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Synthesis of low-complexity stabilizing piecewise affine controllers: A control-Lyapunov function approach

Liang Lu   W. P. M. H. Heemels   Alberto Bemporad

Abstract— **Explicit model predictive controllers computed exactly by multi-parametric optimization techniques often lead to piecewise affine (PWA) state feedback controllers with highly complex and irregular partitionings of the feasible set. In many cases complexity prohibits the implementation of the resulting MPC control law for fast or large-scale system. This paper presents a new approach to synthesize low-complexity PWA controllers on regular partitionings that enhance fast on-line implementation with low memory requirements. Based on a PWA control-Lyapunov function, which can be obtained as the optimal cost for a constrained linear system corresponding to a stabilizing MPC setup, the synthesis procedure for the low-complexity control law boils down to local linear programming (LP) feasibility problems, which guarantee stability, constraint satisfaction, and certain performance requirements. Initially, the PWA controllers are computed on a fixed regular partitioning. However, we also present an automatic refinement procedure to refine the partitioning where necessary in order to satisfy the design specifications. A numerical example show the effectiveness of the novel approach.**

I. Introduction

Piecewise affine (PWA) controllers form a popular and powerful state feedback solution for constrained linear systems and piecewise affine systems. One particularly interesting approach to synthesize PWA controllers is model predictive control (MPC). Indeed, as is well known by now, in case of constrained linear or piecewise affine systems and adopting a PWA performance index, the explicit state feedback laws that are obtained by solving a multi-parametric (mixed-integer) linear programming (mp-(MI)LP) problem result in PWA functions [1]. Due to these results the on-line implementation of the optimal MPC law reduces to a point location problem within a polyhedral partitioning and evaluating the affine control law corresponding to the located region at each sampling time, instead of solving an online optimization problem, as is customary in implicit MPC.

A drawback of the explicit MPC law is that often the number of regions in the partitioning of the feasible set becomes extremely large and that the regions have irregular polyhedral shapes. These two issues might prohibit the use of the explicit MPC law for high sampling rate applications and/or large-scale systems. Additionally, also the storage requirement for the control and region parameters may be considered too large for the device memory of the control unit.

For these reasons, there is recently a growing interest in obtaining suboptimal MPC controllers that result in less regions in the partitioning of the feasible set and in regions of more regular shape (e.g. hypercubes, regular simplices, etc.). The basic objective is to reduce the complexity of the control implementation such that explicit MPC can also be applied to fast and/or large-scale systems [2]–[10].

Many of the existing low-complexity control design methods [2]–[9] are based on first computing the optimal control law and then approximating it by a suboptimal control law in a simple or canonical PWA form, possibly preserving some properties such as constraint satisfaction, stability, and some degree of performance. Canonical PWA functions based on simplicial or hypercubic partitionings are of particular interest as the point location problem can then be efficiently solved. In particular, [5] proposes a mp-QP approximation procedure on quadratic cost based MPC for linear systems. The procedure checks the feasibility of a hierarchical hyper-cubic structure on the vertices. By approximating a stabilizing MPC control law with a properly specified error bound between the approximate cost and the optimal cost, the method in [5] can guarantee the properties of asymptotic stability and constraint satisfaction. In [6] an adaptive multiscale approximation method is proposed based on second order barycentric interpolation, which yields that the approximated control law can be expressed as the convex combination of the control values at the interpolated points. This property of interpolation leads to the evaluation of feasibility and stability conditions for each orthogonal region in the partitioning. One of the main features of the method of [6] is that it can provide stability and performance guarantees, and an automatic refinement procedure. Note that, this approach searches the low-complexity control law in the space of continuous PWA functions due to the fact that interpolation techniques of continuous PWA functions is used. In [7], the result of [6] was extended to nonlinear model predictive control (NMPC). An alternative method was proposed in [8], where an approximation procedure for MPC controllers for constrained linear systems is presented using canonical PWA functions based on regular simplices, with guarantees of local optimality and constraint satisfaction. The method is based on constructing the PWA function with a fixed partitioning that minimizes the approximation

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Liang Lu is currently with the State Key Laboratory of Synthetical Automation for Process Industries, Northeastern University, Shenyang, China, and was with the Department of Mechanical Engineering, Eindhoven University of Technology, the Netherlands liangup@gmail.com

Maurice Heemels is with the Department of Mechanical Engineering, Eindhoven University of Technology, the Netherlands W.P.M.H.Heemels@tue.nl

Alberto Bemporad is with the IMT Institute for Advanced Studies Lucca, Italy alberto.bemporad@imtlucca.it

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error with respect to the explicit optimal MPC law in terms of an $\mathcal{L}_2/\mathcal{L}_\infty$ norm. This method does not provide *a priori* guarantees of stability, which instead can only be verified *a posteriori* via an LP problem by synthesizing (possibly discontinuous) PWA Lyapunov functions. In case the test fails the approximated PWA control law should be redefined somehow. In [9], an approximation method was presented building upon the input-to-state stability (ISS) framework to express the effect of the approximation error on the closed-loop dynamics. Using this property, which often can be inherited from nominal stability, a systematic method to derive an approximate PWA control law with *a priori* guarantees on stability and constraint satisfaction is presented for (open-loop) PWA systems the method does not require any convexity properties of the optimal costs or controller. In [10], instead of directly approximating the optimal control law, the optimal cost function is first approximated with polyhedral cost function using the double-description method. Then a suboptimal controller is obtained corresponding to this approximated polytopic cost function.

In this paper, we present an alternative method, which is a more direct design method, as it does not approximate the optimal MPC control law. Instead, the method is based on the existence of a PWA control-Lyapunov function (CLF) [11]. Such a CLF can, for instance, be obtained as the optimal MPC costs (in case of a linear performance index based on 1 or ∞ norms) corresponding to a stabilizing MPC law for a constrained linear system. Using such a PWA CLF, we provide an LP-based computational method for determining low-complexity controllers using a partitioning that consists of regularly shaped regions, such as orthogonal simplices as in [8] or hypercubes as in [5]. In essence, any desirable polytopic form of the regular regions can be used in our method. As we also provide an automated refinement procedure to refine the regular partitioning, we are actually synthesizing both suitable control gains and a suitable (multi-resolution) partitioning such that constraint satisfaction and stability are guaranteed.

The paper is organized as follows. In Section II, we will introduce some basic notations and definitions that will be used throughout the paper. In Section III, we will formulate the problem that will be studied in this paper, while Section IV will establish the LP problem that has to be solved to construct the low-complexity control law with *a priori* stability and constraint satisfaction guarantees. After that, we will set up a refinement procedure to automatically adjust the partitioning where needed, and will provide a corresponding numerical algorithm for this approach. A numerical example is presented in Section V to demonstrate the effectiveness of the design method in Section IV. Finally, Section VI gives the conclusion of this paper.

II. PRELIMINARIES

For a set $\mathcal{P} \subseteq \mathbb{R}^n$, we denote its interior by int$(\mathcal{P})$. A set $\mathcal{P}$ is called a *polyhedral set* or *polyhedron* if it can be written as the intersection of a finite number of half-spaces. A compact polyhedron is called a *polytope*. We denote the extreme points (vertices) of a polytope $\mathcal{P}$ as extr$(\mathcal{P})$= $\{v_1, \ldots, v_m\}$, which is the minimal set of elements in $\mathcal{P}$ such that co$\{v_1, \ldots, v_m\} = \mathcal{P}$. A set of polytopes $\tilde{\mathcal{P}} = \{\tilde{\mathcal{P}}_i \mid i \in \mathcal{F}_{\tilde{\mathcal{P}}}\}$, where $\mathcal{F}_{\tilde{\mathcal{P}}} \subset \mathbb{Z}_{\geq 1}$ is a finite set of indices, is called a *polytopic partitioning* of a polytope $\mathcal{P}$ if $\cup_i \tilde{\mathcal{P}}_i = \mathcal{P}$ and int$(\tilde{\mathcal{P}}_i) \cap$ int$(\tilde{\mathcal{P}}_j) = \emptyset, \forall i, j \in \mathcal{F}_{\tilde{\mathcal{P}}}, i \neq j$. For a collection $\tilde{\mathcal{P}} = \{\tilde{\mathcal{P}}_i \mid i \in \mathcal{F}_{\tilde{\mathcal{P}}}\}$ of sets with $\tilde{\mathcal{P}}_i \subseteq \mathbb{R}^n$, $i \in \mathcal{F}_{\tilde{\mathcal{P}}}$, and another set $\mathcal{Q} \subseteq \mathbb{R}^n$, the index set $\mathcal{I}(\mathcal{Q}, \tilde{\mathcal{P}})$ is given by

$$\mathcal{I}(\mathcal{Q}, \tilde{\mathcal{P}}) \triangleq \left\{ i \in \mathcal{F}_{\tilde{\mathcal{P}}} \mid \mathcal{Q} \cap \tilde{\mathcal{P}}_i \neq \emptyset \right\}. \tag{1}$$

A function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ *belongs to class* $\mathcal{K}$ *(*$\phi \in \mathcal{K}$*)* if it is continuous, strictly increasing and $\phi(0) = 0$. A function $\phi : \mathbb{R}_+ \to \mathbb{R}_+$ *belongs to class* $\mathcal{K}_\infty$ *(*$\phi \in \mathcal{K}_\infty$*)* if $\phi \in \mathcal{K}$ and $\lim_{s\to\infty} \phi(s) = \infty$. A function $\beta : \mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}_+$ *belongs to class* $\mathcal{KL}$ *(*$\beta \in \mathcal{KL}$*)* if for each fixed $t \in \mathbb{R}_+$, $\beta(\cdot, t) \in \mathcal{K}$ and for each fixed $s \in \mathbb{R}_+$, $\beta(s, \cdot)$ is non-increasing and $\lim_{t\to\infty} \beta(s, t) = 0$.

Consider the discrete-time dynamical system

$$x(t + 1) = \Phi(x(t)), \tag{2}$$

where $x(t) \in \mathbb{R}^n$ is the state at time $t \in \mathbb{Z}_+$ and $\Phi : D \to \mathbb{R}^n$ is a function defined on a domain $D \subset \mathbb{R}^n$. For convenience, we assume that the equilibrium point is at the origin of $\mathbb{R}^n$, i.e. $\Phi(0) = 0$, and $0 \in$int(D).

Definition 1: Let $\lambda \in \mathbb{R}_{[0,1]}$. A set $\mathcal{P} \subseteq D$ is called λ-*contractive* for system (2), if for all $x \in \mathcal{P}$, it holds that $\Phi(x) \in \lambda\mathcal{P}$. In case this property holds for $\lambda = 1$, we call $\mathcal{P}$ a *positively invariant (PI)* set.

Definition 2: We call system (2) *asymptotically stable (AS)* in D if there exists a $\mathcal{KL}$-function β such that, for each initial condition $x(0) = x_0 \in D$, the corresponding state trajectory is defined on $\mathbb{Z}_+$ and satisfies

$$\|x(t)\| \leq \beta(\|x_0\|, t), \quad \forall t \in \mathbb{Z}_+. \tag{3}$$

Lemma 1: [12], [13] Let $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$. Let $D \subseteq \mathbb{R}^n$ with $0 \in$ int(D) be a PI set for system (2) and let $V : D \to \mathbb{R}_+$ be a function such that

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|), \quad \forall x \in D, \tag{4a}$$
$$V(\Phi(x)) - V(x) \leq -\alpha_3(\|x\|), \quad \forall x \in D. \tag{4b}$$

If inequalities (4) hold for all $x \in D$, then system (2) is asymptotically stable (AS) in D.

If a function V satisfies the conditions of Lemma 1, we call V a Lyapunov function for (2).

Consider a system

$$x(t + 1) \quad = \quad \Gamma(x(t), u(t)), \tag{5}$$

where $x(t) \in D \subseteq \mathbb{R}^n$ is the state at time $t \in \mathbb{Z}_+$, and

$$u(t) \in \mathbb{U} \subseteq \mathbb{R}^m \tag{6}$$

is the control input at time $t \in \mathbb{Z}_+$. The set $\mathbb{U}$ represents input constraints and $\Gamma : D \times \mathbb{U} \to \mathbb{R}^n$ is a given function.

Definition 3: A set $\mathcal{P} \subseteq D$ is called a *controlled* λ-*contractive invariant set* for system (5)-(6) with input set $\mathbb{U}$, if for all $x \in \mathcal{P}$ there is a $u \in \mathbb{U}$ such that $\Gamma(x, u) \in \lambda\mathcal{P}$.

In case this property holds for $\lambda = 1$, we call $\mathcal{P}$ a *controlled invariant (CI)* set.

Definition 4: A function $V : D \to \mathbb{R}_+$ is called a *control-Lyapunov function (CLF)* for (5)-(6) and input set $\mathbb{U}$, if there exist $\alpha_1, \alpha_2, \alpha_3 \in \mathcal{K}_\infty$ such that (4a) holds for all $x \in D$, and for all $x \in D$ there is a $u \in \mathbb{U}$ such that

$$V(\Gamma(x,u)) - V(x) \leq -\alpha_3(\|x\|), \text{ and } \Gamma(x,u) \in D. \quad (7)$$

III. Problem Formulation

Consider the discrete-time linear time-invariant system

$$x(t+1) = Ax(t) + Bu(t), \quad (8)$$

where $x(t) \in \mathbb{X} \subseteq \mathbb{R}^n$ and $u(t) \in \mathbb{U} \subseteq \mathbb{R}^m$ are the state and input at time t, respectively. The sets $\mathbb{X}$ and $\mathbb{U}$ represent the state and input constraint sets, which are both assumed to be polytopes containing the origin in their interiors. We write $\mathbb{X}$ and $\mathbb{U}$ as

$$\mathbb{X} = \{x \in \mathbb{R}^n \mid E_x x \leq e_x\}, \ \mathbb{U} = \{u \in \mathbb{R}^m \mid E_u u \leq e_u\} \quad (9)$$

for matrices $E_x \in \mathbb{R}^{n_x \times n}$, $E_u \in \mathbb{R}^{n_u \times m}$ and vectors $e_x \in \mathbb{R}^{n_x}$ $e_u \in \mathbb{R}^{n_u}$ of appropriate dimensions. The goal is to design a stabilizing and well performing state feedback controller for system (8) satisfying the input and state constraints. A suitable technique for this is MPC.

A. Stabilizing MPC setup

A frequently used MPC setup for the linear discrete-time system (8) is solving at sampling instant t the following optimization problem for a fixed horizon $N \in \mathbb{Z}_{\geq 1}$ based on the current state $x(t) \in \mathbb{X}$,

$$\begin{aligned} \min_{U \triangleq [u_0^{\mathrm{T}}, \cdots, u_{N-1}^{\mathrm{T}}]^{\mathrm{T}}} \quad & J(U, x_0) \triangleq \\ & \|Px_N\|_p + \sum_{k=0}^{N-1} \|Qx_k\|_p + \sum_{k=0}^{N-1} \|Ru_k\|_p, \quad (10) \\ \text{s.t.} \quad & x_k \in \mathbb{X}, \quad k = 1, \cdots, N, \\ & u_k \in \mathbb{U}, \quad k = 0, 1, \cdots, N-1, \\ & x_0 = x(t), \\ & x_N \in \mathbb{X}_T, \\ & x_{k+1} = Ax_k + Bu_k, \quad k = 0, 1, \cdots, N-1. \end{aligned}$$

In this paper, we are mainly interested in the case where p is either 1 or ∞ in (10). In addition, we assume that $Q \in \mathbb{R}^{r_Q \times n}$, $R \in \mathbb{R}^{r_R \times m}$ and $P \in \mathbb{R}^{r_P \times n}$ are full-column rank matrices. The set $\mathbb{X}_T \subseteq \mathbb{X}$ is the terminal set, which is chosen to be a polytope containing the origin in its interior.

Under these assumptions, solving this optimization model leads to an optimal control sequence, given by $u_i^*(x(t))$, $i = 0, 1, \cdots, N-1$. This optimal sequence is turned into a feedback control strategy by applying the first control move to the system, i.e.

$$u(t) = \mu^*(x(t)) := u_0^*(x(t)). \quad (11)$$

Closed-loop stability can be guaranteed via the terminal cost and set method [14], provided $\mathbb{X}_T$ and the terminal weight parameterized by P are chosen appropriately. Essentially, this terminal cost and set method typically requires that there exist P, K and a polytopic $\mathbb{X}_T$ such that

$$\|P(A+BK)x\|_p - \|Px\|_p \leq -\|Qx\|_p - \|RKx\|_p, \ \forall x \in \mathbb{X}_T, \quad (12)$$

and $(A+BK)\mathbb{X}_T \subseteq \mathbb{X}_T$ hold. Numerical methods to compute such P, K and $\mathbb{X}_T$ guaranteeing closed-loop stability are presented in [15], which apply even for the case of PWA systems.

Let $\mathbb{X}_f \subseteq \mathbb{X}$ be the set of the feasible states for the optimization problem (10), which is a polyhedron. Therefore, the set $\mathbb{X}_f$ can be represented as

$$\mathbb{X}_f = \{x \in \mathbb{R}^n \mid E_f x \leq e_f\} \quad (13)$$

for a matrix $E_f \in \mathbb{R}^{n_f \times n}$ and vector $e_f \in \mathbb{R}^{n_f}$ of appropriate dimensions. This set $\mathbb{X}_f$ is a positively invariant set for the closed-loop system (8)-(11) due to the choice of P, K and $\mathbb{X}_T$. In addition, as shown in [1], in this case the optimal cost V of (10) is a convex PWA function given by

$$V(x) = \max_{l \in \mathcal{F}_\mathcal{P}} (H_l x + h_l), \quad (14)$$

which is a Lyapunov function for the closed-loop system (8)-(11) on $\mathbb{X}_f$ and a CLF for (8) on $\mathbb{X}_f$ for input set $\mathbb{U}$. Based on (14) we can get the partitioning of $\mathbb{X}_f$ related to V as

$$\mathcal{P}_l := \{x \in \mathbb{R}^n \mid H_l x + h_l \geq H_\ell x + h_\ell, \ell \in \mathcal{F}_\mathcal{P}, \ell \neq l\} \quad (15)$$

with $l \in \mathcal{F}_\mathcal{P}$, i.e.

$$\cup_{l \in \mathcal{F}_\mathcal{P}} \mathcal{P}_l = \mathbb{X}_f \quad (16)$$

and

$$\mathrm{int}(\mathcal{P}_l) \cap \mathrm{int}(\mathcal{P}_\ell) = \emptyset, \quad \forall l, \ell \in \mathcal{F}_\mathcal{P}, l \neq \ell.$$

B. Low-complexity control problem

The problem we want to deal with in this paper is stated as follows:

Problem 1: Given a "regular" polytopic partition $\{\tilde{\mathcal{P}}_i \mid i \in \mathcal{F}_{\tilde{\mathcal{P}}}\}$ of $\mathbb{X}_f \subseteq \mathbb{X}$, where $\mathcal{F}_{\tilde{\mathcal{P}}} \subset \mathbb{Z}_{\geq 1}$ is a finite set of indices, find a PWA state feedback controller given by $\tilde{\mu} : \mathbb{X}_f \to \mathbb{R}^m$ with

$$u(t) = \tilde{\mu}(x(t)) = F_i x(t) + g_i, \quad \text{if } x(t) \in \tilde{\mathcal{P}}_i, \quad (17)$$

where $F_i \in \mathbb{R}^{m \times n}$, $g_i \in \mathbb{R}^m$ for all $i \in \mathcal{F}_{\tilde{\mathcal{P}}}$, such that the closed-loop system (8) and (17) has the following design properties:

(i) The input constraints are satisfied, i.e. $\tilde{\mu}(x) \in \mathbb{U}$ for all $x \in \mathbb{X}_f$.
(ii) The set $\mathbb{X}_f$ is positively invariant for the system (8) and (17), i.e. $x \in \mathbb{X}_f$ implies $Ax + B\tilde{\mu}(x) \in \mathbb{X}_f$, and hence, for any $x_0 \in \mathbb{X}_f$ the solution for (8) and (17) satisfies the state constraints $x(t) \in \mathbb{X}_f \subseteq \mathbb{X}$, $t \in \mathbb{Z}_+$.
(iii) The system (8) and (17) is asymptotically stable in $\mathbb{X}_f$.

Note that, at this point we do not specify exactly what the regular partitioning is that is being used as essentially any partitioning with any form of polytopic regions is allowed in our approach. As advocated in [6] and [8] we can choose the regions to be regular simplices or we can select them to be hypercubes as in [5], [9] which offer interesting implementation advantages as well.

Note also that here we assume the regular polytopic partitioning to be fixed, while later we will actually synthesize it using an automatic refinement procedure.

IV. CLF-based Approach

The main approach will be based on the availability of a CLF $V : \mathbb{X}_f \to \mathbb{R}_+$ for system (8) under (6) for the set $\mathbb{X}_f$. As already remarked, once a stabilizing MPC setup as in (10) is defined, the optimal MPC costs V as in (14) is a CLF of (6) and (8) on the feasible set $\mathbb{X}_f$.

Theorem 1: Consider system (8) with input constraint set $\mathbb{U}$ given by (9) and $\mathbb{X}_f$ as given in (13). Given a "regular" polyhedral partition $\tilde{\mathcal{P}} = \{\tilde{\mathcal{P}}_i \mid i \in \mathcal{F}_{\tilde{\mathcal{P}}}\}$ of $\mathbb{X}_f$, and a scalar $0 \leq \lambda < 1$, if there exist matrices F_i, g_i, $i \in \mathcal{F}_{\tilde{\mathcal{P}}}$ and a function $V : \mathbb{X}_f \to \mathbb{R}_+$ that satisfy (4a) and for all $i \in \mathcal{F}_{\tilde{\mathcal{P}}}$

$$E_u(F_i x + g_i) \leq e_u \text{ for all } x \in \tilde{\mathcal{P}}_i, \tag{18a}$$

$$E_f(Ax + BF_i x + Bg_i) \leq e_f \text{ for all } x \in \tilde{\mathcal{P}}_i, \tag{18b}$$

$$V(Ax + BF_i x + Bg_i) \leq \lambda V(x) \text{ for all } x \in \tilde{\mathcal{P}}_i, \tag{18c}$$

then the properties (i), (ii) and (iii) of Problem 1 are satisfied.

Note that a V satisfying (18a)-(18c) for some control law $\tilde{\mu}$ should be a CLF for (8) and input constraint set $\mathbb{U}$. Therefore, starting the search for a low-complexity PWA control law $\tilde{\mu}$ benefits significantly from the availability of a CLF V, such as the optimal cost of a stabilizing MPC setup as in (10).

Remark 1: Let $\mathcal{F}^0_{\tilde{\mathcal{P}}} := \{i \in \mathcal{F}_{\tilde{\mathcal{P}}} \mid 0 \in \text{cl}(\tilde{\mathcal{P}}_i)\}$. Given that the origin is the desired equilibrium point, we can set $g_i = 0$, for all $i \in \mathcal{F}^0_{\tilde{\mathcal{P}}}$, in order to satisfy (18c).

The the following subsection will be devoted to transform the conditions (18) into a computationally tractable form.

A. Computation of Explicit Control Law

In this section, we will formulate an LP feasibility problem to compute the low-complexity control law $\tilde{\mu}$ solving Problem 1 based on the result of Theorem 1 using an available PWA CLF V as in (14). We will do this step-by-step in the sense that we will show how the individual conditions of Theorem 1 can be turned into as linear constraints in the control parameters F_i, g_i, $i \in \mathcal{F}_{\tilde{\mathcal{P}}}$.

1) Asymptotic stability: In order to solve the stability condition (18c) in Theorem 1 on $\mathbb{X}_f$ in F_i, g_i, $i \in \mathcal{F}_{\tilde{\mathcal{P}}}$, we denote $\tilde{\mathcal{P}}_i$ as Ω for simplicity and also replace F_i, g_i by F, g for notational convenience. We define the vertices of Ω as

$$\text{extr}(\Omega) := \{r_1, \ldots, r_M\}, \text{ for some } M \in \mathbb{Z}_{\geq 1}. \tag{19}$$

We also define the polytopic subregions $\Omega^l := \Omega \cap \mathcal{P}_l$ of Ω, and the vertices of Ω^l, $l \in \mathcal{F}_{\mathcal{P}}$, as

$$\text{extr}(\Omega^l) := \{v^l_1, \ldots, v^l_{M_l}\}, \text{ for some } M_l \in \mathbb{Z}_{\geq 1}. \tag{20}$$

Note that $\{\Omega^l \mid l \in \mathcal{F}_{\mathcal{P}}\}$ is a polytopic partitioning of Ω. In the following theorem, we will formulate an LP feasibility problem that guarantees that the condition (18c) on Ω holds by transforming it into conditions on the vertices in (20).

Theorem 2: Let $V : \mathbb{X}_f \to \mathbb{R}_+$ as in (14) be given and consider a polytope $\Omega \subset \mathbb{X}_f$ with polytopic partitioning $\{\Omega^l \mid l \in \mathcal{F}_{\mathcal{P}}\}$. The following statements are equivalent for $0 \leq \lambda \leq 1$:

(i) For all $x \in \Omega$,

$$V(Ax + BFx + Bg) \quad \leq \quad \lambda V(x). \tag{21}$$

(ii) For all $l \in \mathcal{F}_{\mathcal{P}}$, and all $v \in \text{extr}(\Omega^l)$,

$$V(Av + BFv + Bg) \leq \lambda(H_l v + h_l). \tag{22}$$

Hence, on the basis of Theorem 2, the condition (22) is equivalent to (21) in the unknowns F and g on the region Ω (when the scalar λ is fixed). By using now the form (14) of V, (22) can be rewritten as

$$H_k(Av+BFv+Bg)+h_k \leq \lambda(H_l v+h_l), \quad \text{for all } k \in \mathcal{F}_{\mathcal{P}}, \tag{23}$$

which are affine constraints in F and g, which can be included in an LP. Note that we have to solve (23) for all vertices of $v \in \text{extr}(\tilde{\mathcal{P}}_i \cap \mathcal{P}_l)$, all $l \in \mathcal{F}_{\mathcal{P}}$ and all $i \in \mathcal{F}_{\tilde{\mathcal{P}}}$ to guarantee (21) for all $x \in \mathbb{X}_f = \cup_{i \in \mathcal{F}_{\tilde{\mathcal{P}}}} \tilde{\mathcal{P}}_i$. Clearly, many $\tilde{\mathcal{P}}_i \cap \mathcal{P}_l$ will be empty and therefore do not impose any constraints.

2) Input constraint satisfaction: Denote, as before, $\tilde{\mathcal{P}}_i$ as Ω. By convexity of Ω, for an arbitrary $x \in \Omega = \text{co}\{r_1, \ldots, r_M\}$, there exist constants $\lambda_r \geq 0$ such that $\sum_{r \in \text{extr}(\Omega)} \lambda_r = 1$, and $x = \sum_{r \in \text{extr}(\Omega)} \alpha_r r$. Interestingly, (18a) for all $x \in \Omega$ is equivalent to

$$E_u(Fr + g) \leq e_u \text{ for all } r \in \text{extr}(\Omega). \tag{24}$$

Clearly, (24) is implied by (18a) for all $x \in \Omega$. Conversely, if $x \in \Omega$ then using that $x = \sum_{r \in \text{extr}(\Omega)} \alpha_r r$ leads to

$$\begin{aligned} E_u(Fx + g) &= E_u(F(\sum_{r \in \text{extr}(\Omega)} \alpha_r r) + g) \\ &= E_u(\sum_{r \in \text{extr}(\Omega)} \alpha_r (Fr + g)) \\ &= \sum_{r \in \text{extr}(\Omega)} \alpha_r \underbrace{(E_u(Fr + g))}_{\leq e_u} \leq e_u. \end{aligned} \tag{25}$$

3) Invariant set condition: Similarly as (24) guarantees input constraint satisfaction, it can be shown that condition (18b) can be equivalently characterized by

$$E_f(Ar + BFr + Bg) \leq e_f \text{ for all } r \in \text{extr}(\Omega). \tag{26}$$

4) Overall LP problem: The result of Subsection IV-A.1, IV-A.2 and IV-A.3 all result in LP feasibility conditions with respect to the parameters F_i, g_i, $i \in \mathcal{F}_{\tilde{\mathcal{P}}}$. In fact, we will also search for a local decay factor $0 \leq \lambda_i < 1$ related to $\tilde{\mathcal{P}}_i$, $i \in \mathcal{F}_{\tilde{\mathcal{P}}}$, instead of a global λ, to verify the stability conditions derived in Subsection IV-A.1. Hence, if for any $i \in \mathcal{F}_{\tilde{\mathcal{P}}}$ (23) is satisfied with λ replaced by λ_i for all $v \in \text{extr}(\tilde{\mathcal{P}}_i \cap \mathcal{P}_l)$, $l \in \mathcal{F}_{\mathcal{P}}$, the conditions (21) hold for $0 \leq \lambda = \min_{i \in \mathcal{F}_{\tilde{\mathcal{P}}}} \lambda_i < 1$, as required. To summarize, we get the following LP feasibility problem for all $\Omega = \tilde{\mathcal{P}}_i$, $i \in \mathcal{F}_{\tilde{\mathcal{P}}}$ (assuming $\tilde{\mathcal{P}}_i \subseteq \mathbb{X}_f$) to solve Problem 1:

Find $0 \leq \lambda_i < 1$, and F_i, g_i, $i \in \mathcal{F}_{\tilde{\mathcal{P}}}$ such that

$$E_u(F_i r + g_i) \leq e_u \text{ for all } r \in \operatorname{extr}(\tilde{\mathcal{P}}_i), \tag{27a}$$

$$E_f(Ar + BF_i r + Bg_i) \leq e_f \text{ for all } r \in \operatorname{extr}(\tilde{\mathcal{P}}_i), \tag{27b}$$

$$(H_k(Av + BF_i v + Bg_i) + h_k) \leq \lambda(H_l v + h_l) \tag{27c}$$

$$\text{for all } v \in \operatorname{extr}(\tilde{\mathcal{P}}_i \cap \mathcal{P}_l), k \in \mathcal{F}_{\mathcal{P}},\ \ l \in \mathcal{I}(\tilde{\mathcal{P}}_i, \mathcal{P}),$$

where $g_i = 0$, for all $i \in \mathcal{F}^0_{\tilde{\mathcal{P}}} = \{i \in \mathcal{F}_{\tilde{\mathcal{P}}} \mid 0 \in \operatorname{cl}(\tilde{\mathcal{P}}_i)\}$.

Any feasible solution to (27a)-(27c) provides a stabilizing PWA control law $\tilde{\mu} : \mathbb{X}_f \to \mathbb{U}$ defined on the regular partitioning $\tilde{\mathcal{P}}$ that guarantees properties (i)-(iii) of Problem 1. Note that instead of checking the feasibility of (27) with $0 \leq \lambda_i < 1$, one can also aim at minimizing λ_i.

Interestingly, each region $\tilde{\mathcal{P}}_i$ in the new partitioning and the corresponding control parameters F_i, g_i and λ_i have their own local LP feasibility problem (27) without any relationship to the local LP for $\tilde{\mathcal{P}}_j$, $j \neq i$. In other words, the LP feasibility problems for each $\tilde{\mathcal{P}}_i$, $i \in \mathcal{F}_{\tilde{\mathcal{P}}}$, are decoupled. This indicates the possibility that for the regions $\tilde{\mathcal{P}}_i$'s which resulted in an infeasible local LP, the region can be refined by splitting it into smaller subregions to attempt to make the condition (27) feasible. Note that we know that in case V is a CLF obtained by a stabilizing MPC setup (10), a PWA control law exists, namely an optimal control μ^*, which satisfies the constraints and stability conditions. Therefore, in the next subsection we present an automatic refinement procedure that based on a rough initial partitioning $\tilde{\mathcal{P}}_{init}$, refines the regions where necessary to satisfy the constraints (27).

B. Design Algorithm

1) Refinement procedure: For the regular regions $\tilde{\mathcal{P}}_i$ inside $\mathbb{X}_f$ that need to be refined because (27) is infeasible, we will split the region $\tilde{\mathcal{P}}_i$ into smaller regular subregions. For instance, in case hypercubic regions are used one can perform a dyadic discretization [16] and get 2^n hypercubes, where n is the dimension of the space. A dyadic discretization of a hypercube splits the hypercube into 2^n equal hypercubes by inserting hyperplanes perpendicular to each of the coordinate axes exactly in the middle of each edge of the hypercube (see [16] for more details on dyadic discretization). Based on the smaller subregions, new local problems of the form (27) are generated and solved via LP techniques.

To avoid that the refinement procedure does not terminate or that the number of regular regions becomes too large due to too many refinement steps, a maximum number of regions $n_{\max}$ and/or a maximum level of refinement $h_{\max}$ is added as a stopping criterium to the refinement procedure.

Hence, the refinement procedure provides the means to *synthesize* the partitioning automatically, one can start from a very rough initial partitioning $\tilde{\mathcal{P}}_{init}$.

Algorithm: Automatic Refinement Procedure

Given: A CLF for system (8) $V\colon \mathbb{X}_f \to \mathbb{R}_+$ of the form (14) for input constraint set $\mathbb{U}$ as in (9) is given with $\mathbb{X}_f$ as in (13). The partitioning $\mathcal{P}$ is given according to (15). In addition, an initial (rough) regular partitioning $\tilde{\mathcal{P}}_{init}$, the maximum refinement level $h_{\max} \geq 1$ and the maximal number of cells $n_{\max} \geq 1$ are available.

initialize, Old $:= \tilde{\mathcal{P}}_{init}$, New $:= \emptyset$, $h(\Omega) := 1$ for all $\Omega \in$ Old, $j := 1$
while $j \leq n_{\max}$ **and** Old $\neq \emptyset$ **do**
  select region Ω in Old
  find the overlapping regions $\mathcal{I}(\Omega, \mathcal{P})$
  if (27) has λ_i feasible for Ω **then**
    Old := Old $\setminus \{\Omega\}$
    New := New $\cup \{\Omega\}$
    store control parameters F_j, g_j, corresponding to Ω, obtained from (27)
    $j := j + 1$
  else if $h(\Omega) < h_{\max}$ **then**
    split Ω into subregions $\{\Omega_1, \ldots, \Omega_L\}$
    Old := (Old $\setminus \{\Omega\}) \cup \{\Omega_1, \ldots, \Omega_L\}$
    store $h(\Omega_i) := h(\Omega) + 1, i = 1, \ldots, L$
  else
    output 'warning: maximal level of refinement reached' and terminate algorithm
  end if
end while
if Old $= \emptyset$ **then**
  output 'done'
else
  output 'warning: maximum number of regions reached'
end if
end

In Section V, we will give an example to illustrate the implementation of this algorithm. In particular, in the example we will use hypercubic regions for the regular regions and use dyadic discretization to refine regions when necessary.

V. Example

Consider system (8) with

$$A = \begin{bmatrix} 1.1 & -0.6928 \\ 0.6928 & 1.1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

and input and state constraints given by the sets

$$\mathbb{U} = \{u \in \mathbb{R} \mid -1 \leq u \leq 1\}, \text{ and}$$
$$\mathbb{X} = \left\{x \in \mathbb{R}^2 \mid -10 \leq x \leq 10\right\}.$$

For the MPC setup (10) with horizon $N = 3$, we set the weighting matrices to $Q = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $R = 0.1$, and by using [15] we can compute the terminal weight and the terminal set as

$$P = \begin{bmatrix} 5.9079 & 1.6418 \\ -9.7313 & 6.8375 \end{bmatrix}, \text{and}$$
$$\mathbb{X}_T = \{x \in \mathbb{R}^n \mid \|Px\|_\infty \leq 1.9441\},$$

which guarantee (12) and $(A + BK)\mathbb{X}_T \subseteq \mathbb{X}_T$ for $K = \begin{bmatrix} 0.0059 & -1.8560 \end{bmatrix}$ and thus closed-loop stability of (8) and (11). Hence, the optimal cost $V\colon \mathbb{X}_f \to \mathbb{R}_+$ is a CLF for (8) in $\mathbb{X}_f$ with input constraint $\mathbb{U}$. The partitions corresponding to the optimal costs $V\colon \mathbb{X}_f \rightarrow \mathbb{R}_+$ are computed using the method in [1]. The partitioning of μ^* after simplified is shown in the Fig. 1. As such, the controller partitioning is depicted with 39 irregular regions.

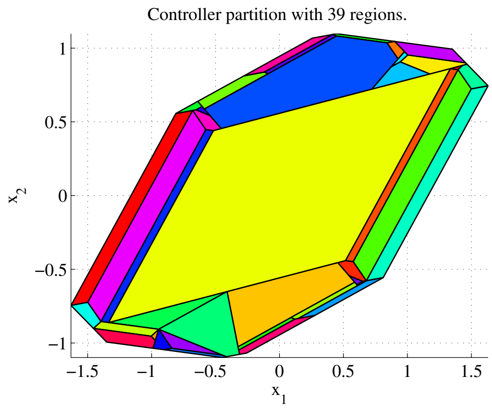

Fig. 1: The optimal partitioning after merging the regions.

We now apply the algorithm in Section IV-B, and obtain a stabilizing low-complexity PWA control law as depicted in Fig. 2, which is defined over 16 regular regions. Obviously, the later control law is much easier for on-line implementation both from perspectives of computation times and memory requirement.

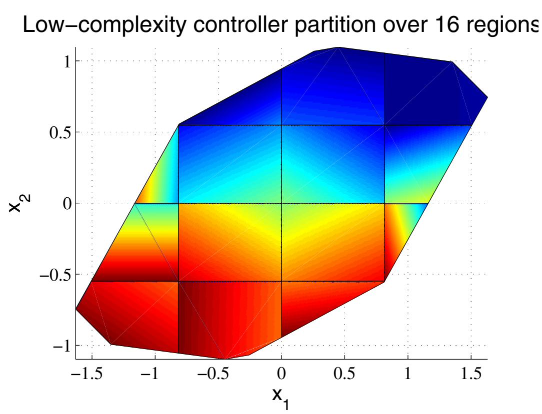

Fig. 2: The control law on the regular partition.

VI. Conclusions

In this paper we presented a control-Lyapunov function (CLF) approach to synthesize low-complexity stabilizing PWA controllers for constrained linear systems. In contrast with PWA control law as obtained through explicit MPC techniques that result in partitionings with a high number of regions and regions of irregular shapes, here the objective was to obtain stabilizing PWA controller defined on regular partitionings in which the regular regions can have any desirable polytopic form (e.g. regular simplices or hypercubes) that enhance fast on-line implementation and reduced memory requirements. The availability of a PWA CLF, which is a prerequisite for our method, can be realized by taking the optimal cost of a stabilizing MPC setup based on linear costs using 1 or ∞ norms. Based on such a PWA CLF a linear programming (LP) feasibility problem was derived that guarantees *a priori* stability and constraints satisfaction. An automatic refinement procedure was also presented that refines the regular partitioning where needed in order to satisfy the design requirements. This refinement procedure has the advantage that it can be used to synthesize the partitioning of the control law, next to the controller gains, automatically by starting from a coarse initial partitioning. Due to the flexibility of the method (any desirable regular partitionings can be used) and allowing to search the space of discontinuous PWA control laws, our method can result in true low-complexity stabilizing PWA control law for constrained linear systems. A numerical example indeed demonstrated the effectiveness of this new approach.

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