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Reference Governors for Linear Systems with Nonlinear Constraints

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***Abstract*— The paper considers the application of reference governors to linear discrete-time systems with constraints on state and output variables given by functional inequalities. Properties and on-line computational procedures which facilitate the implementation of the reference governors for linear systems with convex, convex quadratic, concave, and mixed logical dynamic constraints of *if-then* type are discussed. Two examples are considered. The first example is a three dimensional spacecraft rendezvous and proximity maneuvering problem where constraints are imposed on thrust magnitude, velocity of approach, collision avoidance, and positioning within the line-of-sight cone. While the spacecraft relative motion dynamics in this problem can be treated as linear, two of the constraints in this problem are convex quadratic and one of the constraints is of mixed logical dynamic type. Our second example is an electromagnetically actuated mass spring damper that has linear dynamics, a linear constraint on position, and a concave nonlinear constraint on the force due to nonlinear dependence of force on the distance. The reference governor based on the developments in this paper can elegantly and effectively handle the prescribed constraints.**

I. INTRODUCTION

Reference governors are a class of predictive control algorithms for modifying reference commands to closed-loop systems in order to avoid violation of pointwise-in-time state and control constraints. Referring to Figure 1, the reference governor modifies the reference command, $r(t)$, to a virtual reference command, $v(t)$, whenever it becomes necessary to enforce constraints on closed-loop variables. Set-bounded disturbances or uncertainties, $w(t)$, can also be considered, however, in this paper, to simplify the exposition, we assume that $w(t) = 0$.

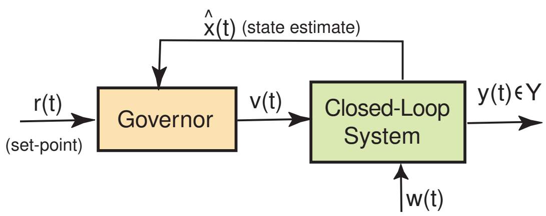

Fig. 1. Reference governor applied to a closed-loop (plant and controller) system subject to constraints.

Reference governors were first proposed as continuous-time algorithms [1] and afterwards developed in a discrete-time framework [2], [3]. Alternative formulations and extensions to linear systems with disturbances and to nonlinear systems have appeared in [4], [5], [6], [7], [8], [9], and references therein.

The reference governors are add-on schemes that, whenever possible, preserve the response of the nominal controller designed by conventional control techniques. In addition, the reference governors lend themselves to computationally efficient implementation for both linear and nonlinear systems subject to disturbances and parameter uncertainties.

Approaches to reference governor design for nonlinear systems exist [7], [10], [11], where the use of on-line simulations or sub-level sets of Lyapunov functions to guard against constraint violation is advocated. A special class of problems that has not received much detailed attention is when the system dynamics are *linear* but the output constraints are specified by requiring that a set of *nonlinear* functional inequalities be satisfied. It is a class of these problems that is considered in this paper.

To motivate the above, we note that control constraints may impede effective implementation of controllers based on feedback linearization where nonlinear dynamics are rendered linear by a coordinate transformation and an appropriately defined feedback law. After the transformation of the dynamics into the linear form, linear constraints on the control input (now a function of the state as determined by this feedback linearizing control law) become nonlinear.

In the paper, the case of convex constraints is considered first (Section II-B), followed by treating a more special case of convex quadratic constraints (Section II-C). The main developments in these sections relate to computing the reference governor parameter $\beta(t) \in [0, 1]$ that links $v(t)$ to $r(t)$,

$$v(t) = v(t-1) + \beta(t)(r(t) - v(t-1)). \quad (1)$$

The reference governor functions by maximizing the value of $\beta(t)$ in (1) subject to $\beta(t)$ being constraint-admissible. We show that, in the case of *convex* constraints, a typical situation is when the constraint-admissible values of $\beta(t)$ form a proper interval $[0, \beta_{max}(t)] \subseteq [0, 1]$. The value of $\beta_{max}(t)$ can be computed using bisections (or other root finding procedures). In the case when the constraints are convex and quadratic, β_{max} can be computed by solving a simple quadratic equation. A similar approach of characterizing the constraint-admissible values of $\beta(t)$ and showing that they form a proper interval is then applied in the case of mixed logical dynamic constraints of *if-then* type (Section II-D). Finally, concave nonlinear constraints are considered by approximating them with dynamically reconfigurable linear constraints (Section II-E).

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Furthermore, an application to a spacecraft rendezvous and proximity maneuvering problem is considered (Section III). While the model for the relative spacecraft motion in this problem is linear, the constraints on thrust magnitude, line-of-sight cone positioning and velocity of approach are of quadratic and of *if-then* type. The reference governor algorithm developed based on the results in this paper is shown to successfully complete a docking maneuver with another spacecraft moving around the Earth in a circular orbit.

Finally, an application to an electromagnetically actuated mass-spring-damper-system is considered (Section IV). The constraint on the maximum force applied by the electromagnet is shown to be concave and is handled using approximations with dynamically reconfigurable linear constraints.

II. Reference Governor for Nonlinear Constraints

We consider an application of reference governors to discrete-time linear systems,

$$\begin{aligned} x(t+1) &= Ax(t) + Bv(t), \\ y(t) &= Cx(t) + Dv(t), \end{aligned} \tag{2}$$

where x is the n-vector state, v is the m-vector reference governor output, and y is the p-vector system output. The system (2) typically represents the combined closed-loop dynamics of the plant and of the controller.

The state and control constraints can be represented by constraints on closed-loop variables, $y(t)$, and expressed as,

$$y(t) \in Y \text{ for all } t \in \mathbb{Z}^+, \tag{3}$$

where $\mathbb{Z}^+$ denotes the set of non-negative integers and Y is a prescribed set. In this paper, we consider the case of Y being specified by nonlinear functional inequalities, *e.g.*,

$$Y = \{y : \ h_i(y) \le 0, \ i = 1, \dots, r\}, \tag{4}$$

where h_i are given functions.

For (2) and (4), the maximum constraint admissible set, $O_\infty(Y)$, is defined as the set of all safe constant reference governor outputs, $v(t) \equiv \rho$, and initial states, $x(0)$, *i.e.*,

$$\begin{aligned} O_\infty(Y) = \{(\rho, x(0)) : y(t|0) \in Y, \\ v(t) = v(0) = \rho, \ \forall t \in \mathbb{Z}^+\}. \end{aligned} \tag{5}$$

Here, $y(t+k|t)$ denotes the predicted output response of (2) k steps ahead with $v(t+k) \equiv v(t)$ for $k \in \mathbb{Z}^+$.

In many cases, $O_\infty(Y)$ has a finite determination property, *i.e.*, it is defined by a finite number of inequalities while similar inequalities end up being redundant for all $t > t^*$. Under appropriate assumptions [3], [6] such that A is Schur, (A, C) is observable, Y is compact, $0 \in intY$, and the allowed range of $v(t)$ is slightly tightened (by auxiliary constraints if necessary) versus the steady-state admissible range, such a t^* is guaranteed to exist, and can be computed or estimated offline, particularly for the convex case. Informally, t^* (the constraint horizon) is such that if the constraints are not violated up to time t^* for a constant $v(t+k) \equiv v(t)$, they are not violated for all future times. Any upper bound on t^* may also be used in place of t^*.

The reference governor maximizes the value of $\beta(t) \in [0, 1]$ subject to the constraints $(v(t), x(t)) \in O_\infty(Y)$. Equivalently, the constraints involved in the optimization of $\beta(t)$ can be expanded as follows,

$$\begin{gathered} \beta(t) \in [0, 1], \\ y(t+k|t) \in Y, \\ v(t+k) = v(t-1) + \beta(t)(r(t) - v(t-1)), \\ k = 0, \dots, t^*. \end{gathered} \tag{6}$$

Ideally, we want the set, Y, in (3) to be a polyhedron, in which case an *explicit* formula for the parameter $\beta(t)$ can be given [6].

We note that the existing results in [3], [6] for treating the constrained problem for system (2) suject to constraints (4) hold for any compact, convex set Y with $0 \in intY$. When Y is not polyhedral, it can be approximated by a polyhedron. Such approximations may not be easy to obtain or accurate (especially when y has multiple dimensions) and it can lead to many linear inequalities and significant on-line computational effort. The approach taken in this paper is to use the linear model (2) to predict the output response but treat the functions $h_i(y)$ directly as being nonlinear, thereby avoiding the need for approximation by polyhedral constraints. Results in [7] apply to the case of linear systems with nonlinear constraints and can be used to guarantee finite time convergence of $v(t)$ to $r(t)$ for several classes of $r(t)$. This is a desirable property indicating that after transients caused by large changes in $r(t)$, the reference governor becomes inactive and nominal closed-loop system performance is recovered. In this paper we therefore focus on issues pertinent to the reference governor implementation for several different classes of nonlinear constraints.

A. Output prediction

We need to predict the state and output of (2) at time, $t+k$, given the state, $x(t)$, at time, t, and assuming a constant $v(t+k) = v(t)$ for $k \ge 0$. We define,

$$\begin{aligned} \Psi &\triangleq (I - A)^{-1}B, \\ \Gamma_x^k &\triangleq CA^k, \\ \Gamma_v^k &\triangleq (C - \Gamma_x^k)\Psi + D. \end{aligned}$$

Then the predicted output k steps ahead of the current time, t, can be expressed using the state transition formula for linear discrete-time systems as,

$$y(t+k|t) = \Gamma_x^k x(t) + \Gamma_v^k v(t). \tag{7}$$

With (1), it follows that,

$$\begin{aligned} &y(t+k|t) \\ &\quad = \Gamma_x^k x(t) + \Gamma_v^k (v(t-1) + \beta(t)(r(t) - v(t-1))), \\ &\quad = \Gamma_x^k x(t) + \Gamma_v^k v(t-1) + \beta(t)\Gamma_v^k (r(t) - v(t-1)). \end{aligned} \tag{8}$$

B. Convex constraints

Suppose that h_i, $i = 1, \dots, r$, in (4) are convex functions satisfying,

$$h_i(\alpha y_1 + (1-\alpha)y_2) \leq \alpha h_i(y_1) + (1-\alpha)h_i(y_2)$$
$$\forall i = 1, \dots, r, \ \forall y_1, y_2 \in \mathbb{R}^p, \ 0 \leq \alpha \leq 1. \quad (9)$$

Consider now the values of $\beta(t)$ in (8) such that $h_i(y(t+k|t)) \leq 0$. By convexity of h_i and linearity of $y(t+k|k)$ in $\beta(t)$, it follows that $h_i(y(t+k|t))$ with $y(t+k|t)$ given by (8) is a convex function of $\beta(t) \in [0,1]$. This in turn shows that the set of allowed values for $\beta(t)$ is either empty or is a connected interval. In what follows, let the set be denoted by

$$K_i^k = [\beta_{i,\min}^k, \beta_{i,\max}^k] \subseteq [0,1]. \quad (10)$$

By intersecting the intervals K_i^k for all $k = 0, \dots, t^*$ and $i = 1, \dots, r$, we obtain an admissible interval for the values of $\beta(t)$, which is denoted by $K(t) = [\beta_{\min}(t), \beta_{\max}(t)]$.

The reference governor guarantees that the output response with $v(t+k) = v(t)$ satisfies the imposed constraints, and hence it guarantees the *recursive feasibility* of $\beta(t) = 0$, *i.e.*, if $v(-1)$ can be chosen for the given $x(0)$, so that $\beta(0) = 0$ at the time instant 0, then there exist a feasible choice for $\beta(t)$, namely $\beta(t) = 0$ for $t \geq 0$. This leads to the following result.

Proposition 1: If $h_i, \ i = 1, \dots, r$ are convex and $\beta(0) = 0$ is feasible at the initial time 0, then an admissible interval for the values of $\beta(t)$ is of the form $K(t) = [0, \beta_{\max}(t)]$, and the reference governor sets $\beta(t) = \beta_{\max}(t)$, $0 \leq \beta_{\max}(t) \leq 1$.

Proposition 1 leads to an easily implementable algorithm to determine $\beta(t)$. Set $\alpha = 1$. For $i = 1, \dots, r$ and $k = 0, \dots, t^*$, repeat: If $h_i(y(t+k|t)) > 0$ where $y(t+k)$ is given by (8) with $\beta(t) = \alpha$ use bisections to find a scalar α^+ on the interval $[0, \alpha]$ such that $h_i(y(t+k|t)) = \Gamma_x^k x(t) + \Gamma_v^k v(t-1) + \beta(t)\Gamma_v^k(r(t) - v(t-1)) = 0$ with $\beta(t) = \alpha^+$. Update $\alpha = \alpha^+$. Typically, only a few bisections need to be performed. Re-ordering the $r \times t^*$ constraints to firstly evaluate the ones active at the previous time instant can practically speed up the computations.

C. Convex Quadratic constraints

Further simplifications in calculating K_i^k occur if h_i are convex quadratic constraints of the form,

$$y^{\mathrm{T}} \tilde{Q} y + \tilde{S} y + \tilde{C} \leq 0, \quad (11)$$

where $\tilde{Q} = \tilde{Q}^{\mathrm{T}} \geq 0$.[1]

When constraints are of the form in (11), the algorithm for determining K_i^k reduces to a simple and explicit formula which is derived from the solution of the quadratic equation.

Suppressing times, superscripts, and subscripts (t, k, and i) for readability, we define,

$$\bar{Q} \triangleq \begin{bmatrix} \Gamma_x^{\mathrm{T}} \tilde{Q} \Gamma_x & \Gamma_x^{\mathrm{T}} \tilde{Q} \Gamma_v \\ \Gamma_v^{\mathrm{T}} \tilde{Q} \Gamma_x & \Gamma_v^{\mathrm{T}} \tilde{Q} \Gamma_v \end{bmatrix}, \quad (12)$$

$$\bar{S} \triangleq \begin{bmatrix} \tilde{S}\Gamma_x & \tilde{S}\Gamma_v \end{bmatrix}. \quad (13)$$

[1]The case where $\tilde{Q} = 0$ is treated more specifically in [3].

In determining feasible values of $\beta = \beta(t)$, *i.e.*, the interval $K = K(t)$, we can set the left hand side of (11) to zero,

$$y^{\mathrm{T}} \tilde{Q} y + \tilde{S} y + \tilde{C} = 0, \quad (14)$$

or

$$\begin{bmatrix} x^{\mathrm{T}} & v^{\mathrm{T}} + \beta(r-v)^{\mathrm{T}} \end{bmatrix} \bar{Q} \begin{bmatrix} x \\ v + \beta(r-v) \end{bmatrix} + \bar{S} \begin{bmatrix} x \\ v + \beta(r-v) \end{bmatrix} + \tilde{C} = 0. \quad (15)$$

Expanding and collecting, this results in a quadratic equation in β. Specifically, define

$$\tilde{q} \triangleq \begin{bmatrix} 0 & (r-v)^{\mathrm{T}} \end{bmatrix} \bar{Q} \begin{bmatrix} 0 \\ r-v \end{bmatrix},$$
$$\tilde{s} \triangleq \left(2 \begin{bmatrix} x^{\mathrm{T}} & v^{\mathrm{T}} \end{bmatrix} \bar{Q} + \bar{S}\right) \begin{bmatrix} 0 \\ r-v \end{bmatrix},$$
$$\tilde{c} \triangleq \begin{bmatrix} x^{\mathrm{T}} & v^{\mathrm{T}} \end{bmatrix} \bar{Q} \begin{bmatrix} x \\ v \end{bmatrix} + \bar{S} \begin{bmatrix} x \\ v \end{bmatrix} + \tilde{C}.$$

Then (15) reduces to a quadratic equation in β, i.e.,

$$\beta^2 \tilde{q} + \beta \tilde{s} + \tilde{c} = 0. \quad (16)$$

If $\tilde{q} \neq 0$, the solution to (16) is,

$$K = \left[\frac{-\sqrt{\tilde{s}^2 - 4\tilde{q}\tilde{c}} - \tilde{s}}{2\tilde{q}}, \frac{\sqrt{\tilde{s}^2 - 4\tilde{q}\tilde{c}} - \tilde{s}}{2\tilde{q}} \right] \bigcap [0,1], \quad (17)$$

and otherwise, if $\tilde{s} \neq 0$, the solution is,

$$K = \left] -\infty, -\frac{\tilde{c}}{\tilde{s}} \right] \bigcap [0,1] \text{ or } \left[-\frac{\tilde{c}}{\tilde{s}}, \infty \right[\bigcap [0,1]. \quad (18)$$

Otherwise, $K = \emptyset$ or $[0,1]$.

D. Mixed Logical Dynamic Constraints

We consider a set of constraints of the *if-then* type,

$$g_i(y) > 0 \rightarrow h_i(y) \leq 0, \quad i = 1, \dots, r, \quad (19)$$

where g_i and h_i are convex functions. Consistently with [8], we refer to the constraint (19) as a mixed logical dynamic (MLD) constraint, since this constraint depends on the output variables that change dynamically.

The treatment of (19) relies on the observation that the set of $\beta(t) \in [0,1]$ for which $g_i(y(t+k|t)) \leq 0$ where $y(t+k|t)$ is given by (8) is a (possibly empty) interval $\hat{K}_i^k \subseteq [0,1]$ and the set of $\beta(t) \in [0,1]$ for which $h_i(y(t|t+k)) \leq 0$ is another (possibly empty) interval $K_i^k \subseteq [0,1]$. Then the set of $\beta(t)$ for which (19) is satisfied with $y = y(t+k|k)$ given by (8) is also an *interval,* $\tilde{K}_i^k = [\beta_{i,\min}^k, \beta_{i,\max}^k]$,

$$\tilde{K}_i^k = [0,1] \cap \left(\hat{K}_i^k \cup \left(\hat{K}_i^{kC} \cap K_i^k \right) \right), \quad (20)$$
$$= [0,1] \cap \hat{K}_i^k \cap K_i^k \quad (21)$$

where $\hat{K}_i^{kC}$ is the complement of $\hat{K}_i^k$. Assuming that the recursive feasibility of $\beta(t) = 0$ is preserved by the reference governor, it follows that $\beta_{i,\min}^k = 0$ and $\beta(t) = \min_{k=0,\dots,t^*, i=1,\dots,r} \quad \beta_{i,\max}^k$.

While handling other classes of MLD constraints with the reference governor is of interest, the confinement of the values of $\beta(t)$ to a single connected interval, which considerably simplifies the treatment and the computations, appears to be a unique property of *if-then* constraints.

E. Concave Constraints

Suppose that the constraints are defined by (3) and (4), with h_i being *concave* functions. In this case, we approximate the constraints $y(t+k|t) \in Y$ by the *affine* (and therefore convex) constraints,

$$y(t+k|t) \in Y_c(t),$$

where

$$Y_c(t) = \{y : h_i(y_{i,*}(t)) + h_i'(y_{i,*}(t))(y - y_{i,*}(t)) \leq 0\},$$
$$i = 1, \ldots, r, \qquad (22)$$

and $Y_c(t) \subset Y$. Here $y_{i,*}(t)$ can depend on t or $x(t)$ so that the linear constraints in (22) are dynamically reconfigurable online. Since h_i are concave functions, $y(t+k|t) \in Y_c(t)$ implies that $y(t+k|t) \in Y$. In addition to computing $\beta(t)$, we now also need to compute $y_{i,*}(t)$. We note that this approach guarantees the recursive feasibility.

Proposition 2: If $y_{i*}(0)$, $i = 1, \ldots, r$ exist such that $\beta(0) = 0$ is feasible, then $\beta(t) = 0$ and $y_{i,*}(t) = y_{i,*}(t-1)$ are feasible for $t > 0$.

While the constraints can be satisfied using the reference governor, the conditions guaranteeing the convergence of $v(t)$ to $r(t)$ for a constant $r(t)$ appear to be considerably more involved.

III. Example 1: Satellite Rendezvous and Proximity Maneuvering

We use an example of spacecraft rendezvous and proximity maneuvering to illustrate the reference governor capability to handle nonlinear convex quadratic constraints and constraints of mixed logical dynamic type. While the constraints are nonlinear, the use of a linear model to represent the spacecraft relative motion dynamics is standard [12], [13]. In our previous work [14], various approximations had to be employed to deal with the same constraints as in this paper, while using computationally effective linear quadratic MPC solutions. The need to make these approximations is avoided altogether with the reference governor, while the nominal unconstrained control strategy need not be replaced by a new controller.

A. Problem formulation

Let there be two spacecraft; one "Chief" and one "Deputy", with the Deputy performing a rendezvous with the Chief, while the Chief orbits around the Earth along the circular orbit. In this problem, we attach the non-inertial Hill frame to the Chief, in which the 1-2-3 axes point respectively in the radial direction away from earth, the along-track direction towards the Chief's motion, and in the cross-track direction towards the Chief's angular momentum. Linearizing and not considering perturbation effects, the discrete Hill-Clohessy-Wiltshire (HCW) equations,

$$x(t+1) = A_0 x(t) + B_0 u(t), \qquad (23)$$

describe the motion of the Deputy in the Hill frame [15], where $x(t) = \begin{bmatrix} x_1 & x_2 & x_3 & \dot{x}_1 & \dot{x}_2 & \dot{x}_3 \end{bmatrix}^{\mathrm{T}}$ is the state vector of the Deputy's positions and velocities in the 3 axes, and $u(t) = \begin{bmatrix} F_1 & F_2 & F_3 \end{bmatrix}^{\mathrm{T}}$ is the vector of thrust forces with entries corresponding to the axes.

Since the reference governor is an add-on control algorithm, we first design a feedback linear-quadratic regulator (LQR), K, to control the Deputy in the Hill frame. For our solution, we choose our Q and R cost matrices to be $Q = \mathrm{diag}(100, 1, 100, 0, 0, 0)$ and $R = I$, penalizing the 1- and 3-directions more than the 2-direction, in which the Deputy approaches the Chief's "dock". We further introduce a feed-forward gain, G, so that $v(t) \in \mathbb{R}^3$ becomes the reference position of the Deputy, with $u(t) = Gv(t)$. The closed loop dynamics are,

$$x(t+1) = (A_0 + BK)x(t) + B_0 G v(t) \qquad (24)$$
$$= \hat{A}x(t) + \hat{B}v(t). \qquad (25)$$

Furthermore, we define the C and D matrices such that the output consists of all the states and reference inputs,

$$y(t) = Cx(t) + Dv(t) = \begin{bmatrix} I \\ 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ I \end{bmatrix} v(t). \qquad (26)$$

We are now able to use the reference governor theory to satisfy constraints. The first is the line-of-sight (LOS) constraint or the requirement that the Deputy stay within a half-cone in the along-track direction so that the Chief can visually detect it. This is described as a half-cone with its center 1m behind the docking point, and within a 15° half-angle. In convex quadratic form, this is written as,

$$h_1(y) = x_1^2 + x_3^2 - \tan^2 15^\circ (x_2 + 1)^2 \qquad (27)$$
$$= x_1^2 - (7 - 4\sqrt{3})x_2^2 + x_3^2$$
$$- 2(7 - 4\sqrt{3})x_2 - (7 - 4\sqrt{3}) \leq 0 \qquad (28)$$

The second constraint is that the Deputy must always stay in front of the docking point in the along-track direction,

$$h_2(y) = -x_2 \leq 0. \qquad (29)$$

The third constraint is that of thrust limitation and is of convex quadratic type: the maximum force allowed is 4N,

$$h_3(y) = u^{\mathrm{T}} u - 4^2 \qquad (30)$$
$$= (Kx + Gv)^{\mathrm{T}}(Kx + Gv) - 4^2 \qquad (31)$$
$$= \begin{bmatrix} x^{\mathrm{T}} & v^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} K^{\mathrm{T}}K & K^{\mathrm{T}}G \\ G^{\mathrm{T}}K & G^{\mathrm{T}}G \end{bmatrix} \begin{bmatrix} x \\ v \end{bmatrix} - 4^2 \leq 0. \qquad (32)$$

We assume that the spacecraft is reoriented instantaneously to provide the thrust vector. The thrust magnitude is realized through the modulation of thruster on/off times [13].

The final constraint is MLD. If the spacecraft approaches to within 1m of the dock in the along-track direction, then its speed must be less than 0.1m/s.

$$g_4(y) = -x_2 + 1 > 0 \rightarrow$$
$$h_4(y) = \dot{x}_1^2 + \dot{x}_2^2 + \dot{x}_3^2 - 0.1^2 \leq 0. \quad (33)$$

B. Simulation Results

In this section, we present an example that uses the above algorithms and guarantees adherence of the system to all constraints.

Using a sampling period of $h = 0.1$s, the constraint-admissible initial conditions for the Deputy are chosen to be (in meters), $x(0) = \begin{bmatrix}100 & 1000 & 200 & 0 & 0 & 0\end{bmatrix}^{\mathrm{T}}$; these imply that, $u(0) = \begin{bmatrix}100 & 1000 & 200\end{bmatrix}^{\mathrm{T}}$. Setting $r(t) = 0,\ \forall t$, we obtain the desired control dynamics.

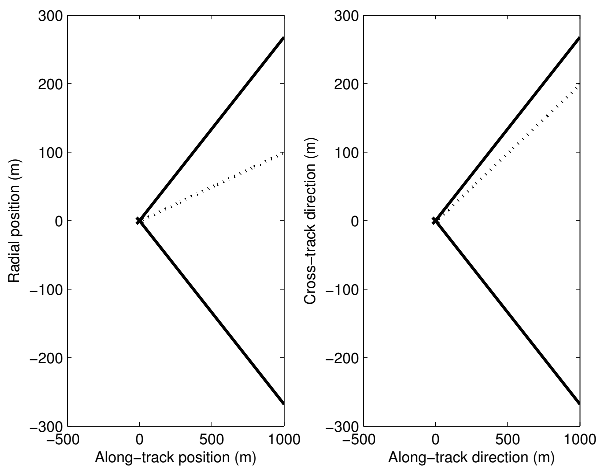

Fig. 2. The trajectory of the Deputy in the 1-2 and 2-3 planes of the Hill frame, resp. (dotted), with constraint boundaries (solid) and docking point (x)

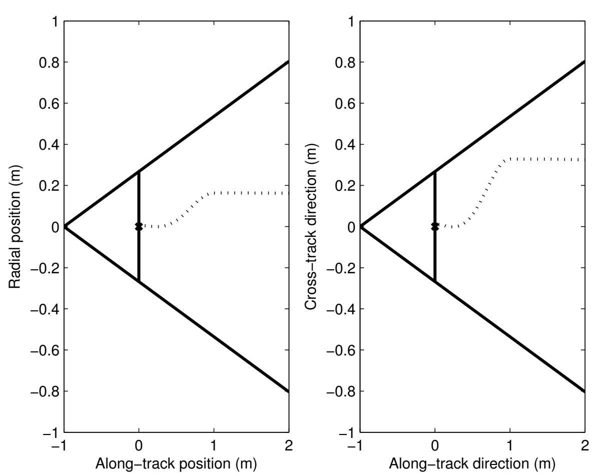

Fig. 3. The close-up view of the trajectory of the Deputy in the 1-2 and 2-3 planes of the Hill frame, resp. (dotted), with constraint boundaries (solid) and docking point (x)

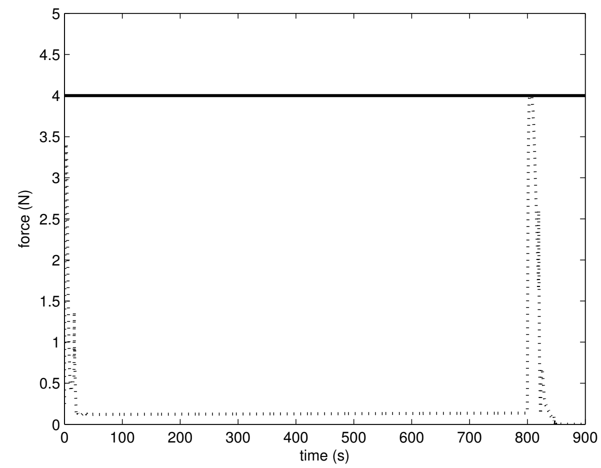

Fig. 4. The thrust force (dotted) and the thrust constraint (solid)

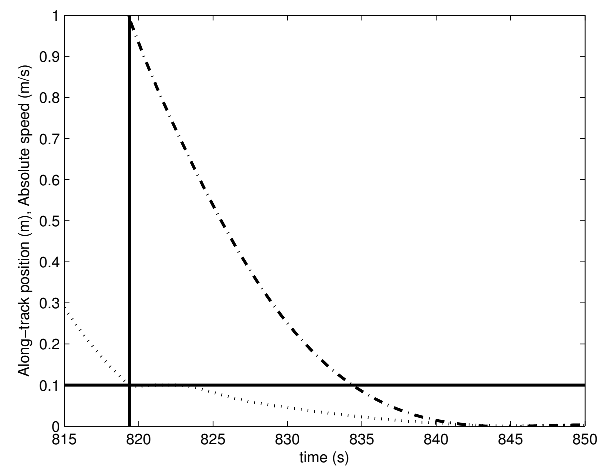

Fig. 5. The along-track position (dot-dashed) and the relative velocity in the Hill frame (dotted) along with soft-docking constraint boundaries (solid)

The resulting trajectory is shown in Fig. 2, showing the Deputy staying within the LOS-cone as it approaches the Chief. Fig. 3 shows the trajectory close up, with the additional constraint that the deputy stay in the positive half-plane in the along-track direction. As it comes close to the Chief, the Deputy moves to the side in order to stay within the LOS-cone and finally perpendicularly docks with the Chief.

The other two constraints are also satisfied, as shown in Figs. 4 and 5. In Fig. 4, the force of the thrust never exceeds 4N and in Fig. 5, when the along-track position is less than 1m (after the 189.4s mark), the Deputy is guaranteed to have a relative velocity of at most 0.1m/s.

IV. Example 2: Electromagentically Actuated Mass-Spring Damper System

In this section, we apply the reference governor to the electromagnetically actuated mass-spring damper system, considered in [16]. A nonlinear reference governor was applied to this example in [7]. Here we demonstrate an alternative treatment of this example using linear system model and nonlinear constraint model.

A. Problem Formulation

The system is described as:

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -k/m & -c/m\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ 1/m\end{bmatrix}u, \tag{34}$$

$$u = \frac{\alpha i^{\mu}}{(d_0 - x_1)^{\gamma}}, \tag{35}$$

where x_1, x_2, and i are the position and velocity of the mass and the applied current, respectively. The rest are parameters with values as in [7]: $\alpha = 4.510^{-5}$, $\mu = 1.92$, $\gamma = 1:99$, $c = 0.6590$, $k = 38.94$, $d_0 = 0.0102$, $m = 1.54$. As in [7], we choose feedback and feedforward gains, $K = [0 \quad -c_d]$ and $G = k$, where $c_d = 4.0$, so that the system is in the form of (24) and $u = kv - c_d x_2$. We delay the choice of the C and D matrices until formulating the constraints.

The first constraint present in the system is that the position of the mass cannot get too close to the actuator,

$$h_1(y) = x_1 - 0.008 \le 0. \tag{36}$$

The other two constraints are related to actuator limitations:

$$0 \le u \le \frac{\alpha i_{\text{max}}^{\mu}}{(d_0 - x_1)^{\gamma}}, \tag{37}$$

where i_{max} is the maximum current available in the electromagnet. The left-hand side of the constraint is a simple linear constraint,

$$h_2(y) = -u = c_d x_2 - kv \le 0. \tag{38}$$

The right-hand side is nonlinear in x_1. To handle this, we linearize the constraint about $(\bar{x}_1, \bar{u})$ at which the constraint is active so that

$$0 \ge (u - \bar{u}) - \frac{\alpha\gamma i_{\text{max}}^{\mu}}{(d_0 - \bar{x}_1)^{\gamma+1}}(x_1 - \bar{x}_1) \tag{39}$$

$$= u - \frac{\alpha i_{\text{max}}^{\mu}}{(d_0 - \bar{x}_1)^{\gamma}} - \frac{\alpha\gamma i_{\text{max}}^{\mu}}{(d_0 - \bar{x}_1)^{\gamma+1}}(x_1 - \bar{x}_1) \tag{40}$$

This approach is visualized in Fig. 6 and satisfies the requirement for Proposition 2 to apply. To deal with this constraint, we define two new variables,

$$\xi_0 \triangleq \frac{\alpha i_{\text{max}}^{\mu}}{(d_0 - \bar{x}_1)^{\gamma}},\ \xi_1 \triangleq \frac{\alpha\gamma i_{\text{max}}^{\mu}}{(d_0 - \bar{x}_1)^{\gamma+1}}. \tag{41}$$

If these, along with $\bar{x}_1$, are treated as constant state variables, then this creates a constraint that is linear with respect to the non-constant states,

$$h_3(y) = -x_1\xi_1 + \bar{x}_1\xi_1 - c_d x_2 - \xi_0 + kv. \tag{42}$$

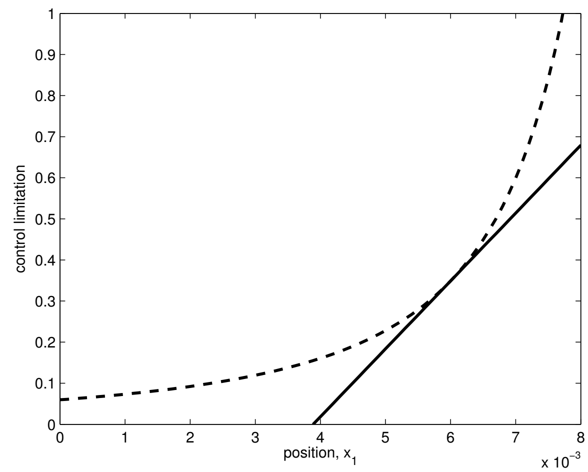

Fig. 6. The upper limit on the control, $u(t)$, with the nonlinear limit (dashed) and the linearized limit (solid) about an equilibrium, 0.006m

B. Simulation Results

Following the example in [7], we choose initial conditions, $x_1(0) = 0$ and $x_2(0) = 0.012$ and a time-step of 0.01s. We then run two simulations, with $i_{\text{max}} = 0.5342$ and with $i_{\text{max}} = 0.365$. The former limit corresponds to that in [7], but the latter is close to the minimum limit that is needed to achieve any equilibrium position within the commanded range in steady-state [17]. The simulations for the two situations, along with the unconstrained case, are presented in Figs. 7-9.

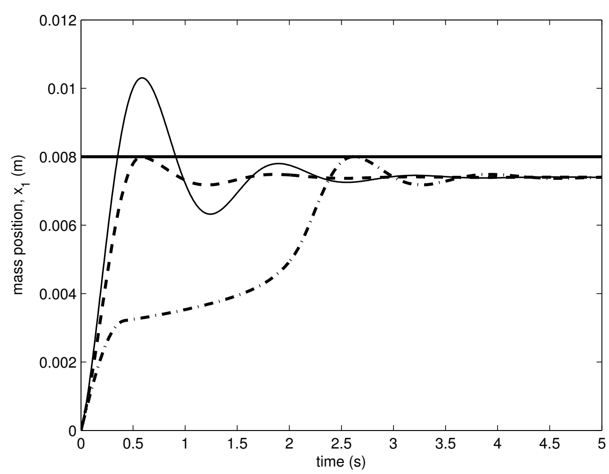

Fig. 7. The mass position response in the three cases: unconstrained (thin-solid), $i_{\text{max}} = 0.5342$ (dashed), and $i_{\text{max}} = 0.365$ (dot-dashed), with constraint shown by a horizontal line.

In Figs. 7 and 8, we see that the reference governor takes two different approaches depending on the current limitation. For the larger limit, it acts similarly to the unconstrained case, the reason for which can be seen in Fig. 9, where $u_{\text{max}}(t)$, the maximum allowed value of the control at time, t, is plotted alongside $u(t)$; the actuator limits in the latter are active for a longer period of time since there is not much

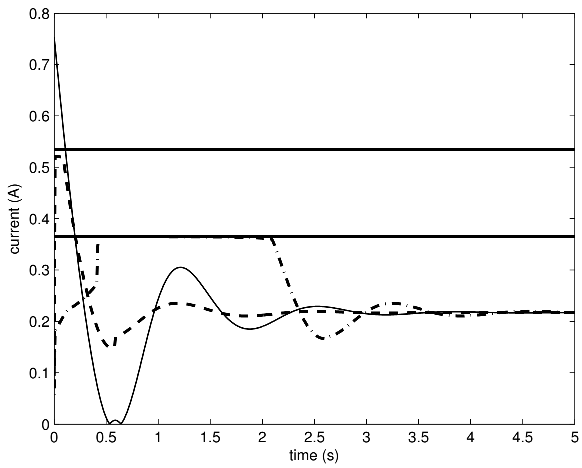

Fig. 8. The current response in the three cases: unconstrained (thin-solid), $i_{\max} = 0.5342$ (dashed), and $i_{\max} = 0.365$ (dot-dashed), with constraints shown by horizontal lines.

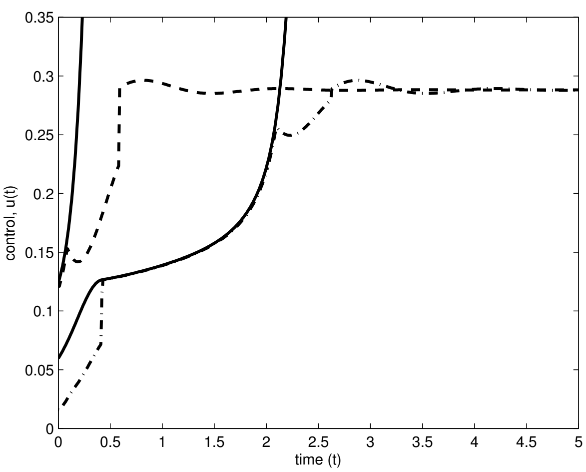

Fig. 9. The control response in two cases: $i_{\max} = 0.5342$ (dashed) and $i_{\max} = 0.365$ (dot-dashed), with respective constraints, $u_{\max}$ (solid)

difference between the available current and the maximum equilibrium current. Furthermore, Fig. 9 shows a seperation between the control limit and the governed input; this is due to sequential linearizations and suggests that decreasing the time-step and therefore more frequent approximations would result in better response.

V. Conclusion

In this paper, we considered an application of the reference governor to linear system models but with nonlinear constraints. This case frequently occurs in practice, including when feedback linearization is used. We discussed different cases: convex, convex quadratic, mixed logical dynamic ("if-then"), and concave constraints. The computations can be arranged elegantly and in the same spirit as the explicit solution [6] for the linear reference governor with linear constraints. Finally, using the developed ideas, we have presented an effective treatment of constraints in the examples of spacecraft rendezvous as well as for an electromagnetically actuated mass spring damper system.

Future work will proceed in two directions: The first direction will be concerned with extending the treatment to the command and extended command governor cases and the second will be to enlarge the classes of nonlinear and mixed logical dynamic constraints that can be effectively treated.

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References

[1] P. Kapasouris, M. Athans, and G. Stein, "Design of Feedback Control Systems for Unstable Plants with Saturating Actuators," *Proc. IFAC IEEE Symp. Nonlinear Control Syst. Design*, Pergamon Press: Oxford, UK, pp. 302–307, 1990.

[2] E. G. Gilbert, I. V. Kolmanovsky and K. T. Tan, "Nonlinear Control of Discrete-Time Linear Systems with State and Control Constraints: A Reference Governor with Global Convergence Properties," in *Proc. 33rd IEEE Conf. Decision Control*, pp. 144–149, 1994.

[3] E. G. Gilbert, I. V. Kolmanovsky, and K. T. Tan, "Discrete-Time Reference Governors and the Nonlinear Control of Systems with State and Control Constraints," *Int. J. Robust Nonlin. Control*, vol. 5, no. 5, pp. 487–504, 1995.

[4] A. Bemporad and E. Mosca, "Constrained Fulfillment in Control Systems via Predictive Reference Management," in *Proc. 33rd IEEE Conf. Decision Control*, pp. 3017–3022, 1994.

[5] A. Bemporad, A. Casavola, and E. Mosca, "Nonlinear Control of Constrained Linear Systems via Predictive Reference Management," *IEEE Trans. Autom. Control*, vol. 42, pp. 340–349, 1996.

[6] E. G. Gilbert and I. V. Kolmanovsky, "Fast Reference Governors for Systems with State and Control Constraints and Disturbance Inputs," *Int. J. Robust Nonlin. Control*, vol. 9, no. 15, pp. 1117–1141, 1999.

[7] E. Gilbert and I. Kolmanovsky, "Nonlinear Tracking Control in the Presence of State and Control Constraints: A Generalized Reference Governor," *Automatica*, vol. 38, no. 12, pp. 2063–2073, 2002.

[8] A. Bemporad and M. Morari, "Control of Systems Integrating Logic, Dynamics and Constraints," *Automatica*, vol. 35, no. 3, pp. 407–427, 1999.

[9] F. Borrelli, P. Falcone, J. Pekar, and G. Stewart, "Reference Governor for Constrained Piecewise Affine Systems," *J. Process Contr.*, vol. 19, no. 8, pp. 1229–1237, 2009.

[10] A. Bemporad, "Reference Governor for Constrained Nonlinear Systems," *IEEE Trans. Autom. Control*, vol. 43, no. 3, pp. 415–419, 1998.

[11] E. G. Gilbert and I. V. Kolmanovsky, "Set-Point Control of Nonlinear Systems with State and Control Constraints: A Lyapunov Function Reference Governor Approach," in *Proc. 39th IEEE Conf. Decision Control*, pp. 2507–2512, 1999.

[12] B. Wie. *Spacecraft Dynamics and Control*, 2nd Ed. Reston, VA: AIAA, 2010.

[13] W. Fehse. *Automated Rendezvous and Docking of Spacecraft*. Cambridge Aerospace Series, 2003.

[14] H. Park, S. Di Cairano, and I. V. Kolmanovsky, "Linear Quadratic Model Predictive Control Approach to Spacecraft Rendezvous and Docking," in *Proc. 21st AAS/AIAA Space Flight Mechanics Meeting*, AAS 11-142, 2011.

[15] W. H. Clohessy and R. S. Wiltshire, "Terminal Guidance System for Satellite Rendezvous," *J. Aerosp. Sci.*, vol. 27, no. 9, pp. 653–658, 1960.

[16] J. Hong, I. A. Cumming, P. D. Washabaugh, and D. S. Bernstein, "Stabilization of a Mass and Spring System with Electromagnetic Actuation," in *Proc. 1997 IEEE Conf. Control Applicat.*, Hartford, CT, pp. 189–194.

[17] R. Miller, I. V. Kolmanovsky, E. G. Gilbert and P. Washabaugh, "Control of Constrained Nonlinear Systems: A Case Study," *IEEE Control Syst. Mag.*, vol. 20, no. 1, pp. 23–32, 2000.