ■

IV. A Tikhonov-based Learning Scheme

Having analyzed the mapping associated with the learning process, we now examine the convergence properties of learning equilibria through such an avenue. In particular, we consider a Tikhonov regularization scheme which necessitates computing a sequence $\{z_k\}$, of which each iterate is defined by the solution of the regularized fixed point problem:

$$z_k = \Pi_K(z_k - \gamma(F(z_k) + \epsilon_k z_k)), \tag{10}$$

where $\{\epsilon_k\}$ is a sequence of regularization parameters converging to zero. It is known that if the mapping F is a continuous $\mathbf{P_0}$-mapping on K and the solution set of $\text{VI}(K, F)$ is nonempty and bounded, then the limit points of the sequence $\{z_k\}$ generated by the Tikhonov algorithm are all solutions to $\text{VI}(K, F)$ [4].

In the context of learning b^*, it is hard to make a conclusion about the "true" convergence, since the mapping is a $\mathbf{P}_0$ mapping and there may be multiple solutions to the fixed point problem. Of these, the one of interest is (x^*, b^*). Developing the convergence theory to such a point requires a result from [6], that uses the following condition.

Condition 1: (Condition A in [6]) There exists a nonempty compact set D such that, for any point $z \in K \backslash D$, there is a point $z' \in K \cap D$ that satisfies $(z - z')^T F(z) \geq 0$.

Lemma 4: (Theorem 3 in [6]) If F is a continuous $\mathbf{P_0}$-mapping, K is nonempty, convex and closed, and Condition 1 is satisfied, then the following assertions hold true:
(a) $\text{VI}(K, F)$ has a solution.
(b) For every $\epsilon > 0$, $\text{VI}(K, F + \epsilon I)$ has a unique solution.
(c) The sequence $\{z_k\}$ generated by the Tikhonov algorithm has limit points, which are all solutions to $\text{VI}(K, F)$.

We also make the following assumption.

Assumption 3 (A3): The aggregate output of all firms has a small positive lower bound, i.e., $\sum_{i=1}^{N} x_{ii} \geq \epsilon$ for some $\epsilon > 0$.

If we learn b^*, the associated $\text{VI}(K, F)$ satisfies Condition 1 by the following lemma, which we can use to conclude the convergence of the Tikhonov scheme.

Lemma 5: Suppose (A1), (A2) and (A3) hold and $F(z)$ and K are defined by (6) and (5), respectively. Then F and K satisfy Condition 1.

Proof: Omitted. ■

However, since F is a $\mathbf{P_0}$-mapping, VI (7) may have more than one solution; However, the following result shows that the Tikhonov sequence converges to (x^*, b^*), the *correct* equilibrium.

Theorem 6: Suppose (A1), (A2) and (A3) hold and $F(z)$ and K are defined by (6) and (5), respectively. Then the sequence generated by (10) converges to (x^*, b^*), where $(x_{ii}^*)_{i=1}^N$ is a solution of the variational inequality (1).

Proof: By Proposition 3 in Section III, the mapping F associated with VI (7) is a $\mathbf{P_0}$-mapping on K. Also, Condition 1 is satisfied by Lemma 5.

By Lemma 4, the solution set $\text{SOL}(K, F)$ of VI (7) is nonempty, and the sequence generated by the Tikhonov algorithm has limit points, which are all solutions to VI (7). Let $(x^*, \tilde{b}^*)$ be such a limit point, where $\tilde{b}^* = (\tilde{b}_1^*, \ldots, \tilde{b}_N^*)^T$. Then, we will show that $\tilde{b}_i^* = b^*$, and $x_{ii}^* = y_i^*$ for all i, where $y^* = (y_1^*, \ldots, y_N^*)^T$ is the solution of VI (1), i.e., the sequence has a unique limit point, which is just the original equilibrium.

The solution (x^k, b^k) to the kth iteration satisfies the following coupled fixed point problems:

$$\begin{aligned} x_{ij}^k &= \Pi_{K_j}\left(x_{ij}^k - \gamma\left(c_j'(x_{ij}^k) + b_i^k x_{ij}^k + b_i^k \sum_{j=1}^N x_{ij}^k - a^* + \epsilon_k x_{ij}^k\right)\right), \\ b_i^k &= \Pi_{\mathbb{R}_+}\left(b_i^k - \gamma\left(\left(a^* - b^* \sum_{i=1}^N x_{ii}^k\right) - \left(a^* - b_i^k \sum_{j=1}^N x_{ij}^k\right) + \epsilon_k b_i^k\right)\right), \end{aligned} \tag{11}$$

for all i. Fix $l \in \{1, \ldots, N\}$. Then, we have

$$\begin{aligned} x_{lj}^k &= \Pi_{K_j}\left(x_{lj}^k - \gamma\left(c_j'(x_{lj}^k) + b_l^k x_{lj}^k + b_l^k \sum_{j=1}^N x_{lj}^k - a^* + \epsilon_k x_{lj}^k\right)\right), \\ b_l^k &= \Pi_{\mathbb{R}_+}\left(b_l^k - \gamma\left(\left(a^* - b^* \sum_{i=1}^N x_{ii}^k\right) - \left(a^* - b_l^k \sum_{j=1}^N x_{lj}^k\right) + \epsilon_k b_l^k\right)\right). \end{aligned} \tag{12}$$

Let $\bar{x}_{ij}^k = x_{lj}^k$ and $\bar{b}_i^k = b_l^k$ for all i, j. Then, for all i, $(\bar{x}_i^k, \bar{b}_i^k)$ is a solution of (12), and thus satisfies the fixed point problem (11). Therefore, $(\bar{x}^k, \bar{b}^k)$ is a solution to (11). Since the mapping associated with (11) is a **P**-mapping, the fixed problem (11) has at most one solution (see Proposition 3.5.10 in [4]), and thus the solution must be $(\bar{x}^k, \bar{b}^k)$, where $\bar{x}_i^k = x_l^k$ and $\bar{b}_i^k = b_l^k$ for all i.

As the limit point of $\{(\bar{x}^k, \bar{b}^k)\}_{k=1}^{\infty}$, $(x^*, \tilde{b}^*)$ must satisfy $x_i^* = x_l^*$ and $\tilde{b}_i^* = \tilde{b}_l^*$ for all i, which implies $\sum_{i=1}^N x_{ii}^* = \sum_{j=1}^N x_{ij}^*$. Since $(x^*, \tilde{b}^*)$ is a solution to VI (7), we have

$$0 = \left(a^* - b^* \sum_{i=1}^N x_{ii}^*\right) - \left(a^* - \tilde{b}_i^* \sum_{j=1}^N x_{ij}^*\right)$$
$$= (\tilde{b}_i^* - b^*) \sum_{i=1}^N x_{ii}^*,$$

and thus $\tilde{b}_i^* = b^*$, which implies $x_{ii}^* = y_i^*$ for all i, where y^* is the solution to VI (1). ■

Remark: Implementing this scheme in a distributed regime requires obtaining solutions to (10) in a distributed fashion. If the mapping F, defined in (6) were monotone (positive semidefinite Jacobians), then distributed projection schemes (gradient-response) may have been an avenue. However, since the mapping is $\mathbf{P_0}$ such an avenue is not immediately available. However, in future work, we intend to examine if a best-response scheme may prove useful.

V. SINGLE-TIMESCALE DISTRIBUTED SCHEME

In this section, we consider a single timescale distributed learning scheme. We propose two distributed schemes for computing equilibria. In the interest of brevity, we only consider the question of learning b^* and both fixed and diminishing steplength schemes will be discussed.

Consider the distributed scheme for learning x^* and b^* (case 2), defined as follows

$$\begin{aligned} x_{ij}^{k+1} &= \Pi_{K_j}\left(x_{ij}^k - \gamma_{1,k}\left(c_j'(x_{ij}^k) + b_i^k x_{ij}^k + b_i^k \sum_{j=1}^N x_{ij}^k - a^*\right)\right), \\ b_i^{k+1} &= \Pi_{K_b}\left(b_i^k - \gamma_{2,k}\left(\left(a^* - b^* \sum_{i=1}^N x_{ii}^k\right) - \left(a^* - b_i^k \sum_{j=1}^N x_{ij}^k\right)\right)\right), \end{aligned} \tag{13}$$

where K_b is a closed convex set in $\mathbb{R}_+$, and $i, j \in \{1, \ldots, N\}$. We make the following additional assumptions on starting points and second derivatives.

Assumption 4 (A4):

1) For all $i, j = 1, \ldots, N$, $b_i^0 = b_j^0$ and $x_{ij}^0 = x_{jj}^0$.
2) $c_i''(\cdot)$ is bounded with some constant $M_i > 0$ on the interval $[0, Cap_i]$ for all i. Let $M = \max\{M_i\}$.

In (A4), (1) states that all firms start from the same point, while (2) imposes some boundedness condition on cost functions. Furthermore, K_x, F_x and F_b are defined as $K_x = \prod_{i=1}^N K_i$,

$$F(z^k) = \begin{pmatrix} F_x(z^k) \\ F_b(z^k) \end{pmatrix}, F_x(z^k) = \begin{pmatrix} c_1' + b^k x_1^k + b^k X^k - a^* \\ \vdots \\ c_N' + b^k x_N^k + b^k X^k - a^* \end{pmatrix},$$

and $F_b(z^k) = (b^k - b^*) \sum_{i=1}^N x_i^k$.

Lemma 7: Suppose (A1), (A2), (A3) and (A4) hold. Then the distributed scheme (13) is equivalent to

$$\begin{aligned} x^{k+1} &= \Pi_{K_x}(x^k - \gamma_{1,k} F_x(z^k)), \\ b^{k+1} &= \Pi_{K_b}(b^k - \gamma_{2,k} F_b(z^k)). \end{aligned} \tag{14}$$

Moreover, we have that $\sum_{i=1}^N x_i^k \geq \epsilon$.

Proof: Omitted. ■

Our main convergence result relies on showing that F_x is strongly monotone and Lipschitz continuous.

Lemma 8: Suppose (A4) holds and $b > 0$. Then $F_x(z) = F_x(x, b)$ is strongly monotone in x with constant b and Lipschitz continuous in x with constant $\sqrt{3(M^2 + b^2 + N^2 b^2)}$.

Proof: Omitted. ■

Since the distributed scheme (13) can be reduced to scheme (14) by Lemma 7, we only need to consider the convergence of the sequence $\{z^k = (x^k, b^k)\}$ generated by (14). Two types of the steplength sequence $\{\gamma_k = (\gamma_{1,k}, \gamma_{2,k})\}$ are considered: (1) γ_k is fixed for all k; (2) $\{\gamma_k\}$ is a diminishing sequence. We only show the proof of the convergence when the steplength is diminishing (Theorem 10). The proof for convergence of the fixed steplength scheme (Theorem 9) is similar and simpler, and thus omitted.

Theorem 9: Suppose (A1), (A2), (A3) and (A4) hold. Let $\{z^k = (x^k, b^k)\}$ be the sequence generated by (14). If $\gamma_k \triangleq (\gamma_1, \gamma_2)$ for all k and $K_b = [\underline{b}, \bar{b}]$, where

$$0 < \gamma_1 < \frac{2\underline{b}}{3(M^2 + \bar{b}^2 + N^2 \bar{b}^2)}, \ \gamma_1 < \frac{\gamma_2 \epsilon}{u}, \ \gamma_2 < \frac{1}{Cap},$$

with $Cap = \sum_{i=1}^N Cap_i > \epsilon$, $u = \|X^* e + x^*\|$, $e^T = (1, \ldots, 1)$, then $z^k \to z^* = (x^*, b^*)$ as $k \to \infty$, where x^* is the solution of VI (1).

Proof: Omitted. ■

Theorem 10: Suppose (A1), (A2), (A3) and (A4) hold. Let $\{z^k = (x^k, b^k)\}$ be the sequence generated by (14). If $\gamma_{1,k} > 0$, $\sum_{k=1}^{\infty} \gamma_{1,k} = \infty$, $\sum_{k=1}^{\infty} \gamma_{1,k}^2 < \infty$, $\gamma_{2,k} > 2\gamma_{1,k} u/\epsilon$, $\sum_{k=1}^{\infty} \gamma_{2,k} = \infty$, and $\sum_{k=1}^{\infty} \gamma_{2,k}^2 < \infty$, where u is defined as in Theorem 9, then $z^k \to z^* = (x^*, b^*)$ as $k \to \infty$, where x^* is the solution of VI (1).

Proof: Since $\gamma_{2,k}$ converges to zero as $k \to \infty$, we have for sufficient large k that $|1 - \gamma_{2,k} \sum_{i=1}^N x_i^k| = (1 - \gamma_{2,k} \sum_{i=1}^N x_i^k) \leq 1 - \gamma_{2,k}\epsilon$,

$$\text{and thus } |b^{k+1} - b^*| \leq |b^k - b^* - \gamma_{2,k}(F_b(z^k) - F_b(z^*))|$$
$$= |1 - \gamma_{2,k} \sum_{i=1}^N x_i^k||b^k - b^*| \leq (1 - \gamma_{2,k}\epsilon)|b^k - b^*|. \tag{15}$$

Noting that $\sum_{k=1}^{\infty}[1 - (1 - \gamma_{2,k}\epsilon)] = \epsilon \sum_{k=1}^{\infty} \gamma_{2,k} = \infty$, we have b^k converges to b^* as $k \to \infty$. The convergence of $\{b^k\}$ implies the boundedness of $\{b^k\}$. That is, for sufficiently large k, $b^*/2 \leq b^k \leq 3b^*/2$. Similarly as in the proof of Theorem 9, we have for sufficiently large k that

$$\|x^{k+1} - x^*\| \leq \alpha_k \|x^k - x^*\| + \gamma_{1,k} u |b^k - b^*|, \tag{16}$$

where $\alpha_k = \sqrt{1 + \gamma_{1,k}^2 L^2 - 2\mu\gamma_{1,k}}$ with $\mu = b^*/2$ and $L^2 = (12M^2 + 27(b^*)^2 + 27N^2(b^*)^2)/4$. Combining (16) and (15), we get

$$\begin{aligned}&\leq \alpha_k\|x^k - x^*\| + \gamma_{1,k} u|b^k - b^*| + (1-\gamma_{2,k}\epsilon)|b^k - b^*|\\&= \alpha_k\|x^k - x^*\| + (1+\gamma_{1,k}u - \gamma_{2,k}\epsilon)|b^k - b^*|\\&\leq \alpha_k\|x^k - x^*\| + (1-\gamma_{1,k}u)|b^k - b^*|\\&\leq q_k(\|x^k - x^*\| + |b^k - b^*|),\end{aligned}$$

where $q_k = \max\{\alpha_k, 1-\gamma_{1,k}u\}$. Note that $\gamma_{1,k}$ converging to 0 implies that there exists some positive integer n such that $\gamma_{1,k} < 2\mu/L^2$ when $k \geq n$. Thus $\sum_{k=1}^{\infty}(1-\alpha_k)$ is

$$\begin{aligned}&= \sum_{k=1}^{n-1}(1-\alpha_k) + \sum_{k=n}^{\infty} \frac{2\mu\gamma_{1,k} - \gamma_{1,k}^2 L^2}{1+\sqrt{1+\gamma_{1,k}^2 L^2 - 2\mu\gamma_{1,k}}}\\&\geq \sum_{k=1}^{n-1}(1-\alpha_k) + \sum_{k=n}^{\infty} \frac{2\mu\gamma_{1,k} - \gamma_{1,k}^2 L^2}{2}\\&= \sum_{k=1}^{n-1}(1-\alpha_k) + \mu\sum_{k=n}^{\infty}\gamma_{1,k} - \frac{L^2}{2}\sum_{k=n}^{\infty}\gamma_{1,k}^2 = \infty,\end{aligned}$$

and $\sum_{k=1}^{\infty}[1-(1-\gamma_{1,k}u)] = u\sum_{k=1}^{\infty}\gamma_{1,k} = \infty$, which implies $\sum_{k=1}^{\infty}(1-q_k) = \infty$. Therefore, $x^k \to x^*$, $b^k \to b^*$ as $k \to \infty$, and thus z^k converges to z^* as $k \to \infty$. ■

VI. Numerical Results

In this section, we provide numerics for learning b^* by using the standard Tikhonov and distributed schemes. For the standard Tikhonov scheme, let $c_i(x_i) = r_i x_i^2 + g_i x_i + h_i$, where r_i and g_i are randomly chosen from the uniform distributions $U[1,10]$ and $U[1,20]$, respectively. Furthermore, $a^* = 100$. When employing the distributed scheme, $r_i = 0$. Also, let Cap_i be chosen from $U[2,20]$ and x_{ij}^0 be randomly chosen from $U[0,10]$ for all i,j. Let b_i^0 be randomly chosen from $U[1,10]$ for learning b^*. In all three instances, the schemes compute the *correct equilibrium*.

		Standard Tikhonov		
N	b^*	No. of iterations	$\|y^* - x^{**}\|$	$\|b - b^* e\|$
5	1	8575	1.2×10^{-4}	0.2×10^{-4}
5	2	5115	0.6×10^{-4}	0.8×10^{-4}
5	3	3988	0.4×10^{-4}	2.1×10^{-4}
10	1	21020	0.9×10^{-4}	0.1×10^{-4}
10	2	10548	0.6×10^{-4}	0.5×10^{-4}
10	3	6012	0.8×10^{-4}	1.6×10^{-4}

For the standard Tikhonov scheme, let $\gamma = 0.01, \epsilon_k = 1/k$ and let y^* denote the solution of original problem and $x^{**} = (x_{ii}^*)_{i=1}^N$ is the diagonal solution of VI (7). For distributed scheme with constant steplength, let $\epsilon = 0.1$, $\underline{b} = 0.2$, $\bar{b} = 10$, $\gamma_{1,k} = 10^{-5}$ and $\gamma_{2,k} = 10^{-2}$.

		Distributed Scheme		
N	b^*	No. of iterations	$\|y^* - x^{**}\|$	$\|b - b^* e\|$
5	1	50146	0	0
5	2	397550	2.1×10^{-3}	0
5	3	278304	1.5×10^{-3}	0
10	1	64803	0	0
10	2	422531	1.3×10^{-3}	0
10	3	292267	9.1×10^{-4}	0

For distributed scheme with diminishing steplength, let $\epsilon = 0.1$. Let $\gamma_{1,k} = 1/k$ and $\gamma_{2,k} = 1000/k$.

		Distributed Scheme		
N	b^*	No. of iterations	$\|y^* - x^{**}\|$	$\|b - b^* e\|$
5	1	44988	0	0
5	2	184800	3.7×10^{-3}	0
5	3	74511	8.2×10^{-4}	0
10	1	55713	0	0
10	2	267447	3.8×10^{-3}	0
10	3	96972	7.2×10^{-4}	0

The termination criteria of the schemes is prescribed as $\|z^k - z^{k-1}\| \leq 10^{-7}$ where $z_k = (x_k, b_k)$.

VII. Concluding Remarks

We have presented a learning framework for computing equilibria in constrained Nash-Cournot games. A characterization of the learning process is provided and the convergence of the associated Tikhonov-based regularization scheme is proven. Furthermore, under a suitable requirement on starting points, we also prove the convergence of constant and diminishing steplength distributed schemes. We contend that the question of computing solutions to optimization/game-theoretic problems while learning unobservable parameters is indeed an important one. The current work represents a first step towards developing such algorithmic schemes.

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