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Sliding Mode Block Control Regulation of the Pendubot

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***Abstract*— In this work, the problem of nonlinear regulation of an underactuated system is treated by means of sliding mode continuous control actions combined with block control technique. The sliding mode state feedback output regulator based on the super twisting algorithm, is applied to the Pendubot system. The transformation of the original system to regular form and then block control technique are used to design a sliding manifold with asymptotically stable motion.To verify the effectiveness of the method, simulations are carried out.**

I. INTRODUCTION

The system Pendubot is an eletromechanical system whose name comes from short for pendulum robot. The device consists of two planar, rigid and rotational links of two degrees of freedom. The Pendubot has an actuator localized on the shoulder (first joint) controlled by mean of a DC motor, while the second joint moves freely along the first joint movement given by the coupling between both joints. The main purpose of the pendulum is research and education within the nonlinear systems framework. With the Pendubot, concepts like nonlinear dynamics, linearization, robotics and control systems design, can be achieved. Some of the principal control problems for the Pendubot are swing-up, stabilization and tracking.

In [1], a swing up and balancing control via model orbit stabilization synthesis is proposed, a modified Van der Pol oscillator is involved into a quasi-homogeneous synthesis as a reference model combined with partial feedback linearization and the so-called homogeneous twisting controller to locally stabilize the Pendubot about the vertical.

In this work we will be centered in the problem of the second link tracking of a sinusoidal shape signal. The trajectory tracking problem plays a central role in output regulation theory [2]. Therefore, it is of great interest the design of output regulators for the Pendubot. In [3], it is presented an approach to achieve trajectory tracking for nonlinear systems, combining linear regulator theory with the Takagi-Sugeno fuzzy methodology, and a real time application of the Pendubot is discussed. In [4], the application of error feedback sliding mode (SM) output regulation technique is applied to the Pendubot, stabilizing the sliding mode equation by means of Jacobian linearization of the *complete* SM dynamics.

In this work, the Pendubot system model is first transformed into regular form such that the SM equation is not affected directly by the control input. Then the Block Control (BC) linearization technique [5] is applied to introduce desired dynamics for a part of the SM equation. The rest SM dynamics part, which coincides with zero dynamics, is linearized via Jacobian linearization. A sliding manifold then is formulated such that the equilibrium point of the complete SM dynamics is locally asymptotically stable. To ensure the designed sliding manifold be stable and eliminate the chattering effect, the continuous SM super-twisting control algorithm [6], is implemented. The effectiveness of the proposed control scheme is verified via simulations. Note the SM regulator problem has been considered in [7] for a class of nonlinear systems with relative degree one only while the control scheme proposed in the present paper can be implemented for systems with arbitrary relative degree.

An important contribution of this work is to provide an algorithm, simpler than the classical regulator theory (based on the full information regulator since most of electromechanical rotational systems include encoders to sense position and velocity of its joints), to permit increase the region of attraction around the desired equilibrium of the Pendubot, then perform a tracking reference signal about its vertically upright position, using a continuous nonlinear controller (to reduce the non desirable chattering issue) and reducing the linearization error making use of the partial BC linearization technique.

II. PENDUBOT MODEL

The Pendubot system consist of two links, the first has a motor, and the second is like a simple pendulum. Consider the system shown in Figure 1, where the generalized coordinates needed to describe the motion of the system are the angular displacement of the first and second link, q_1 and q_2, respectively. The motion equation can be described by the general equation [8]

$$\mathbf{D}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{C}(\mathbf{q},\dot{\mathbf{q}}) + \mathbf{G}(\mathbf{q}) + \mathbf{F}(\dot{\mathbf{q}}) = \tau,$$

where $\mathbf{q} = (\mathbf{q}_1, \mathbf{q}_2^T) \in \mathbb{R}^n$, $\mathbf{q}_1 \in \mathbb{R}^m$ represents the actuated joints, and $\mathbf{q}_2 \in \mathbb{R}^{n-m}$ represents the unactuated ones. $\mathbf{D}(\mathbf{q})$ is the $n \times n$ inertia matrix, $\mathbf{C}(\mathbf{q},\dot{\mathbf{q}})$ is the vector of Coriolis and centripetal torques, $\mathbf{G}(\mathbf{q})$ contains the gravitational terms, $\mathbf{F}(\dot{\mathbf{q}})$ is the vector of viscous frictional terms, and τ is the vector of input torques. For the Pendubot system, the dynamic model is particularized as

$$\begin{pmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{pmatrix}\begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 \end{pmatrix} + \begin{pmatrix} C_1 \\ C_2 \end{pmatrix} + \begin{pmatrix} G_1 \\ G_2 \end{pmatrix} + \begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = \begin{pmatrix} \tau_1 \\ 0 \end{pmatrix},$$

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where $D_{11}(q_2) = m_1 l_{c1}^2 + m_2\left(l_1^2 + l_{c2}^2 + 2l_1 l_{c2}\cos q_2\right) + I_1 + I_2$, $D_{12}(q_2) = D_{21}(q_2) = m_2\left(l_{c2}^2 + l_1 l_{c2}\cos q_2\right) + I_2$, $D_{22} = m_2 l_{c2}^2 + I_2$, $C_1(q_2, \dot{q}_1, \dot{q}_2) = -2m_2 l_1 l_{c2}\dot{q}_1\dot{q}_2 \sin q_2 - m_2 l_1 l_{c2}\dot{q}_2^2 \sin q_2$, $C_2(q_2, \dot{q}_1) = m_2 l_1 l_{c2}\dot{q}_1^2 \sin q_2$, $G_1(q_1, q_2) = m_1 g l_{c1}\cos q_1 + m_2 g l_1 \cos q_1 + m_2 g l_{c2}\cos(q_1 + q_2)$, $G_2(q_1, q_2) = m_2 g l_{c2}\cos(q_1 + q_2)$, $F_1(\dot{q}_1) = \mu_1\dot{q}_1$, $F_2(\dot{q}_2) = \mu_2\dot{q}_2$,

with m_1 and m_2 as the mass of the first and second link of the Pendubot respectively, l_1 and l_2 are the length of the first and second link respectively, l_{c1} and l_{c2} are the distance to the center of mass of link one and two respectively, g is the acceleration of gravity, I_1 and I_2 are the moment of inertia of the first and second link respectively about its centroids, and μ_1 and μ_2 are the viscous drag coefficients.

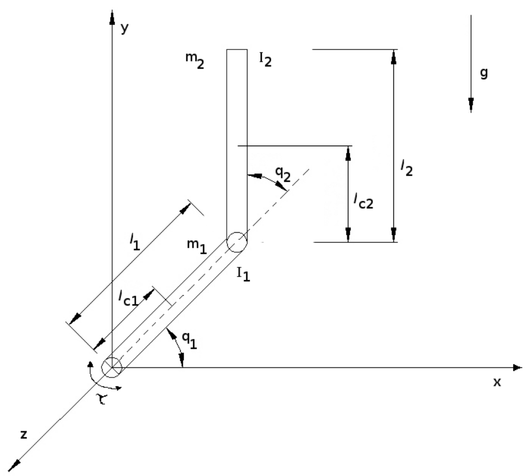

Fig. 1: Schematic diagram of the Pendubot

Choosing variable $\mathbf{x} = \begin{pmatrix} x_1 & x_2 & x_3 & x_4 \end{pmatrix}^T = \begin{pmatrix} q_1 & q_2 & \dot{q}_1 & \dot{q}_2 \end{pmatrix}^T$ as the state vector, $u = \tau_1$ as the control input, and x_2 as the output, the description of the system can be given in state space form as

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})u, \tag{1}$$

$$y = h(\mathbf{x}), \tag{2}$$

where $h(\mathbf{x}) = x_2$ is the system output,

$$\mathbf{f}(\mathbf{x}) = \begin{pmatrix} f_1(\mathbf{x}) \\ f_2(\mathbf{x}) \\ f_3(\mathbf{x}) \\ f_4(\mathbf{x}) \end{pmatrix} = \begin{pmatrix} x_3 \\ x_4 \\ g_3(x_2)p_3(\mathbf{x}) \\ g_4(x_2)p_4(\mathbf{x}) \end{pmatrix},$$

$$\mathbf{g}(\mathbf{x}) = \begin{pmatrix} g_1 \\ g_2 \\ g_3(x_2) \\ g_4(x_2) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \dfrac{D_{22}}{D_{11}(x_2)D_{22} - D_{12}^2(x_2)} \\ \dfrac{-D_{12}(x_2)}{D_{11}(x_2)D_{22} - D_{12}^2(x_2)} \end{pmatrix},$$

$$p_3(\mathbf{x}) = \frac{D_{12}(x_2)}{D_{22}}\left(C_2(x_2, x_3) + G_2(x_1, x_2) + F_2(x_4)\right) - C_1(x_2, x_3, x_4) - G_1(x_1, x_2) - F_1(x_3),$$

$$p_4(\mathbf{x}) = \frac{D_{11}(x_2)}{D_{12}}\left(C_2(x_2, x_3) + G_2(x_1, x_2) + F_2(x_4)\right) - C_1(x_2, x_3, x_4) - G_1(x_1, x_2) - F_1(x_3).$$

And, in a admissible region Ω: $(-\frac{\pi}{2} \leq x_2 \leq \frac{\pi}{2})$

$$|g_4(x_2)| > 0 \tag{3}$$

III. STATE FEEDBACK SLIDING MODE REGULATOR PROBLEM

Consider the nonlinear system (1), with state $\mathbf{x}$, defined on a neighborhood X of the origin of $\mathbb{R}^4$, and $u \in \mathbb{R}^1$, $y \in \mathbb{R}^1$. The vector $\mathbf{f}(\mathbf{x})$ and the columns of $\mathbf{g}(\mathbf{x})$ are smooth vector fields of class $C^{\infty}_{[t,\infty)}$, and in addition, is assumed that $\mathbf{f}(0) = 0$ and $\mathbf{h}(0) = 0$. The output tracking error $\mathbf{e}(\mathbf{x}, \omega)$ is defined as the difference between the output of the system, y, and a reference signal, $q(\omega)$, i.e.

$$e_2 = x_2 - r(\omega), \tag{4}$$

where the reference signal, $r(\omega)$, is generated by a given exosystem described by

$$\dot{\omega} = \mathbf{s}(\omega), \ \mathbf{s}(0) = 0, \tag{5}$$

with $\omega = (\omega_1, \omega_2)^T$, and ω_2 as the reference signal generated by the known exosystem (5), with state ω, defined on a neighborhood W of the origin of $\mathbb{R}^2$. So, $r(\omega) = \omega_2$. The system 5 is characterized by the following assumption:

H1. *The Jacobian matrix* $\mathbf{S} = \left(\frac{\partial \mathbf{s}}{\partial \omega}\right)_{(0)}$ *at the equilibrium point* $\omega = 0$ *has all eigenvalues on the imaginary axis*

Thus,

$$\mathbf{s}(\omega) = \begin{pmatrix} \alpha\omega_2 \\ -\alpha\omega_1 \end{pmatrix}, \ \alpha > 0.$$

Provided that all states of the system are available for measurement. In [2], it has been shown that the control action to system (1) can be provided by a smooth state feedback

$$u = \alpha(\mathbf{x}, \omega).$$

The solvability of the State Feedback Regulator Problem (SFRP), under assumption H1, can be stated in terms of the existence of a pair of mappings

$$\mathbf{x} = \pi(\omega) \text{ and } u = c(\omega), \tag{6}$$

with $\pi(0)=0$ and $c(0)=0$, both defined in a neighborhood $W^\circ \subset W$ of the origin, which solve the partial differential equation (Francis-Isidori-Byrnes equations)

$$\frac{\partial\pi}{\partial\omega}\mathbf{s}(\omega) = \mathbf{f}(\pi(\omega)) + \mathbf{g}(\pi(\omega)\,c(\omega)\,, \qquad 0 = \mathbf{h}(\pi(\omega),\omega)\,. \tag{7}$$

Analogously to SFRP, we consider the State Feedback Sliding Mode Regulation Problem (SFSMRP) which is defined as the problem of finding a sliding manifold

$$\sigma(\mathbf{x},\omega)=0, \quad \sigma(\mathbf{x}) = (\sigma_1,...,\sigma_m)^T\,, \tag{8}$$

and a SM controller, in this case applying the super twisting algorithm [6]

$$u(\mathbf{x}) = -M_1|\sigma|^{1/2}\,\text{sign}(\sigma) + u_1, \tag{9}$$

$$\dot{u}_1(\mathbf{x}) = -M_2\text{sign}(\sigma)\,,\ M_1>0,\ M_2>0, \tag{10}$$

such that the following conditions hold:

- (SMS_{sf}) (Sliding Mode Stability). The state of the closed-loop system (1)-(2), with the controller (9)-(10), converges to the manifold (8) in finite time;
- (S_{sf}). The equilibrium $\mathbf{x}=0$ of the sliding mode dynamics

$$\dot{\mathbf{x}} = \left[\mathbf{f}(\mathbf{x}) + \mathbf{g}(\mathbf{x})\,u_{eq}(\mathbf{x},\omega)\right]\big|_{\sigma(\xi)=0}\,,$$

is asymptotically stable, where u_{eq} is the equivalent control derived from condition $\dot\sigma=0$ [9];

- (R_{sf}). There exists a neighborhood $V\subset X\times W$ of $(0,0)$ such that, for each initial condition $(\mathbf{x}_0,\omega_0)\in V$, the output tracking error (4) goes asymptotically to zero,. i.e. $\lim_{t\to\infty}\mathbf{e}(t)=0$.

IV. CONTROL DESIGN

A. Nonlinear systems in Regular Form

To design a SM controller for the nonlinear system (1), it is more convenient to transform the system via a diffeomorphism

$$\mathbf{x}' = \begin{pmatrix}\varphi_1(\mathbf{x})\\ \varphi_2(\mathbf{x})\end{pmatrix},\ \varphi_1\in\mathbb{R}^3,\ \varphi_2\in\mathbb{R}^1, \tag{11}$$

into so-called regular form [10], which consists of two blocks. The first block not depend on control and the dimension of the second block coincides with the dimension of the control.

$$\dot{\mathbf{x}}_1' = \mathbf{f}_1(\mathbf{x}_1',\mathbf{x}_2')\,, \tag{12}$$

$$\dot{\mathbf{x}}_2' = \mathbf{f}_2(\mathbf{x}') + \mathbf{g}_2(\mathbf{x}')\,u, \tag{13}$$

where $\mathbf{x}'=(\mathbf{x}_1',\mathbf{x}_2')^T$, $\mathbf{x}_1'=(x_1',x_2',x_3')$, $\mathbf{x}_2'=(x_4')$ and $\det(\mathbf{g}_2(\mathbf{x}'))\neq 0$.

The sliding mode control design for systems in the so-called regular form $(12)-(13)$ becomes simpler, since the control input is not appeared in subsystem (12). As result, calculation of the equivalent control u_{eq} to define SM equation, is not needed. The transformation (11) can be obtained from the condition $\frac{\partial\varphi_1(\mathbf{x})}{\partial\mathbf{x}}\mathbf{g}(\mathbf{x})=0$, as follows

$$\begin{pmatrix}x_1'\\x_2'\\x_3'\end{pmatrix} = \varphi_1(\mathbf{x}) = \begin{pmatrix}x_1\\x_2\\x_3-[g_3(x_2)\,g_4^{-1}(x_2)]x_4\end{pmatrix}, \qquad x_4' = \varphi_2(\mathbf{x}) = x_4, \tag{14}$$

where $g_3(x_2)\,g_4^{-1}(x_2) = D_{22}D_{12}^{-1}(x_2')$, more details can be found in [9]. Using (14) and (1), the regular form $(12)-(13)$ becomes

$$\dot{\mathbf{x}}_1' = \begin{pmatrix}f_1(\mathbf{x}')\\f_2(\mathbf{x}')\\f_3(\mathbf{x}')\end{pmatrix}, \tag{15}$$

$$\dot{\mathbf{x}}_2' = f_4(\mathbf{x}') + g_4(x_2')\,u, \tag{16}$$

with

$$\mathbf{f}_1 = \begin{pmatrix} f_1(\mathbf{x}') = x_3' - D_{22}D_{12}^{-1}(x_2')\,x_4' \\ f_2(\mathbf{x}') = x_4' \\ f_3(\mathbf{x}') = F_{31}'\left(\frac{D_{12}(x_2')}{D_{22}} - \frac{D_{11}(x_2')}{D_{12}(x_2')}\right)F_{32}' \end{pmatrix},$$

$$\mathbf{f}_2 = f_4(\mathbf{x}') = g_4(x_2')\,p_4(x')\,,$$

and $F_{31}' = \frac{D_{22}}{D_{11}(x_2')D_{22}-D_{12}^2(x_2')}$, $F_{32}' = C_2(x_2', x_3' - D_{22}D_{12}^{-1}(x_2')\,x_4') + G_2(x_1',x_2') + F_2(x_4')\,.$

B. Nonlinear Block Control Linearization

In order to use the Block Control linearization, the system $(15)-(16)$ is represented as the Nonlinear Block Controllable (NBC)-form $\mathbf{S}_1$, consisting of two blocks, and the residual dynamics $\mathbf{S}_2$:

$$\mathbf{S}_1\begin{cases}\dot{x}_2' = x_4',\\ \dot{x}_4' = f_4(\mathbf{x}') + g_4(x_2')\,u,\end{cases} \tag{17}$$

$$\mathbf{S}_2\begin{cases}\dot{x}_1' = f_1(\mathbf{x}')\,,\\ \dot{x}_3' = f_3(\mathbf{x}')\,.\end{cases} \tag{18}$$

Introducing the zero output manifold $\pi'(\omega) = (\pi_1'(\omega),\pi_2'(\omega),\pi_3'(\omega),\pi_4'(\omega))^T$ and the control error

$$\begin{pmatrix}e_2\\e_4\\e_1\\e_3\end{pmatrix} = \begin{pmatrix}x_2'-\pi_2'(\omega)\\x_4'-\pi_4'(\omega)\\x_1'-\pi_1'(\omega)\\x_3'-\pi_3'(\omega)\end{pmatrix},$$

the system $(17)-(18)$ is represented as

$$\mathbf{S}_1\begin{cases}\dot{e}_2 = e_4 + \pi_4'(\omega) - \frac{\partial\pi_2'(\omega)}{\partial\omega}\mathbf{s}(\omega)\,,\\ \dot{e}_4 = f_4(\mathbf{x}') + g_4(x_2')\,u - \frac{\partial\pi_4'(\omega)}{\partial\omega}\mathbf{s}(\omega)\,,\end{cases} \tag{19}$$

$$\mathbf{S}_2\begin{cases}\dot{e}_1 = f_1(\mathbf{x}') - \frac{\partial\pi_1'(\omega)}{\partial\omega}\mathbf{s}(\omega)\,,\\ \dot{e}_3 = f_3(\mathbf{x}') - \frac{\partial\pi_3'(\omega)}{\partial\omega}\mathbf{s}(\omega)\,.\end{cases} \tag{20}$$

Now, $\pi'(\omega)$ will be calculated with respect to the system $(17)-(18)$, making use of its respective regulator equations

(7):

$$\begin{aligned}
\frac{\partial \pi_2'(\omega)}{\partial \omega}\mathbf{s}(\omega) &= \pi_4'(\omega), && (21)\\
\frac{\partial \pi_1'(\omega)}{\partial \omega}s(\omega) &= \pi_3'(\omega) && (22)\\
&\quad -D_{22}D_{12}^{-1}(\pi_2'(\omega))\,\pi_4'(\omega),\\
\frac{\partial \pi_3'(\omega)}{\partial \omega}\mathbf{s}(\omega) &= F_{31}'\frac{D_{12}(\pi_2'(\omega))}{D_{22}}F_{32}' && (23)\\
&\quad -\frac{D_{11}(\pi_2'(\omega))}{D_{12}(\pi_2'(\omega))}F_{32}',\\
0 &= \pi_2'(\omega)-\omega_2, && (24)
\end{aligned}$$

with

$$\begin{aligned}
F_{31}' &= \frac{D_{22}}{D_{11}(\pi_2'(\omega))D_{22}-D_{12}^2(\pi_2'(\omega))},\\
F_{32}' &= C_2((\pi_2'(\omega)),(\pi_3'(\omega)))\\
&\quad -D_{22}D_{12}^{-1}(\pi_2'(\omega))(\pi_4'(\omega)))\\
&\quad +G_2((\pi_1'(\omega)),(\pi_2'(\omega)))+F_2((\pi_4'(\omega))).
\end{aligned}$$

From equation (24), one directly obtains

$$\pi_2'(\omega)=\omega_2.$$

Then, replacing $\pi_2(\omega)$ in equation (21) yields to

$$\pi_4'(\omega)=-\alpha\omega_1.$$

For calculating $\pi_1'(\omega)$ and $\pi_3'(\omega)$, the solution of equations (22) and (23) is needed. This is in general a difficult task, and it is commonly can be solved proposing an approximated solution as in [4]. Thus, one proposes the following approximated solution for $\pi_1'(\omega)$:

$$\begin{aligned}
\pi_1'(\omega) &= a_0+a_1\omega_1+a_2\omega_2+a_3\omega_1^2+a_4\omega_1^2+a_4\omega_1\omega_2\\
&\quad +a_5\omega_2^2+a_6\omega_1^3+a_7\omega_1^2\omega_2+a_8\omega_1\omega_2^2+a_9\omega_2^3\\
&\quad +O\left(\|\omega\|_1^4\right), && (25)
\end{aligned}$$

where $O(\cdot)$ represents infinite number of terms resulting from approximation proposed. Replacing (25) in (22) yields the approximated solution for $\pi_3'(\omega)$

$$\begin{aligned}
\pi_3'(\omega) &= \alpha\left[a_1\omega_2-a_2\omega_1+2a_3\omega_1\omega_2+a_4\omega_2^2-a_4\omega_1^2\right.\\
&\quad -2a_5\omega_1\omega_2+3a_6\omega_1^2\omega_2+2a_7\omega_1\omega_2^2-a_7\omega_1^3\\
&\quad +a_8\omega_2^3-2a_8\omega_1^2\omega_2-3a_9\omega_1\omega_2^2\\
&\quad \left.-D_{22}D_{12}^{-1}(\omega_2)\,\omega_1\right]+O\left(\|\omega\|_1^4\right). && (26)
\end{aligned}$$

Performing a series Taylor expansion around the equilibrium point $\mathbf{x}_{ep}=\left(\frac{\pi}{2},0,0,0\right)^T$ as in [4] one can find the values $a_i\,(i=0,...9)$.

It is important to mention that there is a natural steady-state constraint for the Pendubot, as we can see in Figure 1, i.e., the sum of two angles q_1 and q_2 equals $\pi/2$

$$\frac{\pi}{2}=\pi_1'(\omega)+\pi_2'(\omega). \quad (27)$$

Using (27) one can obtain an alternate solution to steady-state, $\pi_{1a}'(\omega)=\pi/2-\pi_2'(\omega)$, and replacing $\pi_{1a}'(\omega)$ in equation (22) yields to $\pi_{3a}'(\omega)=\alpha\omega_1$, the subindex a refers an alternative zero output manifold.

C. Sliding Manifold Design

Under the condition (21) the NBC form (17) reduces to

$$\mathbf{S}_1\left\{\begin{array}{l}\dot{e}_2=e_4,\\ \dot{e}_4=f_4(\mathbf{x}')+g_4(x_2')\,u-\frac{\partial\pi_4'(\omega)}{\partial\omega}\mathbf{s}(\omega).\end{array}\right. \quad (28)$$

Following the BC design technique [5], we define first the tracking error as

$$z_2=e_2=x_2'-\pi_2'(\omega). \quad (29)$$

Then involving the desired dynamics for this error as $(-k_2z_2)$, the virtual control e_4 in the first block of (28) is chosen of the form

$$e_4=-k_2z_2+z_4,\ k_2>0, \quad (30)$$

where z_4 is a new variable.

From (30), variable z_4 can be obtained as

$$z_4=e_4+k_2z_2. \quad (31)$$

Using the new variables z_2 (29) and z_4 (31), the system $\mathbf{S}_1$ (28) is represented of the form

$$\mathbf{S}_1\left\{\begin{array}{l}\dot{z}_2=-k_2z_2+z_4,\\ \dot{z}_4=f_4(\mathbf{x}',\omega)+g_4(x_2')\,u,\end{array}\right. \quad (32)$$

where $f_4(\mathbf{x}',\omega)=f_4(\mathbf{x}')-\frac{\partial\pi_4'(\omega)}{\partial\omega}\mathbf{s}(\omega)$.

Now, if we apply directly Block Control

$$u=g_4^{-1}(x_2')\left[(f_4(\mathbf{x}',\omega)-k_4z_4\right],\ k_4>0,$$

or SM control

$$u=u_{eq}(\mathbf{x}',\omega)-k_4\text{sign}(s),$$

with the sliding variable $s=z_4$ and the equivalent control $u_{eq}(\mathbf{x}',\omega)=g_4^{-1}(x_2')\ (f_4(\mathbf{x}',\omega)$, then the system $\mathbf{S}_1$ (32) becomes as a linear system with desired eigenvalues $-k_2$ and $-k_4$,

$$\mathbf{S}_1\left\{\begin{array}{l}\dot{z}_2=-k_2z_2+z_4,\\ \dot{z}_4=-k_4z_4,\end{array}\right.$$

or discontinuous one,

$$\mathbf{S}_1\left\{\begin{array}{l}\dot{z}_2=-k_2z_2+z_4,\\ \dot{z}_4=-k_4g_4(x_2')\,\text{sign}(s),\end{array}\right.$$

respectively.

In both cases the tracking error $z_2(t)$ (29) asymptotically tends to zero providing the zero dynamics $\mathbf{S}_2$ (20) on the manifold $z_2=0$ and $z_4=0$ be unstable.

It can be noted that the zero dynamics, in this case, coincide with uncontrolled SM dynamics.

To ensure the SM dynamics be controllable that permits to stabilize the system $\mathbf{S}_2$, we define first

$$\begin{aligned}
z_1 &= e_1=x_1'-\pi_1'(\omega), && (33)\\
z_3 &= e_3=x_3'-\pi_3'(\omega). && (34)
\end{aligned}$$

Then, the system (32) and (18) is represented in the variables z_2 (29), z_1 (33) and z_3 (34) and z_4 (31) of the form

$$\begin{aligned} \dot{z}_2 &= -k_2 z_2 + z_4, \\ \dot{z}_1 &= f_1(\mathbf{z}, \pi'(\omega), \omega), \\ \dot{z}_3 &= f_3(\mathbf{z}, \pi'(\omega), \omega), \\ \dot{z}_4 &= f_4(\mathbf{z}, \pi'(\omega), \omega) + g_4(\mathbf{z}, \pi'(\omega)) u, \end{aligned} \tag{35}$$

where $\mathbf{z} = (z_2, z_1, z_3, z_4)^T$ and

$$\begin{aligned} f_1(\mathbf{z}, \pi'(\omega), \omega) &= f_1(\mathbf{x}')|_{\mathbf{x}'=\mathbf{z}+\pi'(\omega)} - \frac{\partial \pi_1'(\omega)}{\partial \omega}\mathbf{s}(\omega), \\ f_3(\mathbf{z}, \pi'(\omega), \omega) &= f_3(\mathbf{x}')|_{\mathbf{x}'=\mathbf{z}+\pi'(\omega)} - \frac{\partial \pi_3'(\omega)}{\partial \omega}\mathbf{s}(\omega), \\ f_4(\mathbf{z}, \pi'(\omega), \omega) &= f_4(\mathbf{x}', \omega)|_{\mathbf{x}'=\mathbf{z}+\pi'(\omega)}, \\ g_4(\mathbf{z}, \pi'(\omega)) &= g_4(\mathbf{x}')|_{\mathbf{x}'=\mathbf{z}+\pi'(\omega)}. \end{aligned}$$

Now, we formulate the sliding variable σ (8) as

$$\sigma = z_4 + \mathbf{\Sigma} \begin{pmatrix} z_1 & z_3 \end{pmatrix}^T, \ \ \mathbf{\Sigma} = \begin{pmatrix} k_1 & k_3 \end{pmatrix}, \tag{36}$$

with $k_1 > 0$ and $k_3 > 0$.

Using (36) and (35), the projection motion on the subspace σ can be written as

$$\dot{\sigma} = f_\sigma(\mathbf{z}, \pi'(\omega), \omega) + g_4(\mathbf{z}, \pi'(\omega), \omega) u, \tag{37}$$

where $f_\sigma(\cdot) = f_4(\mathbf{z}, \pi'(\omega), \omega) + k_1 f_1(\mathbf{z}, \pi'(\omega), \omega) + k_3 f_3(\mathbf{z}, \pi'(\omega), \omega)$.

To enforce SM chattering-free motion on the manifold $\sigma = 0$ (36), we apply the super twisting algorithm (9)-(10). Thus, the closed-loop system (37) and (9) becomes of the following form:

$$\begin{aligned} \dot{\sigma} &= f_\sigma - M_1 g_4(\mathbf{z}, \pi'(\omega), \omega) |\sigma|^{1/2} sign(\sigma) \\ &\quad + g_4(\mathbf{z}, \pi'(\omega), \omega) u_1, \\ \dot{u}_1 &= -M_2 sign(\sigma). \end{aligned}$$

In the admissible region Ω

$$|g_4(x_2)| \leq \delta_0,$$

if the controller gains M_1 and M_2 satisfy the following conditions:

$$M_1 > \frac{2\delta}{\delta_0},$$

and

$$M_2 > M_1 \delta_0 \frac{5\delta M_1 \delta_0 + 4\delta^2}{2(M_1 - 2\delta)},$$

then, the closed-loop system state converges to the manifold (36) in finite time [11], ensuring the condition SMS_{sf} is hold.

The sliding mode motion on the manifold $\sigma = 0$ is governed by the following reduced order system:

$$\begin{aligned} \dot{z}_2 &= -k_2 z_2 + z_4, \\ \dot{z}_1 &= f_1(\mathbf{z}, \pi'(\omega), \omega), \\ \dot{z}_3 &= f_3(\mathbf{z}, \pi'(\omega), \omega), \end{aligned} \tag{38}$$

where the desired stabilized dynamics $(-k_2 z_2)$ for the tracking error z_2 was already introduced by the BC design.

To achieve stability of the controlled now sliding mode dynamics (38), where z_4 is considered as the virtual control input, using the linearization of (38) at $\mathbf{x}_{eq} = \left(\frac{\pi}{2}, 0, 0, 0\right)^T$ and $\pi'(\omega) = 0$; the system (38) is represented as:

$$\begin{bmatrix} \dot{z}_2 \\ \dot{z}_1 \\ \dot{z}_3 \end{bmatrix} = \begin{bmatrix} -k_2 & 0 & 0 \\ A_{11} & A_{12} & A_{13} \\ A_{31} & A_{32} & A_{33} \end{bmatrix} \begin{bmatrix} z_2 \\ z_1 \\ z_3 \end{bmatrix} + \begin{bmatrix} 1 \\ b_1 \\ b_3 \end{bmatrix} z_4 + \begin{bmatrix} 0 \\ \phi_1(\mathbf{z}, \pi'(\omega), \omega) \\ \phi_3(\mathbf{z}, \pi'(\omega), \omega) \end{bmatrix}, \tag{39}$$

where

$$\begin{aligned} A_{ij} &= \left[\frac{\partial f_i}{\partial z_j}\right]_{\mathbf{x}=\mathbf{x}_{eq},\ \pi'(\omega)=0}, \quad b_i = \left[\frac{\partial f_i}{\partial z_4}\right]_{\mathbf{x}=\mathbf{x}_{eq},\ \pi'(\omega)=0}, \\ i &= 1, 3, \quad j = 1, 2, 3, \end{aligned}$$

ϕ_2 and ϕ_3 are the residual high order nonlinear terms resulting from linear approximation over (38).

On the sliding manifold $\sigma = 0$ (8), we have

$$z_4 = -k_1 z_1 - k_3 z_3. \tag{40}$$

Now, substituting (40) into (39), the SM motion on this manifold is governed by following linear perturbed system:

$$\begin{bmatrix} \dot{z}_2 \\ \dot{z}_1 \\ \dot{z}_3 \end{bmatrix} = \mathbf{A} \begin{bmatrix} z_2 \\ z_1 \\ z_3 \end{bmatrix} + \begin{bmatrix} 0 \\ \phi_1(\mathbf{z}, \pi'(\omega), \omega) \\ \phi_3(\mathbf{z}, \pi'(\omega), \omega) \end{bmatrix}, \tag{41}$$

with

$$\mathbf{A} = \begin{bmatrix} -k_2 & -k_1 & -k_3 \\ A_{11} & A_{12} - k_1 b_1 & A_{13} - k_3 b_3 \\ A_{31} & A_{32} - k_1 b_1 & A_{33} - k_3 b_3 \end{bmatrix}. \tag{42}$$

Choosing adequate values of k_1, k_2 and k_3, we ensure the matrix $\mathbf{A}$ (42) be Hurwitz. Now, considering that the perturbation terms $\phi_1(\mathbf{z}, \pi'(\omega), \omega)$ and $\phi_3(\mathbf{z}, \pi'(\omega), \omega)$ are continuous and therefore bounded in the admissible compact set, then the equilibrium point $z_i = 0$, $i = 1, 2, 3$ of the system (41) is locally asymptotically stable provided that the condition S_{sf} is also hold. Thus, by continuity, the output tracking error (4) converges to zero, and condition (R_{sf}) holds too.

It is important to note that using combination of the BC and SM control techniques permits to reduce the partial differential equation (7) to the reduced order equation (21) - (23). Moreover, the calculation of the steady state control $c(\omega)$ (6), in this case, is not needed.

V. SIMULATIONS

In order to show the performance of the sliding mode regulator, simulations are carried out. The initial condition for the Pendubot is chosen near the equilibrium point as follows: $x_1(0) = 1.3$, $x_2(0) = 0.2$. The constant $\alpha = 0.3$,, the signal reference amplitude of 1 radian (approximately 57.3 degrees) and $M_1 = -1$, $M_2 = -1.5$.The values of the parameters are taken as follows: $m_1 = 0.829$ Kg,

$m_2 = 0.340$ Kg, $l_1 = 0.203$ m, $l_2 = 0.384$ m,$l_{c1} = 0.155$ m, $l_{c2} = 0.164$ m, $g = 9.81$ m/sec^2, $I_1 = 0.00545$ Kg-m^2,$I_2 = 0.00047$ Kg-m^2. And the coefficients a_i are calculated as follows $a_0 = 1.5708$, $a_1 = -2.5675 \times 10^{-4}$, $a_2 = -0.99828$, $a_3 = 9.0455 \times 10^{-7}$, $a_4 = 1.7821 \times 10^{-8}$, $a_5 = 1.128 \times 10^{-4}$, $a_6 = -1.9783 \times 10^{-9}$, $a_7 = 2.7605 \times 10^{-3}$, $a_8 = -2.4734 \times 10^{-7}$, $a_9 = -8.9727 \times 10^{-4}$.

And $k_1 = 45.131$, $k_2 = 39.517$, $k_3 = 9.5627$ in order to place the sliding mode equation (39) poles at $(-5, -5, -5)$. The constrain (27) was used too for simulation yielding the same results when using the approximated manifold (25) and (26).

VI. CONCLUSIONS

In this work, the State Feedback Sliding Mode Output Regulation Problem via the super-twisting control algorithm has been addressed. The SM control allows straightforward solution to be obtained, i.e., the FIB equation is reduced and the steady-state control needs not to be calculated, simplifying the control design, specially when compared to the classical solutions of the state feedback regulator problem. Additionally, the SM based controller achieves robustness with respect to allowed uncertainties, combining the super twisting algorithm provides the chattering-free motion. The proposed transformation to regular form permits to simplify the deriving of SM equation, since, in this case, calculation of the equivalent control is not needed. Applying the nonlinear block control linearization techniques permits to assign desired stable dynamics for a part of the SM equation while the rest reduced part is stabilized via Jacobian linearization.

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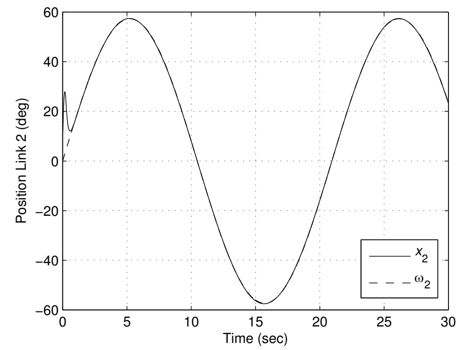
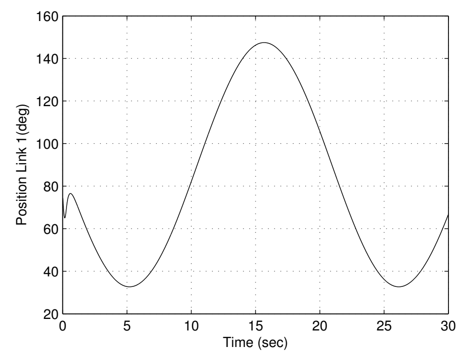


(a)



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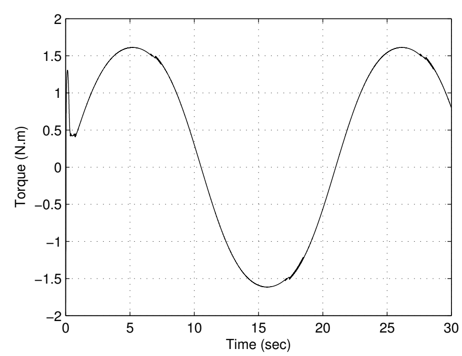

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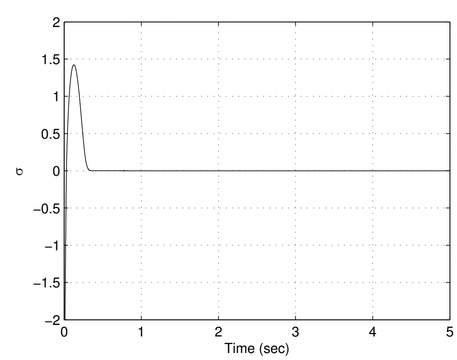

(d)

Fig. 2: a) Comparison of the output reference signal versus the output of the Pendubot controlled by the sliding mode regulator. b) The position angle of the first link. c) The sliding mode control signal. d) The sliding manifold σ.