2011 50th IEEE Conference on Decision and Control and
European Control Conference (CDC-ECC)
Orlando, FL, USA, December 12-15, 2011

On the Minimum Attention Control Problem for Linear Systems: A Linear Programming Approach

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Abstract— **In this paper, we present a novel solution to the minimum attention control problem. In minimum attention control, the objective is to minimise the 'attention' that a control task requires, given certain performance requirements. Here, we interpret 'attention' as the inverse of the time elapsed between two consecutive executions of a control task. Instrumental for the solution will be a novel extension of the notion of a control Lyapunov function. By focussing on linear plants, by allowing for only a finite number of possible intervals between two subsequent executions of the control task and by taking the extended control Lyapunov function to be ∞-norm based, we can formulate the minimum attention control problem as a linear program, which can be solved efficiently online. Furthermore, we provide a technique to construct suitable ∞-norm-based (extended) control Lyapunov functions for our purposes. Finally, we illustrate the theory using a numerical example, showing that minimum attention control can outperform an alternative implementation-aware control law available in the literature.**

I. INTRODUCTION

A current trend in control engineering is to no longer implement controllers on dedicated platforms having dedicated communication channels, but in embedded microprocessors and using (shared) communication networks. Since in such an environment the control task has to share computation and communication resources with other tasks, the availability of these resources is limited and might even be time-varying. Despite the fact that resources are scarce, controllers are typically still implemented in a time-triggered fashion, in which the control task is executed periodically. This design choice is motivated by the fact that it enables the use of the well-developed theory on sampled-data systems to design controllers and analyse the resulting closed-loop systems. This design choice, however, leads to over-utilisation of the available resources and requires over-provisioned hardware, as it might not be necessary to execute the control task every period. For this reason, several alternative control strategies have been developed to reduce the required computation and communication resources needed to execute the control task.

This work is partially supported by the Dutch Science Foundation (STW) and the Dutch Organization for Scientific Research (NWO) under the VICI grant "Wireless controls systems: A new frontier in automation", by the European 7th Framework Network of Excellence by the project "Highly-complex and networked control systems (HYCON2-257462)", and by the project "Decentralised and Wireless Control of Large-Scale Systems (WIDE-224168)", and by the National Science Foundation (NSF) award 0834771.

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One of such approaches is self-triggered control, see, e.g., [1]–[3]. In self-triggered control, the control law consists of two elements: namely, a feedback controller that computes the control input, and a triggering mechanism that determines when the control task should be executed. Current design methods for self-triggered control are emulation-based approaches, by which we mean that the feedback controller is designed for an ideal implementation, while subsequently the triggering mechanism is designed (based on the given controller). Since the feedback controller is designed before the triggering mechanism, it is difficult, if not impossible, to obtain an optimal design of the combined feedback controller and triggering mechanism in the sense that the minimum number of controller executions is achieved while guaranteeing a certain level of closed-loop performance.

In this paper, we consider minimum attention control (MAC), see [4], in which the objective is to minimise the attention the control loop requires, i.e., MAC maximises the next execution instant, while guaranteeing a certain level of closed-loop performance. Note that this control strategy is similar to self-triggered control, where also the objective is to have as few control task executions as possible, given a certain closed-loop performance requirement. However, contrary to self-triggered control, MAC is typically not designed using emulation-based approaches in the sense that it does not require a separate feedback controller to be available before the triggering mechanism can be designed. Clearly, this joint design procedure is more likely to yield a (close to) optimal design than a sequential design procedure would.

The control problem studied in this paper is similar to one of the problems studied in [5]. However, by focussing on linear systems, we will propose an alternative approach to solve the control problem at hand. In the solution strategy we propose, we focus on linear plants, as already mentioned, and consider only a finite number of possible interexecution times. Furthermore, we will employ control Lyapunov functions (CLFs) that can be seen as an extension of the CLFs for sampled-data systems, which will enable us to guarantee a certain level of performance. These extended CLFs will first be formulated for general sampled-data systems and will later be particularised to ∞-norm-based functions, see, e.g., [6], [7]. Namely, by using ∞-norm-based extended CLFs, we can formulate the MAC problem as a linear program (LP), which can be efficiently solved online, thereby alleviating the computational burden as experienced in [5]. Furthermore, we provide a technique to construct suitable ∞-norm-based (extended) control Lyapunov functions for the control objective under consideration. We will also discuss in Remark III.6

that the proposed method can be applied to the anytime attention control problem that was studied in [5]. Finally, we will illustrate the theory using a numerical example, showing that the proposed methodology outperforms the self-triggered control strategy of [3].

A. Nomenclature

The following notational conventions will be used. For a vector $x \in \mathbb{R}^n$, we denote by $[x]_i$ its i-th element and by $\|x\|_p := \sqrt[p]{\sum_{i=1}^n |x_i|^p}$ its p-norm, $p \in \mathbb{N}$, and by $\|x\|_\infty = \max_{i\in\{1,\dots,N\}} |x_i|$, its ∞-norm. For a matrix $A \in \mathbb{R}^{n\times m}$, we denote by $[A]_{ij}$ its i,j-th element, by $A^\top \in \mathbb{R}^{m\times n}$ its transposed and by $\|A\|_p := \max_{x\neq 0} \frac{\|Ax\|_p}{\|x\|_p}$, its induced p-norm, $p \in \mathbb{N}\cup\{\infty\}$. In particular, $\|A\|_\infty := \max_{i\in\{1,\dots,N\}} \sum_{j=1}^m |[A]_{ij}|$. Finally, we denote the set of nonnegative real numbers by $\mathbb{R}_+ := [0,\infty)$ and a set-valued function F from $\mathbb{R}^n$ to $\mathbb{R}^m$ by $F : \mathbb{R}^n \hookrightarrow \mathbb{R}^m$, meaning that $F(x) \subseteq \mathbb{R}^m$ for each $x \in \mathbb{R}^n$.

II. Problem Formulation

In this section, we formulate the minimum attention control problem. To do so, let us consider a linear time-invariant (LTI) plant given by

$$\tfrac{\mathrm{d}}{\mathrm{d}t}x = Ax + Bu, \tag{1}$$

where $x \in \mathbb{R}^{n_x}$ denotes the state of the plant and $u \in \mathbb{R}^{n_u}$ the input applied to the plant. The plant is controlled in a sampled-data fashion, using a zero-order hold (ZOH), which leads to

$$u(t) = \hat{u}_k, \quad \text{for all } t \in [t_k, t_{k+1}), \tag{2}$$

where the discrete-time control inputs $\hat{u}_k$, $k \in \mathbb{N}$, and the strictly increasing sequence of execution instants $\{t_k\}_{k\in\mathbb{N}}$ are given by the solutions to the following control problem.

Problem II.1 (Minimum Attention Control (MAC)) Find a set-valued function $F_{\mathrm{MAC}} : \mathbb{R}^{n_x} \hookrightarrow \mathbb{R}^{n_u}$ and a function $h : \mathbb{R}^{n_x} \to \mathbb{R}_+$, such that

$$\begin{cases} \hat{u}_k \in F_{\mathrm{MAC}}(x(t_k)) \\ t_{k+1} = t_k + h(x(t_k)), \end{cases} \tag{3}$$

for all $k \in \mathbb{N}$, renders the plant (1) with ZOH (2) stable and guarantees a certain level of performance, both defined in an appropriate sense, while, for each $x \in \mathbb{R}^{n_x}$, $h(x)$ is as large as possible.

To make the preceding problem well defined we need to give a precise meaning to the terms stability and performance qualifying the solutions of the closed-loop system given by (1), (2), with (3).

Definition II.2 *The closed-loop system, given by* (1), (2) *and* (3)*, is said to be* globally exponentially stable *(GES) with a convergence rate $\alpha > 0$ and a gain $c > 0$, if for any initial condition $x(0)$, the corresponding solutions satisfy*

$$\|x(t)\| \leqslant ce^{-\alpha t}\|x(0)\|, \tag{4}$$

for all $t \in \mathbb{R}_+$.

The notion of performance used in this paper is explicitly expressed in terms of the convergence rate α as well as the gain c. Only requiring a desired convergence rate α could yield a very large gain c and, thus, could yield unacceptable closed-loop behaviour. As we will show below (see Lemma III.2), the guaranteed gain c typically becomes large when the time between two controller executions, i.e., $t_{k+1} - t_k$, is large. Therefore, special measures have to be taken to prevent the gain c from becoming unacceptably large, while still maximising $t_{k+1} - t_k$ for all $k \in \mathbb{N}$.

III. Formulating the MAC Problem using Control Lyapunov Functions

In this section, we will propose a solution to the MAC problem by formulating it as an optimisation problem. In this optimisation problem, we will use an extension to the notion of a control Lyapunov function (CLF). Before doing so, we will briefly revisit some existing results on CLFs, see, e.g., [8], [9], and show how they can be used to design control laws that render the plant (1) with ZOH (2) GES with a certain convergence rate $\alpha > 0$ and a certain gain $c > 0$.

A. Preliminary Results on CLFs

Let us now introduce the notion of a CLF, which has been applied to discrete-time systems in [9] and will now be applied to periodic sampled-data systems, given by the plant (1) with ZOH (2), in which $t_{k+1} = t_k + h$, $k \in \mathbb{N}$, for some fixed $h > 0$.

Definition III.1 *Consider the plant* (1) *with ZOH* (2)*. The function $V : \mathbb{R}^{n_x} \to \mathbb{R}$ is said to be a* control Lyapunov function *(CLF) for* (1) *and* (2)*, a convergence rate $\alpha > 0$, a control gain bound $\beta > 0$ and an interexecution time $h > 0$, if there exist constants $\underline{a}, \overline{a} \in \mathbb{R}_+$ and $q \in \mathbb{N}$, such that for all $x \in \mathbb{R}^{n_x}$*

$$\underline{a}\|x\|^q \leqslant V(x) \leqslant \overline{a}\|x\|^q, \tag{5}$$

and, for all $x \in \mathbb{R}^{n_x}$, there exists a control input $\hat{u} \in \mathbb{R}^{n_u}$, satisfying $\|\hat{u}\| \leqslant \beta\|x\|$ and

$$V(e^{Ah}x + \textstyle\int_0^h e^{As}\mathrm{d}sB\hat{u}) \leqslant e^{-\alpha q h}V(x). \tag{6}$$

Based on a CLF for a convergence rate $\alpha > 0$, a control gain bound $\beta > 0$ and an interexecution time $h > 0$, as in Definition III.1, the control law

$$\begin{cases} \hat{u}_k \in F(x) := \{u \in \mathbb{R}^{n_u} \mid f(x,u,h,\alpha) \leqslant 0, \\ \qquad\qquad\qquad\qquad \text{and } \|u\| \leqslant \beta\|x\|\}, \\ t_{k+1} = t_k + h, \end{cases} \tag{7}$$

in which

$$f(x,u,h,\alpha) := V(e^{Ah}x + \textstyle\int_0^h e^{As}B\mathrm{d}s\, u) - e^{-\alpha q h}V(x), \tag{8}$$

renders the plant (1) with ZOH (2) GES with a convergence rate $\alpha > 0$ and a certain gain $c > 0$, as we will show in the following lemma.

Lemma III.2 *Assume there exists a CLF for* (1) *with* (2)*, a convergence rate $\alpha > 0$, a control gain bound $\beta > 0$ and an interexecution time $h > 0$, in the sense of Definition III.1.*

Then, the control law (7) *renders the plant* (1) *with ZOH* (2) *GES with the convergence rate* α *and the gain* $c = \bar{c}(\alpha, \beta, h)$, *where*

$$\bar{c}(\alpha, \beta, h) := \sqrt[q]{\frac{\overline{a}}{\underline{a}}}\Big(e^{\|A\|h} + \beta \int_0^h e^{\|A\|s}\mathrm{d}s\|B\|\Big)e^{\alpha h}. \quad (9)$$

Proof: This lemma is a special case of Lemma III.4 that we will present below. ■

Lemma III.2 illustrates why it is important to express the notion of performance both in terms of the convergence rate α as well as the gain c, as was mentioned at the end of Section II. Namely, even though a CLF could guarantee GES with a certain convergence rate α, for some control gain bound β and for any arbitrarily large h, by using a corresponding CLF in the control law (7), the consequence is that the guaranteed gain c becomes extremely large, see Lemma III.2. In particular, c grows exponentially as h becomes larger, which (potentially) yields undesirably large responses for large interexecution times $h = t_{k+1} - t_k$, $k \in \mathbb{N}$. To avoid having such unacceptable behaviour, we propose a control design methodology that is able to guarantee a desired convergence rate α, as well as a desired gain c, even for large h. This requires an extension of the CLF defined above.

B. Extended CLFs

The observation that the interexecution time h influences the gain c is important to allow the MAC problem to be formalised using CLFs. Namely, in order to achieve sufficiently high performance (meaning a sufficiently large α and a sufficiently small c), Lemma III.2 indicates that the interexecution time h has to be selected sufficiently small. This, however, contradicts the MAC problem, in which the interexecution time is to be maximised. We therefore propose an extended control Lyapunov function (eCLF), which we will subsequently use to solve the MAC problem. Roughly speaking, the eCLF is such that it does not only decrease from t_k to t_{k+1}, but also from t_k to intermediate time instants $t_k + \hbar_l$, for some $\hbar_l > 0$ satisfying $t_{k+1} - t_k > \hbar_l$, $k \in \mathbb{N}$, $l \in \{1, \ldots, L-1\}$. The existence of such an eCLF guarantees high performance, even though the interexecution time $\hbar_L := t_{k+1} - t_k$, $k \in \mathbb{N}$, can be large, as we will show after giving the formal definition of the eCLF.

Definition III.3 *Consider the plant* (1)*, with ZOH* (2)*. The function* $V : \mathbb{R}^{n_x} \to \mathbb{R}$ *is said to be an* extended control Lyapunov function *(eCLF) for* (1) *and* (2)*, a convergence rate* $\alpha > 0$*, a control gain bound* $\beta > 0$*, and a set* $\mathcal{H} := \{\hbar_1, \ldots, \hbar_L\}$*,* $L \in \mathbb{N}$*, satisfying* $\hbar_{l+1} > \hbar_l > 0$ *for all* $l \in \{1, \ldots, L-1\}$*, if there exist constants* $\underline{a}, \overline{a} \in \mathbb{R}_+$ *and* $q \in \mathbb{N}$*, such that for all* $x \in \mathbb{R}^{n_x}$

$$\underline{a}\|x\|^q \leqslant V(x) \leqslant \overline{a}\|x\|^q \quad (10)$$

and, for all $x \in \mathbb{R}^{n_x}$*, there exists a control input* $\hat{u} \in \mathbb{R}^{n_u}$*, satisfying* $\|\hat{u}\| \leqslant \beta\|x\|$ *and*

$$V\big(e^{A\hbar_l}x + \textstyle\int_0^{\hbar_l} e^{As}\mathrm{d}sB\hat{u}\big) \leqslant e^{-\alpha q \hbar_l}V(x) \quad (11)$$

for all $l \in \{1, \ldots, L\}$.

As before, based on an eCLF for a convergence rate $\alpha > 0$, a control gain bound $\beta > 0$, and a set $\mathcal{H}$ as in Definition III.3, the control law

$$\begin{cases} \hat{u}_k \in F(x) := \big\{u \in \mathbb{R}^{n_u} \,|\, f(x, u, \hbar_l, \alpha) \leqslant 0 \\ \qquad\qquad \forall\, l \in \{1, \ldots, L\} \text{ and } \|u\| < \beta\|x\|\big\}, \\ t_{k+1} = t_k + \hbar_L \end{cases} \quad (12)$$

with $f(x, u, \hbar_l, \alpha)$ as defined in (8), renders the plant (1) with ZOH (2) GES with a convergence rate $\alpha > 0$ and a certain gain $c > 0$ that is typically smaller than the gain obtained using an ordinary CLF, as we will show in the following lemma.

Lemma III.4 *Assume there exists an eCLF for* (1) *with* (2)*, a convergence rate* $\alpha > 0$*, a control gain bound* $\beta > 0$ *and a set* $\mathcal{H} := \{\hbar_1, \ldots, \hbar_L\}$*, in the sense of Definition III.3. Then, the control law* (12) *renders the plant* (1) *with ZOH* (2) *GES with the convergence rate* α *and the gain* $c = \bar{c}(\alpha, \beta, \Delta_\hbar, \hbar_L)$*, where*

$$\bar{c}(\alpha, \beta, \Delta_\hbar, \hbar_L) :=$$
$$\sqrt[q]{\frac{\overline{a}}{\underline{a}}}\Big(e^{\|A\|\Delta_\hbar} + \beta e^{\alpha(\hbar_L - \Delta_\hbar)} \int_0^{\Delta_\hbar} e^{\|A\|s}\mathrm{d}s\|B\|\Big)e^{\alpha\Delta_\hbar}, \quad (13)$$

with $\Delta_\hbar := \max_{l\in\{1,\ldots,L\}}(\hbar_l - \hbar_{l-1})$*, in which* $\hbar_0 := 0$.

Proof: The proof can be found in [10], [11]. ■

The existence of an eCLF for a well-chosen set $\mathcal{H}$ (i.e., realising a sufficiently small $\Delta_\hbar$) guarantees high performance in terms of the convergence rate α and the gain c, while still allowing for large interexecution times $\hbar_L = t_{k+1} - t_k$, $k \in \mathbb{N}$. Indeed, by using the intermediate time instants $t_k + \hbar_l$, the gain c in Lemma III.4 is generally much smaller than the gain c in Lemma III.2. However, making $\Delta_\hbar$ too small might lead to infeasibility of the control law, as decreasing $\Delta_\hbar$ for a fixed interexecution time $t_{k+1} - t_k$ means taking more intermediate times $\hbar_l$ and, thus, that more inequality constraints are added to the set-valued function F in (12), which, besides resulting in a much more complicated control law, might cause $F(x) = \emptyset$ for some $x \in \mathbb{R}^{n_x}$. Hence, a tradeoff can be made between the magnitude of the gain c and the number of constraints in $F(x)$ and we will exactly exploit this fact in the solution to the MAC problem, as we will show below.

C. Solving the MAC Problem using eCLFs

We will now propose a solution to the MAC problem. As a starting point, we consider the control law (12), which is based on an eCLF. Indeed, the existence of an eCLF for a convergence rate $\alpha > 0$, a control gain bound $\beta > 0$ and a set $\mathcal{H}$ implies GES with convergence rate α and gain c of the plant (1), with ZOH (2) and the control law (12), according to Lemma III.4. However, given the function V, a convergence rate α, a control gain bound β and a set $\mathcal{H}$, it might not always be possible to ensure that $F(x) \neq \emptyset$ for all $x \in \mathbb{R}^{n_x}$. To resolve this issue, we take subsets of $\mathcal{H}$ of the form $\mathcal{H}_{\bar{L}} := \{\hbar_1, \ldots, \hbar_{\bar{L}}\}$, for $\bar{L} \in \{1, \ldots, L\}$, such that $\mathcal{H}_1 \subseteq \mathcal{H}_2 \subseteq \ldots \subseteq \mathcal{H}_L = \mathcal{H}$, and propose MAC, in which the objective is to maximise $\bar{L} \in \{1, \ldots, L\}$ for each given $x \in \mathbb{R}^{n_x}$. In other words, for each given $x \in \mathbb{R}^{n_x}$, $\bar{L}$ is maximised such that $F_{\bar{L}}(x) \neq \emptyset$, in which

$$F_{\bar{L}}(x) := \{u \in \mathbb{R}^{n_u} \,|\, f(x,u,\hbar_l,\alpha) \leqslant 0 \\ \forall\, l \in \{1,\dots,\bar{L}\} \text{ and } \|u\| \leqslant \beta\|x\|\}, \quad (14)$$

with $f(x,u,\hbar_l,\alpha)$ as defined in (8). We maximise $\bar{L}$ to make the interexecution times $\hbar_{\bar{L}} = t_{k+1} - t_k$ maximal, yielding that the control law requires minimum attention. Hence, this MAC law is given by (3), in which we take

$$\begin{cases} F_{\mathrm{MAC}}(x) := F_{\bar{L}^\star(x)}(x) \\ h(x) := \hbar_{\bar{L}^\star(x)} \end{cases} \quad (15)$$

and

$$\bar{L}^\star(x) := \max\{l \in \{1,\dots,L\} \,|\, F_l(x) \neq \emptyset\}. \quad (16)$$

Indeed, the control law (3), with (15) and (16) is a solution to the MAC problem, as every control input $\hat{u}_k$ is chosen such that the interexecution time $t_{k+1} - t_k = \hbar_{\bar{L}^\star(x(t_k))}$ is the largest one in the set $\mathcal{H}$ for which $F_{\bar{L}^\star(x(t_k))}(x(t_k)) \neq \emptyset$. Note that this control law is well defined if $F_{\mathrm{MAC}}(x) \neq \emptyset$, for all $x \in \mathbb{R}^{n_x}$. This condition is equivalent to requiring that $F_1(x) \neq \emptyset$ for all $x \in \mathbb{R}^{n_x}$. Namely, for each $x \in \mathbb{R}^{n_x}$, it holds that $F_1(x) \supseteq F_2(x) \supseteq \dots \supseteq F_L(x)$, which gives that, for each $x \in \mathbb{R}^{n_x}$, $F_{\mathrm{MAC}}(x) \neq \emptyset$ implies that $F_1(x) \neq \emptyset$, while the fact that $F_1(x) \neq \emptyset$ implies that $F_{\mathrm{MAC}}(x) \neq \emptyset$ follows directly from (15) and (16). Hence, (15) is well defined if $F_1(x) \neq \emptyset$ for all $x \in \mathbb{R}^{n_x}$, which is guaranteed if the function V is an ordinary CLF for (1) with (2), a convergence rate $\alpha > 0$, a control gain bound $\beta > 0$ and an interexecution time $\hbar_1$, in the sense of Definition III.2.

We will now formally show that the proposed control law renders the plant (1) with ZOH (2) GES with convergence rate α and a certain gain c.

Theorem III.5 *Assume there exist a set $\mathcal{H} := \{\hbar_1,\dots,\hbar_L\}$, satisfying $\hbar_{l+1} > \hbar_l > 0$ for all $l \in \{1,\dots,L-1\}$, $L \in \mathbb{N}$, and an ordinary CLF for* (1) *with* (2)*, a convergence rate $\alpha > 0$, a control gain bound $\beta > 0$, and the interexecution time $\hbar_1$, in the sense of Definition III.1. Then, the MAC law* (3)*, with* (8)*,* (14)*,* (15) *and* (16)*, renders the plant* (1) *with ZOH* (2) *GES with the convergence rate α and the gain $c = \bar{c}(\alpha,\beta,\Delta_\hbar,\hbar_L)$ as in* (13).

Proof: The proof can be found in [10], [11]. ■

Remark III.6 The eCLF-based solution to the MAC problem can also be used to solve the anytime attention control (AAC) problem, which was introduced in [5]. In AAC, it is assumed that at each execution of the control task, the next execution instant of the control task is determined by the real-time scheduler. This situation is realistic in many embedded and networked systems, where the computation and/or communication resources have to be distributed among different tasks, meaning that the resources available for control are time-varying and the interexecution times becomes time varying. The objective in AAC is to maximise the performance of the closed-loop system (in terms of the convergence rate α and the gain c), given a certain interexecution time. As it was observed in [5], the same solution strategy used for the MAC problem can also be used for solving the AAC problem. Owing to space limitations, we will not present the solution to the AAC problem in this paper, but refer the interested reader to [10], [11].

IV. Obtaining a Well-Defined Solution

In this section, we will address the issue of how to guarantee that the solution to the MAC problem is well defined, i.e., that $F_{\mathrm{MAC}}(x) \neq \emptyset$ for all $x \in \mathbb{R}^{n_x}$. As was observed in the previous section, the existence of an ordinary CLF for (1) with (2), a convergence rate α, a control gain bound β and an interexecution time h ensures that the MAC law is well defined. To obtain such a CLF, and to guarantee that the control problem can be solved efficiently (as we will show in the next section), we focus in this section on ∞-norm-based CLFs of the form

$$V(x) = \|Px\|_\infty, \quad (17)$$

with $P \in \mathbb{R}^{m \times n_x}$ satisfying $\mathrm{rank}(P) = n_x$. Note that (17) is a suitable candidate CLF, in the sense of Definition III.3, with $q = 1$, since (5) is satisfied with

$$\overline{a} = \|P\|_\infty, \quad (18a)$$

$$\underline{a} = \max\{a > 0 \,|\, a\|x\|_\infty \leqslant \|Px\|_\infty \ \forall\, x \in \mathbb{R}^{n_x}\}. \quad (18b)$$

In fact, $\mathrm{rank}(P) = n_x$ ensures that $\overline{a} > 0$.

We will now provide a two-step procedure to obtain a suitable CLF. The first step is to consider an auxiliary control law of the form

$$u(t) = Kx(t) \quad (19)$$

that renders the plant (1) GES. To avoid any misunderstanding, (19) is not the control law being used; it is just an auxiliary control law that is useful to construct a candidate CLF. The actual MAC law will be given by (3), with (15) and (16) and does not use a matrix K.

Using the auxiliary control law, we can find a Lyapunov function for the plant (1) with control law (19) (without ZOH (2)) by employing the following intermediate result. This intermediate result can be seen as a slight extension of the results presented in [6], [7] to allow GES to be guaranteed, instead of only global asymptotic stability.

Lemma IV.1 *Assume that there exist a matrix $P \in \mathbb{R}^{m \times n_x}$, with $\mathrm{rank}(P) = n_x$, a matrix $Q \in \mathbb{R}^{m \times m}$ and a scalar $\hat{\alpha} > 0$ satisfying*

$$P(A + BK) - QP = 0 \quad (20a)$$

$$[Q]_{ii} + \sum_{j \in \{1,\dots,m\}\setminus\{i\}} |[Q]_{ij}| \leqslant -\hat{\alpha}, \quad (20b)$$

for all $i \in \{1,\dots,m\}$. Then, control law (19) *renders the plant* (1) *GES with convergence rate $\hat{\alpha}$ and gain $\hat{c} = \overline{a}/\underline{a}$, with $\overline{a}$ and $\underline{a}$ as in* (18).

Proof: The proof can be found in [10], [11]. ■

Note that it is always possible, given stabilisability of the pair (A,B), to find a matrix P satisfying the hypotheses of Lemma IV.1, and constructive methods to obtain a matrix P

are given in [6], [7]. The second step in the procedure is to show that a matrix P satisfying the conditions of Lemma IV.1, renders the plant (1) with ZOH (2) GES in case the auxiliary control law is given, for all $k \in \mathbb{N}$, by

$$\begin{cases} \hat{u}_k = Kx(t_k) \\ t_{k+1} = t_k + h \end{cases} \tag{21}$$

provided that $h > 0$ is well chosen.

Lemma IV.2 *Suppose the conditions of Lemma IV.1 are satisfied. Then, for each $\alpha > 0$ satisfying $\alpha < \hat{\alpha}$, the system given by* (1), (2) *and* (21) *is GES with convergence rate α and gain $c = \bar{c}(\alpha, \|K\|, h)$ as in* (9)*, for all $h < h_{\max}(\alpha)$ with*

$$h_{\max}(\alpha) = \min\big\{\hat{h} > 0 \,\big|\, \|Pe^{A\hat{h}}(P^\top P)^{-1}P^\top + P\textstyle\int_0^{\hat{h}} e^{As}\mathrm{d}sBK(P^\top P)^{-1}P^\top\|_\infty > e^{-\alpha\hat{h}}\big\}. \tag{22}$$

Proof: The proof can be found in [10], [11]. ■

Using the matrix P and the function $h_{\max}(\alpha)$ obtained from Lemmas IV.1 and IV.2, we can now formally state the conditions under which the proposed solution to the MAC problem is well defined and how to achieve a desired convergence rate α and a desired gain c.

Theorem IV.3 *Assume there exist matrices $P \in \mathbb{R}^{m\times n_x}$, $K \in \mathbb{R}^{n_u\times n_x}$, and a scalar $\hat{\alpha} > 0$ satisfying the conditions of Lemma IV.1, and let $0 < \alpha < \hat{\alpha}$ and $c > \hat{c}$. If the control gain bound β satisfies $\beta \geqslant \|K\|_\infty$ and the set $\mathcal{H} := \{\hbar_1, \dots, \hbar_L\}$, $L \in \mathbb{N}$, is such that $\hbar_1 < h_{\max}(\alpha)$ as in* (22)*, and $c \geqslant \bar{c}(\alpha, \beta, \Delta_\hbar, \hbar_L)$ as in* (13)*, then the MAC law* (3)*, with* (8)*,* (14)*,* (15)*,* (16) *and* (17)*, is well defined and renders the plant* (1) *with ZOH* (2) *GES with the convergence rate α and the gain c.*

Proof: The proof can be found in [10], [11]. ■

This theorem formally shows how to choose the scalar β, and the set $\mathcal{H}$ to make the proposed solution to the MAC problem well defined and how to achieve a desired convergence rate α and a desired gain c.

V. MAKING THE SOLUTION TO THE MAC PROBLEM COMPUTATIONALLY TRACTABLE

As a final step in providing a complete solution to the MAC problem, we will now propose a computationally efficient algorithm to compute the control inputs generated by the MAC law using online optimisation. To do so, note that the ∞-norm-based CLFs as in (17) allow us to rewrite (8) as

$$f(x,u,h,\alpha) = \big\|Pe^{Ah}x + \textstyle\int_0^h Pe^{As}B\mathrm{d}s\,u\big\|_\infty - e^{-\alpha h}\|Px\|_\infty. \tag{23}$$

We can now observe that the constraint $f(x,u,h,\alpha) \leqslant 0$ that appears in (15) is equivalent to $\bar{f}(x,u,h,\alpha) \leqslant 0$, where

$$\bar{f}(x,u,h,\alpha) := \begin{bmatrix} Pe^{Ah}x + P\int_0^h e^{As}\mathrm{d}sBu \\ -Pe^{Ah}x - P\int_0^h e^{As}\mathrm{d}sBu \end{bmatrix} - e^{-\alpha h}\|Px\|_\infty \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \tag{24}$$

and the inequality is assumed to be taken elementwise, which results in $2m$ linear scalar constraints for u.

Equation (24) reveals that ∞-norm-based CLFs convert the considered problem into a feasibility problem with linear constraints, allowing us to propose an algorithmic solution to the MAC problem. The algorithm is based on solving the maximisation that appears in (16) by incrementally increasing $\bar{L}$.

Algorithm V.1 Let the matrix $P \in \mathbb{R}^{m\times n_x}$, the scalars $\alpha, \beta > 0$, and the set $\mathcal{H} := \{\hbar_1, \dots, \hbar_L\}$, satisfying the conditions of Theorem IV.3, be given. At each t_k, $k \in \mathbb{N}$, given state $x(t_k)$:

1) $l := 0$
2) $\mathcal{U}_0^{\mathrm{MAC}} := \{u \in \mathbb{R}^{n_u} \mid \begin{bmatrix} u \\ -u \end{bmatrix} - \beta\|x(t_k)\|_\infty \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} \leqslant 0\}$
3) While $\mathcal{U}_l^{\mathrm{MAC}} \neq \emptyset$, and $l < L$
   - $\mathcal{U}_{l+1}^{\mathrm{MAC}} := \mathcal{U}_l^{\mathrm{MAC}} \cap \{u \in \mathbb{R}^{n_u} \mid \bar{f}(x(t_k), u, \hbar_{l+1}, \alpha) \leqslant 0\}$
   - $l := l + 1$
4) If $l = L$ and $\mathcal{U}_L^{\mathrm{MAC}} \neq \emptyset$, take $\hat{u}_k \in \mathcal{U}_L^{\mathrm{MAC}}$, and $t_{k+1} = t_k + \hbar_L$
5) Or else, if $\mathcal{U}_l^{\mathrm{MAC}} = \emptyset$, take $\hat{u}_k \in \mathcal{U}_{l-1}^{\mathrm{MAC}}$, and $t_{k+1} = t_k + \hbar_{l-1}$.

Remark V.2 Since verifying that $\mathcal{U}_l^{\mathrm{MAC}} \neq \emptyset$, for some $l \in \{1, \dots, L\}$, is a feasibility test for linear constraints, the algorithm can be efficiently implemented online using existing solvers for linear programs.

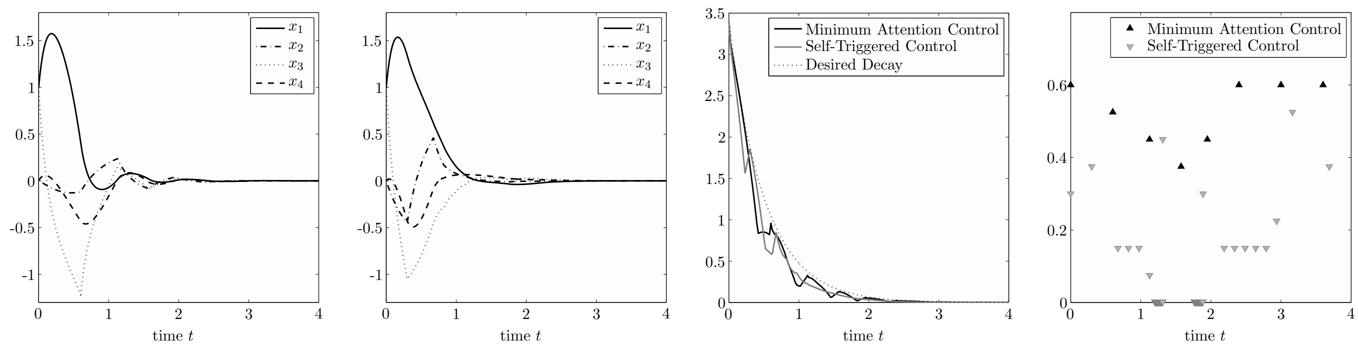
VI. ILLUSTRATIVE EXAMPLES

In this section, we illustrate the presented theory using a well-known example in the NCS literature, see, e.g., [12], consisting of a linearised model of a batch reactor. The details of the linearised model of the batch-reactor model and the controller can be found in the aforementioned reference. In order to solve the MAC problem, we need a suitable CLF. To obtain such a CLF, we use the results from Section IV and use an auxiliary control law (19), with

$$K = \begin{bmatrix} 0.0360 & -0.5373 & -0.3344 & -0.0147 \\ 1.6301 & 0.5716 & 0.8285 & -0.2821 \end{bmatrix} \tag{25}$$

yielding that the eigenvalues $A + BK$ are all real valued, distinct and smaller than or equal to -2. This allows us to find a Lyapunov function of the form (17) using Lemma IV.1, with P being the inverse of the matrix consisting of the eigenvectors of $A + BK$, Q being a diagonal matrix consisting of the eigenvalues of $A + BK$, $\hat{\alpha} = 2$ and $\hat{c} \approx 23.9$. This Lyapunov function will serve as an eCLF used in the MAC problem.

Given this eCLF, we can solve the MAC problem using Algorithm V.1. Before doing so, we use the result of Theorem IV.3 to guarantee that the MAC law is well defined and renders the closed-loop system GES with desired convergence rate $\alpha = 0.98\hat{\alpha} = 1.96$ and desired gain $c = 4\hat{c} \approx 95.7$. According to Theorem IV.3, this convergence



(a) Evolution of the states of the plant using MAC.

(b) Evolution of the states of the plant using self-triggered control.

(c) The decay of the Lyapunov function using MAC and self-triggered control.

(d) The interexecution times using MAC and self-triggered control.

Fig. 1: Figures corresponding to the illustrative example.

rate α and this gain c can be achieved by taking $\beta = \|K\|_\infty \approx 3.1$ and

$$\begin{aligned}\mathcal{H} &= \{\hbar_1, \dots, \hbar_{10}\} \\ &= \{\tfrac{1.5}{1000}, \tfrac{7.5}{100}, \tfrac{15}{100}, \tfrac{22.5}{100}, \tfrac{30}{100}, \tfrac{37.5}{100}, \tfrac{45}{100}, \tfrac{52.5}{100}, \tfrac{60}{100}, \tfrac{67.5}{100}\}, \quad (26)\end{aligned}$$

because it holds that $\hbar_1 < h_{\max}(\alpha)$ and that $\bar{c}(\alpha, \beta, \Delta_\hbar, \hbar_L) \leqslant c$. To implement Algorithm V.1 in MATLAB, we use the routine `polytope` of the MPT-toolbox [13], to create the sets $\mathcal{U}_l^{\text{MAC}}$, to remove redundant constraints and to check if the set $\mathcal{U}_l^{\text{MAC}}$, $l \in \{1, \dots, 10\}$, is nonempty.

When we simulate the response of the plant with the resulting MAC law for the initial condition $x(0) = [1\;0\;1\;0]^\top$, we can observe that the closed-loop system is indeed GES, see Figure 1a, and satisfies the required convergence rate α, see Figure 1c. To show the effectiveness of the theory, we compare our results with the self-triggered control strategy in the spirit of [3], however tailored to work with ∞-norm-based Lyapunov functions, resulting (by using the notation used in this paper) in a control law (2) with $\hat{u}_k = Kx(t_k)$, and $t_{k+1} = t_k + \hbar_{\bar{L}(x(t_k))}$, where

$$\begin{aligned}\bar{L}(x(t_k)) = \max\{\hat{L} \in \{1, \dots, L\} \mid \\ f(x(t_k), Kx(t_k), \hbar_l, \alpha) \leqslant 0 \ \forall\, l \in \{1, \dots, \hat{L}\}\}. \quad (27)\end{aligned}$$

To illustrate that also this control strategy renders the plant (1) GES, we show the response of the plant to the initial condition $x(0) = [1\;0\;1\;0]^\top$ in Figure 1b, and the decay of the Lyapunov function in Figure 1c. Note that the decay of the Lyapunov function for MAC is comparable to the decay of the Lyapunov function for self-triggered control. However, when we compare the resulting interexecution times as depicted in Figure 1d, we can observe that the MAC yields much larger interexecution times. Hence, from a resource utilisation point of view, the proposed MAC outperforms the self-triggered control law.

VII. Conclusion

In this paper, we proposed a novel way to solve the minimum attention control problem. Instrumental for the solutions is a novel extension to the notion of a control Lyapunov function. We solved the control problem by focussing on linear plants, by considering only a finite number of possible intervals between two subsequent executions of the control task and by choosing the extended control Lyapunov function (eCLF) to be ∞-norm-based, which allowed the control problem to be formulated as a linear program. We provided a technique to obtain suitable eCLFs that renders the MAC problem solvable with a guaranteed upper bound on the attention, i.e., a lower bound on the inter-execution times. We illustrated the theory using a numerical example, showing that the proposed methodology outperforms a self-triggered control strategy that is available in the literature.

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