2011 50th IEEE Conference on Decision and Control and
European Control Conference (CDC-ECC)
Orlando, FL, USA, December 12-15, 2011

The Roots of Unity and a Direct Method for the Computation of Stable Internal Positive Representations of Linear Systems

Filippo Cacace, Alfredo Germani, Costanzo Manes

Abstract— **This paper presents a new technique for the construction of Internal Positive Representations (IPRs) of discrete time linear systems. The proposed method overcomes the limitations of a previously proposed technique, which provides stable IPRs of systems under a restrictive assumption on the position of the eigenvalues in the complex plane. The new method here presented exploits a suitable representation of complex vectors and matrices by means of nonnegative combinations of the roots of unity, and provides a stable IPR for any stable system. The position of the eigenvalues in the complex plane only affects the state-space dimension of the IPR.**

I. INTRODUCTION

The interest in the realization problem of dynamic systems by means of positive state-space representations lies in the fact that there are several technological frameworks which allow high-speed implementation of positive systems (Charge Routing Networks [2], [3], [10], fiber optic filters [5]). In [4] the input-output behavior of a class of Single-Input-Single-Output transfer functions is obtained as the difference of two positive filters, and some extensions have been studied in [12], [13]. The computation of the positive filters is based on the quite complicated realization theory of positive systems [1], [6], and the positive matrices of the state-space realization are computed by numerically solving an optimization problem.

The more general concept of *Internal Positive Representation* (IPR) of Multi-Input-Multi-Output systems has been introduced in [8], [9], where also a simple and straightforward algorithm for its construction has been presented. The only drawback of the method is that stable IPRs are obtained if and only if the poles of the original system belong to a given subset of the open unit disk of the complex plane (a square subset denoted $\mathcal{P}_4$). In this paper, such a limitation is overcome: the presented algorithm provides a stable IPR for any stable system. This result is obtained thanks to a new positive representation of complex numbers which exploits the N-th roots of the unity, for suitable values of the integer N.

The paper is organized as follows. The main concepts of the algebra of positive N-representations of complex matrices and vectors are illustrated in Section II. The new methodology of IPR construction is presented in Section III, in a rather general form, while two algorithms for the construction of stable IPR are described in Sections IV and IV. Conclusions follow.

Alfredo Germani and Costanzo Manes are with the Dipartimento di Ingegneria Elettrica e dell'Informazione, Università degli Studi dell'Aquila, Via G. Gronchi 18, Nucleo Industriale di Pile, 67100 LAquila – Italy. e-mail: alfredo.germani@univaq.it, costanzo.manes@univaq.it.

Filippo Cacace is with Università Campus Bio-Medico di Roma, Via Álvaro del Portillo, 21, 00128 Roma - Italy, e-mail: f.cacace@unicampus.it.

A. Notations

$\mathbb{R}_+^n$ and $\mathbb{R}_+^{r\times c}$ denote the sets of nonnegative n-vectors and $r \times c$ matrices, respectively. For a given $x \in \mathbb{R}^n$, the symbols $x^+ \in \mathbb{R}_+^n$, $x^- \in \mathbb{R}_+^n$, and $|x| \in \mathbb{R}_+^n$ denote its positive and negative parts, and the componentwise modulus, respectively (i.e., such that $x = x^+ - x^-$ and $|x| = x^+ + x^-$). Similarly, A^+ and A^- denote the positive and negative parts of a real matrix A, and $|A|$ the componentwise modulus. $\|x\|_1 = \sum_{i=1}^n |x_i|$ denotes the 1-norm of $x \in \mathbb{R}^n$. Given two integers h and N, the symbol $|h|_N$ denotes h *modulo* N, i.e. the remainder of the integer division of h by N. The symbols $\sigma(A)$ and $\rho(A)$ denote the *spectrum* and the *spectral radius* of the square matrix A, respectively. A is said to be stable if $\rho(A) < 1$. The superscript * denotes componentwise conjugation of matrices and vectors (without transposition).

II. POSITIVE N-REPRESENTATION OF COMPLEX VECTORS AND MATRICES

In this section a formalism is introduced for the representation of real or complex vectors and matrices by means of positive combinations of the N-th roots of unity.

A. Positive N-representation of complex numbers

For a given positive integer N, let $q_k^{(N)} = e^{j\frac{2\pi}{N}k}$, with $k \in \mathbb{Z}$, denote the N-th roots of unity in the complex plane, i.e. the solutions of the equation $z^N = 1$, which are also the vertices of $\mathcal{P}_N$, the regular polygon with N sides inscribed in the unit circle. When the integer N is clear from the context, the simpler notation q_k will be used instead of $q_k^{(N)}$. Let $Q^{(N)}$ be the following row vector in $\mathbb{C}^N$:

$$Q^{(N)} = \begin{bmatrix} q_0 & q_1 & \dots & q_{N-1} \end{bmatrix}. \tag{1}$$

For $N > 2$ ($N \geq 2$) any $z \in \mathbb{C}$ ($z \in \mathbb{R}$) can be expressed as a nonnegative linear combination of the components of $Q^{(N)}$.

Definition 1: Given $z \in \mathbb{C}$ and an integer $N > 2$ (or, given $z \in \mathbb{R}$ and $N \geq 2$) a vector $\tilde{z} \in \mathbb{R}_+^N$ is said to be a *positive N-representation* of z if it is such that

$$z = Q^{(N)}\tilde{z} = \sum_{k=0}^{N-1} q_k\, \tilde{z}_k. \tag{2}$$

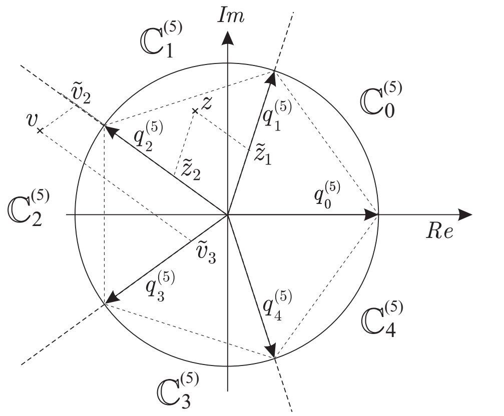

Fig. 1. **Min-positive 5-representations of complex numbers z and v.**

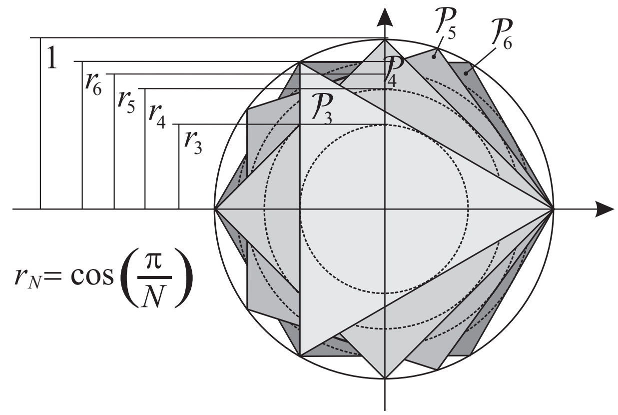

Fig. 2. **The sequence $\mathcal{P}_k$ asymptotically covers the open unit disk.**

Obviously, a complex (real) number admits infinite positive N-representations for $N > 2$ ($N \geq 2$). For instance, all positive 2-representations of given a real number z are

$$\tilde{z}_\alpha = \begin{bmatrix} z^+ + \alpha \\ z^- + \alpha \end{bmatrix}, \qquad \alpha \geq 0, \tag{3}$$

(being $Q^{(2)} = [1 \;\; -1]$, trivially, $Q^{(2)}\tilde{z}_\alpha = z^+ - z^- = z$). The simplest positive 2-representation is the one with $\alpha = 0$, and will be named the *min-positive 2-representation* of z and denoted with the symbol $\Pi_2(z)$, i.e.

$$\Pi_2(z) = \begin{bmatrix} z^+ \\ z^- \end{bmatrix} \tag{4}$$

Note that at most one component of $\Pi_2(z)$ is nonzero.

The concept of min-positive representation can be extended to complex numbers for $N > 2$, by choosing positive N-representations characterized by a minimal number of nonzero components (at most two).

Definition 2: For a given integer $N > 2$, the *min-positive N-representation* of a real or complex number z is the unique positive N-representation $\tilde{z} \in \mathbb{R}_+^N$ such that at most two components are *non-zero* and *consecutive*. The notation for the min-positive N-representation of z is $\Pi_N(z)$.

By definition, $\tilde{z} = \Pi_N(z)$ is such that for some $h \in [0, N-1]$:

$$z = Q^{(N)}\Pi_N(z) = q_h\tilde{z}_h + q_{|h+1|_N}\tilde{z}_{|h+1|_N}. \tag{5}$$

Fig. 1 provides a geometrical interpretation of min-positive N-representations. The N-th roots of unity q_h, $h \in [0, N-1]$, allow to partition the complex plane into N disjoint sectors, denoted $\mathbb{C}_h^{(N)}$, numbered from 0 to $N-1$, defined as

$$\mathbb{C}_h^{(N)} = \{z \in \mathbb{C}\colon z = \alpha q_h + \beta q_{|h+1|_N}, \text{ for some } \alpha > 0,\ \beta \geq 0\}. \tag{6}$$

Note that $q_h \in \mathbb{C}_h^{(N)}$, while $q_{|h+1|_N} \notin \mathbb{C}_h^{(N)}$, and

$$z \in \mathbb{C}_h^{(N)} \quad \Rightarrow \quad \|\Pi_N(z)\|_1 = \tilde{z}_h + \tilde{z}_{|h+1|_N}. \tag{7}$$

In Fig. 1, $z \in \mathbb{C}_1^{(5)}$ and $v \in \mathbb{C}_2^{(5)}$. Thus, $\|\Pi_5(z)\|_1 = \tilde{z}_1 + \tilde{z}_2$, and $\|\Pi_5(v)\|_1 = \tilde{v}_2 + \tilde{v}_3$.

Remark 1: The function $\Pi_N(\cdot)$ is nonlinear: in general $\Pi_N(z) + \Pi_N(v) \neq \Pi_N(z+v)$ unless z and v belong to the same sector. Even so, it must be noted that the function $Q^{(N)}\Pi_N(\cdot)$ is *linear* (i.e., $z + v = Q^{(N)}\Pi_N(z+v)$).

For a given integer $N > 2$, the open regular polygon $\mathcal{P}_N \subset \mathbb{C}$ with vertices q_h, $h \in [0, N-1]$, and the set $\partial\mathcal{P}_N$ of its frontier points can be defined as follows

$$\begin{aligned} \mathcal{P}_N &= \{z \in \mathbb{C} : \|\Pi_N(z)\| < 1\}, \\ \partial\mathcal{P}_N &= \{z \in \mathbb{C} : \|\Pi_N(z)\| = 1\}. \end{aligned} \tag{8}$$

In Fig. 1, $z \in \mathcal{P}_5$, and therefore $\|\Pi_5(z)\|_1 < 1$, while $v \notin (\mathcal{P}_5 \cup \partial\mathcal{P}_5)$, so that $\|\Pi_5(v)\|_1 > 1$.

Theorem 1: Let $z \in \mathbb{C}$ be such that $|z| < 1$. Then, $z \in \mathcal{P}_N$ for all $N > \pi/\arccos(|z|)$.

The main consequence of Theorem 1 is that if $|z| < 1$, then $\|\Pi_N(z)\|_1 < 1$ for N *large enough*. This property is visually demonstrated in Fig. 2.

B. N-Representation of the product of complex numbers

For $h \in [0, N-1]$, let $S_h^{(N)} \in \mathbb{R}^{N \times N}$ denote *circular shift matrices*, i.e. such that $S_0^{(N)} = I_N$ and

$$S_h^{(N)} = \begin{bmatrix} 0_{h\times(N-h)} & I_h \\ I_{N-h} & 0_{(N-h)\times h} \end{bmatrix}, \quad h \in [1, N-1]. \tag{9}$$

Definition 3: Let $p \in \mathbb{C}$, and $N > 2$, (or let $p \in \mathbb{R}$ and $N \geq 2$) and let $\tilde{p}$ be a positive N-representation of p (i.e., $p = Q^{(N)}\tilde{p}$). The following $N \times N$ Toeplitz circulant matrix

$$\widetilde{\mathcal{P}} = \sum_{h=0}^{N-1} S_h^{(N)}\tilde{p}_h = \begin{bmatrix} \tilde{p}_0 & \tilde{p}_{N-1} & \cdots & \tilde{p}_1 \\ \tilde{p}_1 & \tilde{p}_0 & \cdots & \tilde{p}_2 \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{p}_{N-1} & \tilde{p}_{N-2} & \cdots & \tilde{p}_0 \end{bmatrix}. \tag{10}$$

is the *positive circulant N-representation* of p associated to $\tilde{p}$. If $\tilde{p}$ is the min-positive N-representation of p, i.e. $\tilde{p} = \Pi_N(p)$, then $\widetilde{\mathcal{P}}$ is the *min-positive circulant N-representation* of p, and will be denoted as $\widetilde{\Pi}_N(p)$.

Proposition 2: If $\tilde{p}$ and $\tilde{x}$ are a positive N-representations of complex numbers p and x, respectively, the product $\widetilde{\mathcal{P}}\tilde{x}$, where $\widetilde{\mathcal{P}}$ is a positive circulant N-representation of p, is a positive N-representation of the product $p\,x$.

The proof is achieved by demonstrating the identity

$$p\,x = Q^{(N)}\widetilde{\mathcal{P}}\tilde{x}. \tag{11}$$

Remark 2: If p and x are real numbers, then

$$\begin{aligned} \widetilde{\Pi}_2(p) &= \begin{bmatrix} p^+ & p^- \\ p^- & p^+ \end{bmatrix}, \quad \Pi_2(x) = \begin{bmatrix} x^+ \\ x^- \end{bmatrix}, \\ \widetilde{\Pi}_2(p)\Pi_2(x) &= \begin{bmatrix} p^+x^+ + p^-x^- \\ p^+x^- + p^-x^+ \end{bmatrix}. \end{aligned} \tag{12}$$

It can be easily checked that $Q^{(2)}\widetilde{\Pi}_2(p)\Pi_2(x) = p\,x$, although $\widetilde{\Pi}_2(p)\Pi_2(x) \neq \widetilde{\Pi}_2(p\,x)$. This is the kind of representation used in [8], [9].

C. Example: positive representation of a simple system

Consider a transfer function $W(z)$ characterized by a pair of conjugate poles (p, p^*), written in the form

$$W(z) = \frac{r}{z-p} + \frac{r^*}{z-p^*}. \tag{13}$$

It admits the following complex state-space representation:

$$x(t+1) = p\,x(t) + r\,u(t) \tag{14}$$

$$y(t) = 2\Re\big(x(t)\big), \tag{15}$$

Now assume that the input sequence $u(t)$ is nonnegative. For a given integer N, let $\widetilde{\mathcal{P}} \in \mathbb{R}_+^{N\times N}$ be a positive circulant N-representation of p, and $\tilde{r} \in \mathbb{R}_+^N$ a positive N-representation of r. Then the state-transition equation (14) admits the following positive representation:

$$\tilde{x}(t+1) = \widetilde{\mathcal{P}}\tilde{x}(t) + \tilde{r}\,u(t), \tag{16}$$

in that $\tilde{x}(t) \in \mathbb{R}_+^N$ is a sequence of positive N-representations of the complex state $x(t)$, i.e. $x(t) = Q^{(N)}\tilde{x}(t)$.

The main result of this paper is that, for the given integer N, the choice $\widetilde{\mathcal{P}} = \widetilde{\Pi}_N(p)$ (min-positive circulant N-representation of p) is stable if and only if $p \in \mathcal{P}_N$. It follows, thanks to Theorem 1, that if $W(z)$ is stable (i.e. $|p| < 1$) it is always possible to choose N such that $p \in \mathcal{P}_N$, thus ensuring the stability of the positive representation (16).

D. Positive N-Representations of vectors

Consider a vector $x = [x_1\ \dots x_n]^T \in \mathbb{C}^n$, and an integer $N > 2$ (or $x \in \mathbb{R}^n$ and $N \geq 2$). Let $\xi_i \in \mathbb{R}_+^N$, $i = 1, \dots, n$, denote some positive N-representations of the components x_i. By definition, the vectors $\xi_i = \begin{bmatrix} \xi_{i,0} & \cdots & \xi_{i,N-1} \end{bmatrix}^T$ are such that $x_i = Q^{(N)}\xi_i$, and x can be written as

$$x = \sum_{h=0}^{N-1} q_h \hat{x}_h, \quad \text{where } \hat{x}_h = \begin{bmatrix} \xi_{1,h} \\ \vdots \\ \xi_{n,h} \end{bmatrix}. \tag{17}$$

Definition 4: The vector $\tilde{x} = [\hat{x}_0^T\ \dots\ \hat{x}_{N-1}^T]^T \in \mathbb{R}_+^{nN}$, where the subvectors $\hat{x}_h$ are defined in (17), is called a *positive N-representation* of x. If $\xi_i = \Pi_N(x_i)$, the vector $\tilde{x}$ is said to be the *min-positive N-representation* of x.

From now on, the symbol $\Pi_N(\cdot)$ will be applied indifferently to scalars and vectors, so that if $x \in \mathbb{C}^n$, then $\Pi_N(x) \in \mathbb{R}_+^{nN}$. Defining

$$Q_n^{(N)} = Q^{(N)} \otimes I_n = \begin{bmatrix} q_0 I_n & q_1 I_n & \cdots & q_{N-1} I_n \end{bmatrix}, \tag{18}$$

the sum (17) can be written as

$$x = Q_n^{(N)}\tilde{x}, \tag{19}$$

E. Positive circulant N-Representation of matrices

As it has been done for vectors in section II-D, the positive N-representations of the components of a matrix $A \in \mathbb{C}^{r\times n}$ can be organized into N matrices $\widehat{A}_h \in \mathbb{R}_+^{r\times n}$, such that

$$A = \sum_{h=0}^{N-1} q_h \widehat{A}_h. \tag{20}$$

Definition 5: Given a complex matrix $A \in \mathbb{C}^{r\times n}$ and $N > 2$ (or $A \in \mathbb{R}^{r\times n}$ and $N \geq 2$), and given N positive matrices $\widehat{A}_h \in \mathbb{R}_+^{r\times n}$ such that (20) holds, the following $rN \times nN$ Toeplitz block-circulant matrix:

$$\widetilde{A} = \sum_{h=0}^{N-1} S_h^{(N)} \otimes \widehat{A}_h = \begin{bmatrix} \widehat{A}_0 & \widehat{A}_{N-1} & \cdots & \widehat{A}_1 \\ \widehat{A}_1 & \widehat{A}_0 & \cdots & \widehat{A}_2 \\ \vdots & \vdots & \vdots\vdots\vdots & \vdots \\ \widehat{A}_{N-1} & \widehat{A}_{N-2} & \cdots & \widehat{A}_0 \end{bmatrix}. \tag{21}$$

is called a *positive circulant N-representation* of A. If the elements of the matrices $\widehat{A}_h$ are the min-positive N-representations of the components of A, then $\widetilde{A}$ is the *min-positive circulant N-representation* of A, and is indicated as $\widetilde{A} = \widetilde{\Pi}_N(A)$.

Proposition 3: Consider $A \in \mathbb{C}^{r\times n}$, $x \in \mathbb{C}^n$ and an integer $N > 2$ (or $N \geq 2$, if both A and x are real). Let $\widetilde{A}$ be a positive circulant N-representation of A, and let $\tilde{x}$ be a positive N-representation of x. Then, the product $\widetilde{A}\tilde{x}$ is a positive N-representation of the product Ax.

As in Proposition 2, the proof is achieved by showing that

$$Ax = Q_r^{(N)}\widetilde{A}\tilde{x}. \tag{22}$$

If A is a square and stable matrix (i.e. $\rho(A) < 1$), in general its positive circulant N-representations are not guaranteed to be stable. A very useful result is the following:

Theorem 4: Given $A \in \mathbb{C}^{n\times n}$ and $N > 2$ (or $A \in \mathbb{R}^{n\times n}$ and $N \geq 2$), let $\widetilde{A}$ be any positive circulant N-representation, as defined in (21). Let

$$M(\widetilde{A}) = \sum_{k=0}^{N-1} \widehat{A}_k. \tag{23}$$

Then $\widetilde{A}$ and $M(\widetilde{A})$ have the same spectral radius, i.e. $\rho(\widetilde{A}) = \rho\big(M(\widetilde{A})\big)$.

As a particular case, $\rho\big(\widetilde{\Pi}_N(A)\big) = \rho\big(M\big(\widetilde{\Pi}_N(A)\big)\big)$

Remark 3: The min-positive 2-representations of a real vector x and matrix A are, as in [8], [9],

$$\Pi_2(x) = \begin{bmatrix} x^+ \\ x^- \end{bmatrix}, \quad \widetilde{\Pi}_2(A) = \begin{bmatrix} A^+ & A^- \\ A^- & A^+ \end{bmatrix}. \tag{24}$$

F. Positive N-Representation of products of complex matrices with real vectors

Consider a complex matrix $B \in \mathbb{C}^{n\times p}$ and a positive circulant N-representation $\widetilde{B} \in \mathbb{R}_+^{nN\times pN}$, a real vector $u \in \mathbb{R}^p$ and its min-positive N-representation $\Pi_N(u) \in \mathbb{R}_+^{pN}$. Then, $\tilde{x} = \widetilde{B}\Pi_N(u) \in \mathbb{R}_+^{nN}$ is a positive N-representation of the product $x = Bu \in \mathbb{C}^n$. Being u real, a suitable nonnegative matrix $H_p^{(N)} \in \mathbb{R}^{pN\times 2p}$ can be defined such that

$$\Pi_N(u) = H_p^{(N)}\Pi_2(u), \tag{25}$$

From this and from $\tilde{x} = \widetilde{B}\Pi_N(u)$ it follows

$$\tilde{x} = \overline{B}\,\Pi_2(u), \qquad \text{where} \quad \overline{B} = \widetilde{B}H_p^{(N)}. \tag{26}$$

Note that $\overline{B} \in \mathbb{R}^{nN\times 2p}$, while $\widetilde{B} \in \mathbb{R}^{nN\times pN}$.

G. Computation of the real part of a complex product

Let $C \in \mathbb{C}^{q\times n}$ and $x \in \mathbb{C}^n$. Let $\widetilde{C} \in \mathbb{R}_+^{qN\times nN}$ be a positive circulant N-representation of C and $\tilde{x} \in \mathbb{R}_+^{nN}$ be a positive N-representation of x, so that $Cx = Q_q^{(N)}\widetilde{C}\tilde{x}$. Let $y = \Re(Cx)$ denote the real part of the product Cx. A positive 2-representation of y can be readily computed as

$$\tilde{y} = R_q^{(N)}\widetilde{C}\tilde{x}, \quad \text{with } R_q^{(N)} = \begin{bmatrix} \Re\big(Q_q^{(N)}\big)^+ \\ \Re\big(Q_q^{(N)}\big)^- \end{bmatrix} \in \mathbb{R}_+^{2q\times qN}. \tag{27}$$

III. Internal Positive Representations (IPRs) of Linear Systems

In this paper the symbol $S = \{A, B, C, D; U, X, Y\}$ denotes a discrete-time state-space system of the form

$$S: \begin{cases} x(t+1) = Ax(t) + Bu(t), & u(t) \in U,\ x(t) \in X \\ \quad y(t) = Cx(t) + Du(t), & y(t) \in Y, \end{cases} \tag{28}$$

where U, X, Y are the input, state and output spaces.
S is a *Complex System* if $A \in \mathbb{C}^{n\times n}$, $B \in \mathbb{C}^{n\times p}$, $C \in \mathbb{C}^{q\times n}$, $D \in \mathbb{C}^{q\times p}$; $U = \mathbb{C}^p$, $X = \mathbb{C}^n$, $Y = \mathbb{C}^q$;
S is a *Real System* if $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times p}$, $C \in \mathbb{R}^{q\times n}$, $D \in \mathbb{R}^{q\times p}$; $U = \mathbb{R}^p$, $X = \mathbb{R}^n$, $Y = \mathbb{R}^q$;
S is a *Positive System* if $A \in \mathbb{R}_+^{n\times n}$, $B \in \mathbb{R}_+^{n\times p}$, $C \in \mathbb{R}_+^{q\times n}$, $D \in \mathbb{R}_+^{q\times p}$; $U = \mathbb{R}_+^p$, $X = \mathbb{R}_+^n$, $Y = \mathbb{R}_+^q$.
Given a real system $S = \{A, B, C, D; \mathbb{R}^p, \mathbb{R}^n, \mathbb{R}^q\}$ and a nonsingular $T \in \mathbb{C}^{n\times n}$, let $S(T)$ denote the system $\{TAT^{-1}, TB, CT^{-1}, D; \mathbb{R}^p, T\mathbb{R}^n, \mathbb{R}^q\}$ (denoted *Complex Representation of a Real System*).

A. IPRs of Real and Complex Systems

Definition 6: Let $S = \{A, B, C, D; U, X, Y\}$ be a system, with input, state, and output denoted $u(t), x(t), y(t)$. An *Internally Positive Representation* (IPR) of S is a Positive System $\mathcal{S} = \{\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}; \mathbb{R}_+^{\tilde{p}}, \mathbb{R}_+^{\tilde{n}}, \mathbb{R}_+^{\tilde{q}}\}$ together with four transformations:

$$\begin{aligned} &\mathcal{T}_X^f : X \to \mathbb{R}_+^{\tilde{n}}, && \mathcal{T}_X^b : \mathbb{R}_+^{\tilde{n}} \to X, \\ &\mathcal{T}_U : U \to \mathbb{R}_+^{\tilde{p}}, && \mathcal{T}_Y : \mathbb{R}_+^{\tilde{q}} \to Y, \end{aligned} \tag{29}$$

such that, denoting with $\tilde{u}(t), \tilde{x}(t), \tilde{y}(t)$ the positive input, state, and output sequences of $\widetilde{S}$, for any pair $(x_0, t_0) \in X \times \mathbb{Z}$, and for any input sequence $u(t) \in U$ starting at t_0, by setting $\tilde{x}(t_0) = \mathcal{T}_X^f(x_0)$ and $\tilde{u}(t) = \mathcal{T}_U(u(t))$ in $\widetilde{S}$, then

$$x(t) = \mathcal{T}_X^b(\tilde{x}(t)), \quad y(t) = \mathcal{T}_Y(\tilde{y}(t)), \quad \forall t \geq t_0. \tag{30}$$

Fig. 3 depicts an IPR. By Definition 6, the input-output behavior of the IPR and of the original system are the same.

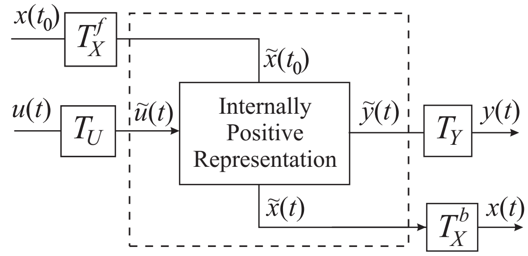

Fig. 3. **Block diagram of an Internally Positive Representation.**

B. N-IPRs of Real and Complex Systems

The main result of this paper is that a straightforward application of the algebra of positive N-representations of vector and matrices presented in Section II can easily provide IPRs of real and complex systems.

Theorem 5: Consider a discrete time real system $S = \{A, B, C, D; \mathbb{R}^p, \mathbb{R}^n, \mathbb{R}^q\}$ and an integer $N \geq 2$ (or a complex system $S = \{A, B, C, D; \mathbb{C}^p, \mathbb{C}^n, \mathbb{C}^q\}$ and $N > 2$), let $(\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D})$ be positive circulant N-representations of the system matrices (A, B, C, D). Then, the positive system

$$\widetilde{S} = \{\widetilde{A}, \widetilde{B}, \widetilde{C}, \widetilde{D}; \mathbb{R}_+^{pN}, \mathbb{R}_+^{nN}, \mathbb{R}_+^{qN}\}, \tag{31}$$

together with the four transformations

$$\begin{aligned} &\mathcal{T}_X^f(x) = \Pi_N(x), && \mathcal{T}_X^b(\tilde{x}) = Q_n^{(N)}\tilde{x}, \\ &\mathcal{T}_U(u) = \Pi_N(u), && \mathcal{T}_Y(\tilde{y}) = Q_q^{(N)}\tilde{y}, \end{aligned} \tag{32}$$

define an Internal Positive Representation of S (denoted N-IPR of system S).

The proof is quite straightforward and exploits the properties of the positive N-representations discussed in the previous section. Note that when the state of the system S has dimension n, the state of the N-IPR has dimension nN.

Remark 4: The approach described in [8], [9] for the construction of IPRs coincides with the method presented in Theorem 5 when $N = 2$.

An IPR with matrices of smaller size can be obtained when the input an output spaces are real (as in the case of Real Systems or of Complex Representations of Real Systems).

Theorem 6: Let $S = \{A, B, C, D; \mathbb{R}^p, X, \mathbb{R}^q\}$, with $X \subset \mathbb{C}^n$, be a Real System or a Complex Representation of a Real System. For a given $N > 2$, let the matrices $\widetilde{A} \in \mathbb{R}_+^{nN\times nN}$, $\widetilde{B} \in \mathbb{R}_+^{nN\times pN}$ and $\widetilde{C} \in \mathbb{R}_+^{qN\times nN}$ be positive circulant N-representations of A, B, C, respectively. Then, the positive system $\mathcal{S} = \{\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}; \mathbb{R}_+^{2p}, \mathbb{R}_+^{nN}, \mathbb{R}_+^{2q}\}$ where

$$\begin{aligned} &\mathcal{A} = \widetilde{A}, && \mathcal{B} = \widetilde{B}H_p^{(N)}, \\ &\mathcal{C} = R_q^{(N)}\widetilde{C}, && \mathcal{D} = \begin{bmatrix} D^+ & D^- \\ D^- & D^+ \end{bmatrix}, \end{aligned} \tag{33}$$

(the matrix $H_p^{(N)} \in \mathbb{R}^{pN\times 2p}$ is defined in section II-F, while $R_q^{(N)}$ is defined in (27)) together with the four transformations

$$\begin{aligned} &\mathcal{T}_X^f(x) = \Pi_N(x), && \mathcal{T}_U(u) = \Pi_2(u) = \begin{bmatrix} u^+ \\ u^- \end{bmatrix}, \\ &\mathcal{T}_X^b(\tilde{x}) = Q_n^{(N)}\tilde{x}, && \mathcal{T}_Y(\tilde{y}) = Q_q^{(2)}\tilde{y} = \begin{bmatrix} I_q & -I_q \end{bmatrix}\tilde{y}, \end{aligned} \tag{34}$$

make up an IPR of S.

In general, the stability of an IPR of a stable system is not guaranteed in general (see Theorem 4), and changes of coordinates may help in finding stable IPRs.

IV. Construction of stable IPRs

This section is aimed to exploit the method of IPR construction of Theorems 5 and 6 to overcome the limitations of the technique in [8], [9], that provides stable IPRs only of systems with eigenvalues in $\mathcal{P}_4$.

It is shown that if a system is stable, then there always exist a change of coordinates and an integer N such that the constructions of Theorems 5 and 6 provide a stable IPR. The method of stable IPR construction takes advantage of the following theorem.

Theorem 7: Let $J \in \mathbb{C}^{n\times n}$ be a stable matrix in the Jordan canonical form, and let $N > 0$ such that $\sigma(J) \in \mathcal{P}_N$. Then, $\widetilde{\Pi}_N(J)$ is stable.

(*Sketch of the Proof*) By Theorem 1, there always exists $N \in \mathbb{Z}$ (large enough) such that $\sigma(J) \in \mathcal{P}_N$. Let $\widetilde{J} = \widetilde{\Pi}_N(J)$, and let $\widehat{J}_k$ be the blocks that make up $\widetilde{J}$. Then, by Theorem 4, the matrix

$$M(\widetilde{J}) = \sum_{k=0}^{N-1} \widehat{J}_k \tag{35}$$

is such that $\rho\big(M(\widetilde{J})\big) = \rho(\widetilde{J})$. It can be shown that $M(\widetilde{J})$ has the same block diagonal structure of J, and that whereas the terms on the diagonal of J are the eigenvalues λ_h, the terms on the diagonal of $M(\widetilde{J})$ are $\|\Pi_N(\lambda_h)\|_1$. By assumption, $\lambda_h \in \mathcal{P}_N$ implies that $\|\Pi_N(\lambda_h)\|_1 < 1$, which in turn implies $\rho(\widetilde{J}) < 1$. ■

The first step of the proposed algorithm for the construction of a stable IPR of real systems is to change the system coordinates and to transform the system in the Jordan canonical form. To this aim, given a stable real system $S = \{A, B, C, D; \mathbb{R}^p, \mathbb{R}^n, \mathbb{R}^q\}$, let n_r denote the number of real eigenvalues of A, and n_c the number of pairs of conjugate complex eigenvalues, each one counted with its own algebraic multiplicity, so that $n = n_r + 2n_c$. It is known that there exists a nonsingular matrix $T \in \mathbb{C}^{n\times n}$, with the structure $T = \mathrm{col}(T_r, T_c, T_c^*)$, $T_r \in \mathbb{R}^{n_r\times n}$, and $T_c \in \mathbb{C}^{n_c\times n}$, such that $J = TAT^{-1}$ is in the following Jordan canonical form:

$$J = \mathrm{diag}(J_r, J_c, J_c^*), \tag{36}$$

where $J_r \in \mathbb{R}^{n_r\times n_r}$ denotes the Jordan block associated to the real eigenvalues, while $J_c \in \mathbb{C}^{n_c\times n_c}$ denotes the Jordan block associated to the complex eigenvalues with positive imaginary part. Let $\bar{T} = T^{-1}$, with the structure $\bar{T} = \mathrm{row}(\bar{T}_r, \bar{T}_c, \bar{T}_c^*)$. Then the matrices $\bar{B} = TB$ and $\bar{C} = CT^{-1}$ have the following structure

$$\begin{aligned} \bar{B} &= \mathrm{col}(\bar{B}_r, \bar{B}_c, \bar{B}_c^*),\\ \bar{C} &= \mathrm{row}(\bar{C}_r, \bar{C}_c, \bar{C}_c^*), \end{aligned} \tag{37}$$

where $\bar{B}_r = T_r\bar{B},\ \bar{B}_c = T_c\bar{B},\ \bar{C}_r = \bar{C}\bar{T}_r,\ \bar{C}_c = \bar{C}\bar{T}_c$.

J-IPR Algorithm

Given a stable real system $S = \{A, B, C, D; \mathbb{R}^p, \mathbb{R}^n, \mathbb{R}^q\}$,

- find the smallest $N \in \mathbb{Z}$ such that $\sigma(A) \subset \mathcal{P}_N$;
- find $T \in \mathbb{C}^{n\times n}$, such that the matrices of the transformed system $S(T)$ have the structure (36)-(37);
- compute the min-positive circulant representations:

$$\begin{aligned} &\widetilde{J}_r = \widetilde{\Pi}_2(J_r), \quad \widetilde{B}_r = \widetilde{\Pi}_2(\bar{B}_r), \quad \widetilde{C}_r = \widetilde{\Pi}_2(\bar{C}_r),\\ &\widetilde{J}_c = \widetilde{\Pi}_N(J_c), \quad \widetilde{B}_c = \widetilde{\Pi}_N(\bar{B}_c), \quad \widetilde{C}_c = \widetilde{\Pi}_N(\bar{C}_c); \end{aligned} \tag{38}$$

- define the following matrices:

$$\begin{aligned} &\mathcal{A} = \mathrm{diag}(\widetilde{J}_r, \widetilde{J}_c), && \mathcal{B} = \mathrm{col}(\widetilde{B}_r, \widetilde{B}_c H_p^{(N)}),\\ &\mathcal{C} = \mathrm{row}(\widetilde{C}_r, 2R_q^{(N)}\widetilde{C}_c), && \mathcal{D} = \widetilde{\Pi}_2(D), \end{aligned} \tag{39}$$

- define the following transformations:

$$\begin{aligned} &\mathcal{T}_X^f(x) = \mathrm{col}\big(\Pi_2(T_r x), \Pi_N(T_c x)\big),\\ &\mathcal{T}_X^b(\tilde{x}) = T^{-1}\mathrm{diag}\Big(Q_{n_r}^{(2)}, \mathrm{col}(Q_{n_c}^{(N)}, (Q_{n_c}^{(N)})^*)\Big)\tilde{x},\\ &\mathcal{T}_U(u) = \Pi_2(u), \qquad \mathcal{T}_Y(\tilde{y}) = Q_q^{(2)}\tilde{y}. \end{aligned} \tag{40}$$

The system $\mathcal{S} = \{\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}; \mathbb{R}_+^{2p}, \mathbb{R}_+^{\tilde{n}}, \mathbb{R}_+^{2q}\}$, where $\tilde{n} = 2n_r + Nn_c$, together with the four transformations (40), is a stable IPR of the system S.

The proof that $\mathcal{S}$ provided by this algorithm is an IPR of system S can be worked out by straightforward computations involving the algebra of the positive N-representations presented in Section II. The stability is a direct consequence of Theorem 7.

This algorithm has been used to construct the IPR for the transfer function $W(z)$ in the example of Sec. II-C (it has been applied to the diagonal pole-residual state-space representation of $W(z)$).

Note that Theorem 1 guarantees that if $\rho(A) < 1$ then there exists $N \in \mathbb{Z}$ (large enough) such that $\sigma(A) \subset \mathcal{P}_N$. The size of the state space of the IPR provided by this algorithm is $2n_r + n_cN$, where N depends on the position of the complex eigenvalues of A in the complex plane (there may be situations where N must rather large in order to satisfy $\sigma(A) \subset \mathcal{P}_N$).

V. Construction of stable IPRs of smaller size

In this section a method is described for achieving stable IPRs of smaller size than those produced by the J-IPR Algorithm of the previous section. Due to lack of space, this method is only sketched. The idea behind the method is to regard the system in the Jordan coordinates as the parallel of a set of subsystems (each one associated to the Jordan block associated to a single eigenvalue), to separately construct IPRs of each block. In this way, different choices of the integer N can be made for each Jordan block, according to the position of the eigenvalues in the complex plane.

The following definitions are needed for the description of this IPR construction method:

Polygon unions $\overline{\mathcal{P}}_k$, $k = 2, 3, \ldots$:

$$\overline{\mathcal{P}}_2 = \mathcal{P}_2, \quad \text{and} \quad \overline{\mathcal{P}}_k = \bigcup_{i=1}^{k} \mathcal{P}_i, \; k > 2; \tag{41}$$

Polygon innovations $\widetilde{\mathcal{P}}_k$, $k = 2, 3, \ldots$:

$$\widetilde{\mathcal{P}}_2 = \mathcal{P}_2, \quad \text{and} \quad \widetilde{\mathcal{P}}_k = \mathcal{P}_k \setminus \overline{\mathcal{P}}_{k-1}, \; k > 2. \tag{42}$$

Given a stable matrix $A \in \mathbb{R}^{n \times n}$, the eigenvalues can be grouped into subsets Λ_k, $k = 2, 3, \ldots$, defined as

$$\Lambda_k = \{z \in \sigma(A) : \; z \in \widetilde{\mathcal{P}}_k\}. \tag{43}$$

Thus, Λ_2 will contain all real eigenvalues of A, if any, and Λ_k, for $k > 2$, will contain all complex eigenvalues (conjugate pairs) contained in $\mathcal{P}_k$ and not in $\overline{\mathcal{P}}_{k-1}$. Let m denote the number of non empty sets Λ_k associated to complex eigenvalue pairs, and let $k_1, k_2, \ldots, k_m$ denote the indexes of such sets. Let n_r (possibly zero) denote the number of real eigenvalues, counted with their algebraic multiplicity, and let n_{k_i}, $i = 1, \ldots, m$ denote the number of complex pairs in Λ_{k_i}, counted with their algebraic multiplicity, so that

$$n = n_r + \sum_{i=1}^{m} n_{k_i}. \tag{44}$$

Then, consider the change of coordinate T that puts the system in the Jordan canonical form, where the Jordan blocks of the matrix $J = TAT^{-1}$ are organized according to the indexes k_i, $i = 1, \ldots, m$ as follows

$$\begin{aligned} &J = \operatorname{diag}(J_r, J_{k_1}, J_{k_1}^*, \ldots, J_{k_m}, J_{k_m}^*), \\ &J_r \in \mathbb{R}^{n_r \times n_r}, \qquad \text{with} \;\; \sigma(J_r) \subset \mathcal{P}_2 = (-1, 1), \\ &J_k \in \mathbb{C}^{n_k \times n_k}, \qquad \text{with} \;\; \sigma(J_k) \subset \widetilde{\mathcal{P}}_k, \\ &\quad k \in \{k_1, \ldots . k_m\}. \end{aligned} \tag{45}$$

The matrix T and its inverse $\bar{T} = T^{-1}$ have the following structures

$$\begin{aligned} T &= \operatorname{col}(T_r, T_{k_1}, T_{k_1}^*, \ldots, T_{k_m}, T_{k_m}^*) \\ \bar{T} &= \operatorname{row}((\bar{T}_r, \bar{T}_{k_1}, \bar{T}_{k_1}^*, \ldots, \bar{T}_{k_m}, \bar{T}_{k_m}^*). \end{aligned} \tag{46}$$

The transformed matrices $\bar{B} = TB$ and $\bar{C} = CT^{-1}$ admit the following block partitions, of compatible dimensions with the blocks of J:

$$\begin{aligned} \bar{B} &= \operatorname{col}(\bar{B}_r, \bar{B}_{k_1}, \bar{B}_{k_1}^*, \ldots, \bar{B}_{k_m}, \bar{B}_{k_m}^*), \\ \bar{C} &= \operatorname{row}(\bar{C}_r, \bar{C}_{k_1}, \bar{C}_{k_1}^*, \ldots, \bar{C}_{k_m}, \bar{C}_{k_m}^*). \end{aligned} \tag{47}$$

where $\bar{B}_r = T_r B$, $\bar{B}_{k_i} = T_{k_1} B$, $\bar{C}_r = C\bar{T}_r$, $\bar{C}_{k_i} = C\bar{T}_{k_i}$.

Applying the J-IPR Algorithm to the real subsystem $S_r = \{J_r, \bar{B}_r, \bar{C}_r, D; \mathbb{R}^p, \mathbb{R}^{n_r}, \mathbb{R}^q\}$ we get $N = 2$, while applying it to each complex subsystem S_i

$$\begin{aligned} S_i = \{\operatorname{diag}(J_{k_i}, J_{k_i}^*), &\operatorname{col}(\bar{B}_{k_i}, \bar{B}_{k_i}^*), \\ &\operatorname{row}(\bar{C}_{k_i}, \bar{C}_{k_i}^*), 0; \mathbb{R}^p, X_{k_i}, \mathbb{R}^q\}, \end{aligned}$$

with $X_{k_i} = \operatorname{col}(T_{k_i}, T_{k_i}^*)\mathbb{R}^n \subset \mathbb{C}^{n_{k_i}}$, we get $N = k_i$. It follows that we get an IPR of dimension $2n_r$, plus m IPRs, each one of dimension given by the product $n_{k_i} k_i$, $i = 1, \ldots, m$. The parallel of these $m + 1$ IPRs makes up an IPR for the original system whose dimension is

$$\tilde{n} = 2n_r + \sum_{i=1}^{m} k_i n_{k_i}, \tag{48}$$

which, in general, is smaller than the dimension of the IPR provided by the J-IPR Algorithm, which is $\tilde{n} = 2n_r + Nn_c$ (note that $n_c = \sum_{i=1}^{m} n_{k_i}$, and $N \geq \max\{k_1, \ldots, k_m\}$). Further details on this method can not be reported here due to lack of space.

VI. Concluding Remarks

The technique of positive N-representation of complex matrices and vectors by means of the N-th roots of unity has been developed and used in this paper for the construction of stable IPRs of stable (real or complex) systems. The presented algorithms of IPRs construction is straightforward and is characterized by a very low computational cost, unlike other methods available in the literature where the matrices of the IPR are computed by solving optimization problems. The approach can also be extended to classes of nonlinear systems, like polynomial systems [7].

References

[1] B.D.O. Anderson, M. Deistler, L. Farina, and L. Benvenuti, "Nonnegative realization of a linear system with nonnegative impulse response", *IEEE Trans. Circuits Syst. I*, Vol. 43, No. 2, pp. 134–142, 1996.

[2] L. Benvenuti, L. Farina, "On the class of linear filters attainable with charge routing networks," *IEEE Trans. on Circ. and Syst.-II,* Vol. 43, pp. 618–622, 1996.

[3] L. Benvenuti, L. Farina, "Discrete-time filtering via charge routing networks," *Signal Proc.,* Vol. 49, pp. 207–215, 1996.

[4] L. Benvenuti, L. Farina, B.D.O. Anderson, "Filtering through combination of positive filters," *IEEE Trans. on Circ. and Syst.-I: Fundamental Theory and Appl-,* Vol. 46, No. 12, pp. 1431–1440, 1999.

[5] L. Benvenuti, L. Farina, "The design of fiber-optic filters," *J. of Lightwave Technology,* Vol. 19, No. 9, pp. 1366–1375, 2001.

[6] L. Benvenuti, L. Farina, "A tutorial on the positive realization problem," *IEEE Trans. Automat. Contr.*, Vol. 49, No. 5, pp. 651–664, 2004.

[7] F. Cacace, A. Germani, and C. Manes, "Representation of a Class of Polynomial MIMO systems via Positive Realizations", *Proc. European Control Conference*, Sept. 2009.

[8] A. Germani, C. Manes, P. Palumbo, "State representation of a class of MIMO systems via positive systems," *in Proc. IEEE Conf. on Decision and Control*, New Orleans, Louisiana, Dec. 2007.

[9] A. Germani, C. Manes, and P. Palumbo, "Representation of a Class of MIMO Systems via Internally Positive Realization," *European Journal of Control*, Vol. 16, No. 3, pp. 291–304, 2010.

[10] A. Gersho, B. Gopinath, "Charge-Routing Networks," *IEEE Trans. on Circ. and Syst,* Vol. 26, No. 2, pp. 81–92, 1979.

[11] F. I. Karpelevich, "On the characteristic roots of matrices with nonnegative elements", in *Eleven Papers Translated From Russian*. Providence, RI: American Math. Soc., vol. 140, 1988.

[12] B. Nagy, M. Matolcsi, "Minimal positive realizations of transfer functions with nonnegative multiple poles," *IEEE Trans. on Autom. Contr.,* Vol. 50, pp. 1447–1450, 2005.

[13] B. Nagy, M. Matolcsi, M. Szilvási, "Order bound for realization of a combination of positive filters," *IEEE Trans. on Autom. Contr.,* Vol. 52, pp. 724–729, 2007.