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Distributed Clock Synchronization: Joint Frequency and Phase Consensus

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***Abstract*— Distributed synchronization has gradually gained importance over the last two decades. The ad-hoc nature of new applications has increased the need for robust and scalable distributed algorithms that are capable of generating high precision timing information. However, current solutions usually produce phase errors when the frequencies are heterogeneous. This paper proposes a distributed synchronization procedure that can achieve consensus in both frequency and phase. The algorithm uses only local information and is robust to frequency heterogeneity and network topology. A sufficient condition for global convergence is shown by leveraging recent results on coupled oscillators. We further characterize an invariant constant of the algorithm that relates the limiting frequency ω^* with the harmonic mean of the clocks' natural frequencies. Simulations are provided to illustrate and verify these properties.**

I. INTRODUCTION

The need of a common time reference among network nodes has always been an important issue in communication networks. Historically, it was primarily used to allow coherent data communication among telecommunication nodes and coordination for medium access control in cellular networks. These solutions usually require a centralized clock distribution architecture and depend on highly stable clocks with relative frequency offsets of less than 10^{-10} [17].

Nowadays, synchronization is used in a vast diversity of applications. Examples of these include data fusion of time sensitive measurements in distributed estimation or tracking [5], energy efficient MAC protocols with sleep periods [18], and collaborative transmission using space-time coding [4].

Unfortunately, traditional synchronization architectures have become increasingly unsuitable for these applications due to several reasons. First, the synchronization of the entire network relies on a few number of nodes. This implies that the whole system is fragile to the failure of those nodes. Second, in order to achieve high precision, expensive clocks are usually needed and cannot be placed in every node of the network. And finally, the centralized nature of the solution makes it not scalable since errors will accumulate when the number of clocks grows.

Essentially, there are three requisites that an ideal synchronization protocol should satisfy. It should be distributed and independent of network topology, i.e., each node only uses neighbors' timing information to adjust it own time. It should be robust to high variance in clock's frequency distribution, and it should minimize the phase error as much as possible.

Several synchronization algorithms have been proposed along this line of thoughts, see e.g. [13], [14] and references therein. One possible solution is to use discrete time PLLs (Phase Lock Loop). The resulting algorithms can be shown to globally converge but they are either sensitive to heterogenous frequencies [16], or can only be analyzed for the two node scenario [10]. There have also been studies on frequency and phase estimation with noisy measurements. However, the techniques involved usually only cover large number of nodes asymptotics [15] while guaranteeing $O(1)$ phase errors.

This paper builds upon related work on coupled oscillators, e.g., [8], [7]. These systems usually need to introduce phase mismatch to compensate the frequency differences. We solve this problem by adding a new integrator in the loop together with a linear consensus term. Moreover, we also provide a global convergence result under certain conditions on the topology, i.e. connectivity, and coupling.

The rest of the paper is organized as follows. Section II introduces the model. In Section III, we use an invariant property of the system to characterize the final achieved frequency ω^* in term of initial conditions and system parameters. Global convergence is established in Section IV. Simulations are used to illustrate our findings in Section V and conclusions are presented in Section VI.

II. MOTIVATION AND MODEL

A. Modeling Clocks

We consider a network of N nodes. The connectivity of the network is described by a graph $G = (V, E)$ where two nodes, $i, j \in V$, are allowed to interchange timing information if and only if there is some edge $ij \in E$. This exchange of information can be done by explicit transmission or implicit estimation, and it is assumed to have negligible delay.

Each node contains a clock of natural frequency $\frac{1}{T_i}$ which is assumed to be implemented by a continuous counter $n_i \in [0, 1]$ that increases its count according to

$$\dot{n}_i = \frac{1}{T_i} - \delta(n_i - 1), \quad \forall i \in V. \tag{1}$$

The Dirac's delta function δ forces the counter to restart once it reaches the value 1. Notice T_i is also the total time needed for n_i to go through the interval $[0, 1]$.

The main goal of this paper is to find a control strategy that bring all the clocks to a time consensus using only neighbors' information, i.e.,

$$n_i(t) \to \frac{1}{T*}t + n^* \pmod 1, \quad \forall i \in V, \tag{2}$$

as $t \to +\infty$, with T^* being the final common period.

Despite the discontinuities that (1) generates, the periodicity of the trajectories allows a transformation from counters n_i to phases ϕ_i in the unit circle $\mathbb{S}^1$ such that (1) becomes

$$\dot{\phi}_i = \omega_i, \quad \forall i \in V, \tag{3}$$

with $\omega_i = \frac{2\pi}{T_i}$ and whose corresponding trajectories are smooth. The transformation follows from identifying the two extremes of the interval $[0,1]$ to the same point in $\mathbb{S}^1$ and the change of variable $\phi_i = 2\pi n_i$.

Remark 1: There are other possible implementations that also have a phase model representation, e.g. voltage controlled oscillators generating sinusoidal signals. The results of this paper are applicable to such systems if a suitable phase and frequency estimation is feasible.

Remark 2: This transformation also provides an interesting interpretation for the time consensus problem. Since the state space of the phase model is the N-torus $\mathbb{T}^N$, the time consensus problem is equivalent to the *Second Order N-torus Consensus* which seeks convergence in both phase and frequency, i.e., $\|\phi_j(t) - \phi_i(t)\| \to 0$ and $\left\|\dot{\phi}_j(t) - \dot{\phi}_i(t)\right\| \to 0$, for all $i, j \in V$ as $t \to +\infty$.

The system is said to reach *frequency consensus* if the trajectories converge to limit cycles of the form

$$\phi_i(t) = \omega^* t + \phi_i^* \quad \forall i \in V, \tag{4}$$

where ω^* denotes the synchronizing frequency. Furthermore, the system achieves *phase consensus* if $\phi_i^* = \bar{\phi}$, $\forall i \in V$.

To illustrate the challenge of this problem, we first consider a standard model of coupled oscillators

$$\dot{\phi}_i = \omega_i + \sum_{j \in \mathcal{N}_i} f_{ij}(\phi_j - \phi_i) \tag{5}$$

in which each node i corrects its own frequency by an additive term depending on the phase difference with its neighbors. $\mathcal{N}_i$ denotes the set of neighbors of i and the function f_{ij} is 2π-periodic and usually odd, e.g. see Figure 2. Even when the frequencies are homogeneous among the nodes, (5) presents several limit cycles of the form of (4). Their existence and stability depend on several factors such as topology and coupling [8], and most of the existing works are constrained to study either local stability or fixed topologies [8], [9].

Only recently, global convergence results have been obtained by adding constraints on f_{ij} [11], [7]. In spite of these global convergence results, all of them assume that every oscillator has the same natural frequency ω_i.

In fact, once the frequencies are different, phase consensus breaks. This is mainly due to the fact that in order for synchronization to occur

$$\omega^* = \omega_i + \sum_{j \in \mathcal{N}_i} f_{ij}(\phi_j^* - \phi_i^*),$$

must hold $\forall i \in V$ and thus the system needs to compensate the frequency mismatch by introducing a certain phase difference.

B. Combining Synchronization of Coupled Oscillators with Consensus Algorithms

We now show how the limitation of coupled oscillators in achieving phase consensus when the frequencies are different can be overcome by combining ideas from coupled oscillators and linear consensus literature. Instead of additively changing the frequency as in (5), we propose to control the clock speed using a multiplicative scalar γ_i, i.e.

$$\dot{\phi}_i = \omega_i \gamma_i, \quad \forall i \in V. \tag{6}$$

This can be done, for example, in our clock implementation by multiplying the counter value n_i times γ_i. In this way, only when $\gamma_i = 1$, the ith clock will run at its own natural frequency.

The problem now reduces to how to define a control law for γ_i. Since our aim is to obtain consensus in both frequency, $\gamma_i \omega_i$, and phase, ϕ_i, then the adaptation $\dot{\gamma}_i$ should accept such desired solution.

For instance, a first try to solve this problem might be to use

$$\dot{\gamma}_i = \sum_{j \in \mathcal{N}_i} f_{ij}(\phi_j - \phi_i), \quad \forall i \in V,$$

which amounts to adding an integrator to the dynamics. Formally, we can express the dynamics in vector from as,

$$\dot{\gamma} = -BF(B^T\phi) \quad \text{and} \quad \dot{\phi} = \Omega\gamma, \tag{7}$$

where $\Omega = \operatorname{diag}[\omega_i]$, B is the oriented incidence matrix of G [1], i.e.

$$B(k, ij) := \begin{cases} 1 & \text{if } k = j, \\ -1 & \text{if } k = i, \\ 0 & \text{otherwise,} \end{cases} \tag{8}$$

and $F(\cdot)$ is the column vector valued function

$$F(y) := [f_{ij}(y_{ij})]_{ij \in E}.$$

What it is interesting of (7) is that even though the frequencies ω_i might be different, the system still allows phase and frequency consensus. In fact, by setting $\gamma_i = \frac{\omega^*}{\omega_i}$, $\phi_i^* = \bar{\phi}$, and integrating (7) we obtain the consensus orbit

$$\phi(t) = \omega^* t \mathbf{1}_N + \bar{\phi}\mathbf{1}_N, \forall i \in V,$$

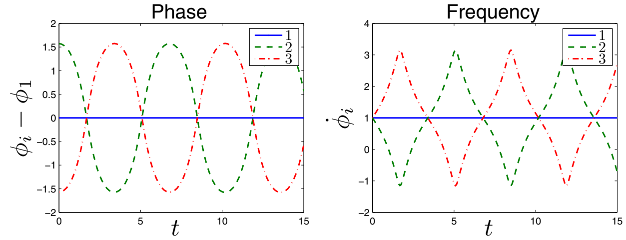
where $\mathbf{1}_N \in \mathbb{R}^N$ is the column vector of all ones.

However, a more detailed study of (7) unveils an additional oscillatory behavior that this system exhibits. To see this consider the function $W : \mathbb{T}^N \times \mathbb{R}^N \to \mathbb{R}$,

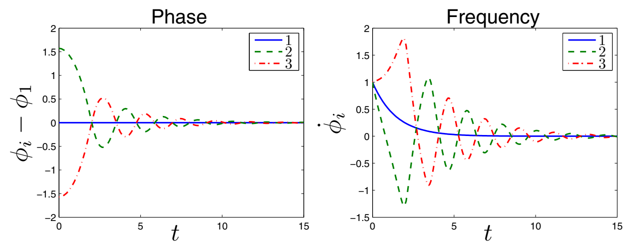
$$W(\phi, \gamma) = \frac{1}{2}\gamma^T \Omega \gamma + V(B^T\phi), \tag{9}$$

where $V(y) = \sum_{ij \in E} \int_0^{y_{ij}} f_{ij}(s)ds$.

The function $W(\phi, \gamma)$ can be interpreted as the energy function of (7). In fact, it is easy to see that $\dot{\phi} = \frac{\partial W}{\partial \gamma}$ and $\dot{\gamma} = -\frac{\partial W}{\partial \phi}$ which means that the system (7) is Hamiltonian and that the energy $W(\phi, \gamma)$ remains constant along trajectories, i.e. $\dot{W} \equiv 0$.



(a) Nonlinear oscillations of (7): Phases plotted relative to ϕ_1



(b) Adding a positive damping

Fig. 1. Oscillations and Damping

This suggests that one could possibly find trajectories in which energy changes from kinetic ($\frac{1}{2}\gamma^T\Omega\gamma$) to potential $V(B^T\phi)$ and back again over time. In Figure 1(a) we illustrates one of these trajectories. We simulated a fully connected network of 3 nodes with $\omega_i = 1$, $\forall i \in V$ and with initial condition $\phi = (0, \frac{\pi}{2}, -\frac{\pi}{2})^T$, and $\gamma = \mathbf{1}_3^T$.

Therefore, although (7) allows the type of solutions we are seeking, the additional integration introduced does not guarantee its convergence. A standard technique to overcome this oscillatory nonlinear behavior [12], [6] is to introduce a damping term in (7) that disspates energy. For instance, consider

$$\dot{\gamma} = -BF(B^T\phi) - \nu\Omega\gamma \quad \text{and} \quad \dot{\phi} = \Omega\gamma, \tag{10}$$

where ν is a positive scalar.

Figure 1(b) shows how now the trajectories with the same initial conditions as before converge. Unfortunately, as Figure 1(b) suggests, (10) can only admit limit cycles with $\omega^* = 0$ which is unsuitable for our application.

The problem is that the term $-\nu\Omega\gamma$ in $\dot{\gamma}$ is behaving similarly to the system $\dot{x} = -\nu x$ which clearly has a unique equilibrium in $x = 0$. However, if we consider instead,

$$\dot{x}_i = \sum_{ij} a_{ij}(x_j - x_i),$$

it is well known from linear consensus literature that under mild conditions on $a = [a_{ij}]_{ij \in V\times V}$, the trajectories with given initial condition x^0 always converge to $x_i(t) \to \frac{1}{n}\sum_{i=1}^n x_i^0$ $\forall i \in V$. More precisely, this occurs whenever $a_{ij} \geq 0$ and the induced graph $G_a = (V, E_a)$, with $E_a = \{ij \in V \times V | a_{ij} > 0\}$, is *connected*.

Therefore, it seems promising to study

$$\dot{\gamma} = -BF(B^T\phi) - L(a)\Omega\gamma \quad \text{and} \quad \dot{\phi} = \Omega\gamma, \tag{11}$$

where $L(a) = B_a \mathrm{diag}[a_{ij}] B_a^T$ is the weighted Laplacian [1] of the possibly different graph $G_a = (V, E_a)$ and B_a denotes the incidence matrix of G_a as defined in (8).

In the Euclidean counterpart of this problem it is possible to guarantee convergence even when only two nodes share speed information [2]. In our case, we need to assume that the undirected graph G_a is connected.

One interpretation of the two terms of $\dot{\gamma}$ in (11) is the following. The term $-BF(B^T\phi)$ seeks phase consensus, although it cannot achieve it by itself. And the term $-L(a)\Omega\gamma$ seeks frequency consensus and in fact it can achieve, but it fails to guarantee phase consensus. Thus, the term $-L(a)$ acts as a damping term for the phase consensus algorithm, or equivalently $-BF(B^T\phi)$ acts as a correction term of the frequency consensus algorithm.

III. Synchronization Frequency

In this section we compute the value ω^* achieved by (11). We start by providing a general characterization for ω^*.

Proposition 1: Given initial conditions (ϕ^0, γ^0). If the system (11) converges to an orbit like (4), then the achieved frequency can be computed using

$$\omega^* = \frac{\sum_{i=1}^N \gamma_i^0}{\sum_{i=1}^N \frac{1}{\omega_i}}. \tag{12}$$

Proof: A well know property of B (or B_a) is that $\ker[B^T] = \mathrm{span}[\mathbf{1}_N]$ whenever G (or G_a) is connected. Using this property, it is straightforward to show that $\mathbf{1}_N^T\dot{\gamma} \equiv 0$. Then, given initial condition γ^0 we have

$$\sum_{i=1}^N \gamma_i(t) = \mathbf{1}_N^T\gamma(t) = \mathbf{1}_N^T(\gamma^0 + \int_{s=0}^t \dot{\gamma}(s)ds)$$
$$= \mathbf{1}_N^T\gamma^0 + 0 = \mathbf{1}_N^T\gamma^0.$$

Thus, the quantity $\sum_{i=1}^N \gamma_i(t) = \sum_{i=1}^N \gamma_i^0$ is an invariant of the system.

Suppose now that the system converges to a limit cycle, or equivalently that $\gamma_i(t) \to \frac{\omega^*}{\omega_i}$. Then it follows

$$\sum_{i=1}^N \gamma_i^0 = \sum_{i=1}^N \gamma_i(t) \to \sum_{i=1}^N \frac{\omega^*}{\omega_i} = \omega^* \sum_{i=1}^N \frac{1}{\omega_i}.$$

Solving for ω^* gives the desired result. ■

When every clock starts with initial frequency equal to its own natural frequency ($\gamma_i = 1$), ω^* will be the harmonic mean, i.e.,

$$\frac{1}{\omega^*} = \frac{1}{N}\sum_{i=1}^N \frac{1}{\omega_i}. \tag{13}$$

The reason why the system does not achieve the average of $\{\omega_i\}$ is that the system is in fact averaging a different quantity. This can be seen by substituting ω_i with $\frac{2\pi}{T_i}$ in (13) which gives, $T^* = \frac{2\pi}{\omega^*} = \frac{1}{N}\sum_{i=1}^N \frac{2\pi}{\omega_i} = \frac{1}{N}\sum_{i=1}^N T_i$. Thus, the achievable frequency is such that the cycle duration T^* is the average cycle duration among all the oscillators when running with their natural frequencies $\frac{1}{T_i}$'s.

Notice also that this property is still preserved if every time a new clock is added to the network, its γ_i is initialized to 1, since then we will still have $\mathbf{1}_{N+1}^T\gamma_{N+1}^0 = \mathbf{1}_N^T\gamma_N^0 + 1 = N + 1$.

IV. Global Synchronization

A. Frequency Concensus

We first present our global convergence result for frequency consensus.

Theorem 1 (Frequency Consensus): Consider the system (11) running over connected graphs G and G_a, with f_{ij} being symmetric, odd and continuously differentiable. Then, for every initial condition, the trajectories converge to a limit cycle as in (4) with ω^* as in (12).

Proof: Consider the Lyapunov candidate function $W(\phi,\gamma)$ as defined in (9). Notice that the domain of W is composed by the cross product ($\times$) of a compact space $\mathbb{T}^N$ and the unbounded space $\mathbb{R}^N$. Therefore, to apply the global version of Lassale's Invariance Principle we only need W to be radially unbounded with respect to γ which is true since Ω is positive definite.

Thus, for any given initial condition (ϕ^0,γ^0) with $W(\phi^0,\gamma^0)=c$ we can always find a scalar $r>0$ such that for every γ not in a ball $\mathcal{B}_r\subset\mathbb{R}^N$ of radius r and center 0, $W(\phi,\gamma)>c$ for any $\phi\in\mathbb{T}^N$. Therefore, the set $\Psi_c:=\{(\phi,\gamma):W(\phi,\gamma)\le c\}\subset\mathbb{T}^N\times\mathcal{B}_r$ si compact.

We start by taking the derivative of W along the trajectories. This gives

$$\begin{aligned}\dot W(\phi,\gamma)&=\gamma^T\Omega\dot\gamma+\left\langle B\nabla V(B^T\phi),\dot\phi\right\rangle\\&=\gamma^T\Omega[-BF(B^T\phi)-BB^T\Omega\gamma]\\&\quad+\left\langle B\nabla V(B^T\phi),\Omega\gamma\right\rangle\\&=-\gamma^T\Omega L(a)\Omega\gamma-\gamma^T\Omega BF(B^T\phi)\\&\quad+\gamma^T\Omega BF(B^T\phi)\\&=-(\gamma\Omega)^TL(a)(\Omega\gamma)\le 0\end{aligned}$$

where in the first two steps we use the chain rule for gradients $\nabla(V\circ B^T)(\phi)=B\nabla V(B^T\phi)$ and (11), in the third step we use the identity $\nabla V(y)=F(y)$, and in the last step we used the fact that $L(a)$ is positive semidefinite, i.e. $x^TL(a)x\ge 0$ $\forall x$.

Thus, we have shown that Ψ_c is a compact positively invariant set since $\dot W(\phi,\gamma)\le 0$ $\forall(\phi,\gamma)\in\Psi_c$. Lassale Invariance Principle then implies that the system converges to the largest invariant M set inside $\{\dot W\equiv 0\}\cap\Psi_c$. Now, since G_a connected implies that $\mathbf{1}_N$ is the only eigenvector of $L(a)$ with zero eigenvalue, then $\dot W\equiv 0$ implies

$$\Omega\gamma(t)\equiv\omega(t)\mathbf{1}_N.$$

Differentiating both sides, we get $\Omega\dot\gamma(t)\equiv\dot\omega(t)\mathbf{1}_N$ which is also restricted to $\mathrm{span}[\mathbf{1}_N]$. However, we already know that $\dot\gamma(t)\in\ker[\mathbf{1}_N^T]$. Then, since

$$\Omega^{-1}\mathrm{span}[\mathbf{1}_N]\cap\ker[\mathbf{1}_N^T]=\{0\},$$

we must have $\dot\gamma\equiv 0$, which implies $\gamma(t)\equiv\omega^*\Omega^{-1}\mathbf{1}_N$ for some constant scalar ω^*. Therefore we must have $M=M_{\mathbb{T}^N}\times\{\omega^*\Omega^{-1}\mathbf{1}_N\}$ and the system converges to an orbit like (4). Proposition 1 shows that ω^* is as in (12). ■

Remark 3: Theorem 1 guarantees that the system will synchronize to the frequency harmonic mean of the nodes (provided $\gamma_i^0=1$) but it does not guarantee phase consensus. The main problem is that, as in the classical couple oscillators system, there might be other attractive orbits besides consensus. In the next section we show that certain conditions on coupling functions can guarantee that only the phase consensus orbit is attractive.

B. Phase Consensus

In this section we focus on studying the stability of the limit cycles. We know from Theorem 1 that (11) converges for every initial condition to an orbit like (4), where ω^* is characterized by (12). Also, since $\gamma(t)\to\gamma^*$ with $\gamma_i^*=\frac{\omega^*}{\omega_i}$, then from (11) we get

$$\begin{aligned}0&=-BF(B^T\phi^*)-L(a)\Omega\gamma^*\\&=-BF(B^T\phi^*)-B_a\mathrm{diag}[a_{ij}]B_a^T\Omega\Omega^{-1}\omega^*\mathbf{1}_N\\&=-BF(B^T\phi^*)\end{aligned}$$

where in the last step we used again $\ker[B_a^T]=\mathrm{span}[\mathbf{1}_N]$. Thus, ϕ^* must be a solution to $BF(B^T\phi^*)=0$.

These orbits are exactly the same that would be achieved by the system of coupled oscillators (5) if $\omega_i=\omega^*$ and f_{ij} is as in Theorem 1. Their stability, when using (5), depends on the locations of the eigenvalues of the Laplacian

$$L(w(\phi^*))=B\mathrm{diag}[f'_{ij}(\phi_j^*-\phi_i^*)]B^T,$$

which is the negation of the Jacobian of (5). Thus, if there is at least one negative eigenvalue of $L(w(\phi^*))$, then the orbit defined by ϕ^* is unstable.

The challenge in the coupled oscillators case was finding conditions on f_{ij} that guarantee the instability of every non-consensus orbit since their locations are typically unknown. In [7] it was shown that a sufficient condition for phase consensus is that f_{ij} belongs to the family of functions $\mathcal{F}_b$, with $b\in(0,\frac{2\pi}{N-1}]$, such that f_{ij} is:

- Symmetric: $f_{ij}=f_{ji}$ $\forall ij$
- Odd: $f_{ij}(-\theta)=-f_{ij}(\theta)$
- Continuously differentiable: $f_{ij}\in C^1$
- $f'_{ij}(\theta;b)>0$, $\forall\theta\in(0,b)\cup(2\pi-b,2\pi)$, and
- $f'_{ij}(\theta;b)<0$, $\forall\theta\in(b,2\pi-b)$.

See Figure 2 for an illustration with $b=\frac{\pi}{2}$ and $\frac{\pi}{6}$.

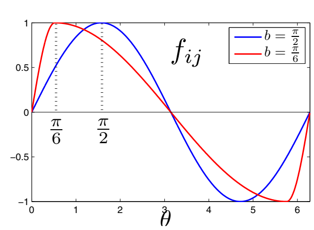


Fig. 2. Coupling function $f_{ij}\in\mathcal{F}_b$ for $b=\frac{\pi}{2}$ and $b=\frac{\pi}{6}$

Although the Jacobian matrix of (11),

$$J_{\phi^*}=\begin{bmatrix}0&\Omega\\-L(w(\phi^*))&-L(a)\Omega\end{bmatrix},$$

now depends on other terms like $L(a)$ and Ω, we will show that, provided $L(a)$ is positive definite, induces a connected graph G_a and $\omega_i > 0\ \forall i \in V$, the eigenvalues of $L(w(\phi^*))$ still control the stability.

In order to see this property, consider small perturbation $\delta\phi$, $\delta\gamma$ around a certain orbit (4) and the following change of variable

$$x = T^T \delta\phi, \quad z = T^T \Omega \delta\gamma$$

where $T \in \mathbb{R}^{N\times(N-1)}$ is the matrix whose columns $\{T_j\}$ are orthonormal and span $\ker[\mathbf{1}_N^T]$. Notice that by definition, TT^T is the orthogonal projection onto $\ker[\mathbf{1}_N^T]$ and $T^T T = I_{N-1}$, the identity matrix of dimension $N-1$.

The transformation T is clearly not invertible, but it is quite useful to keep track of the disagreement of $\delta\phi$ and $\Omega\delta\gamma$. This is because given $x = T^T v$, x becomes zero only when $v \in \text{span}[\mathbf{1}_N]$. .

In other words, the change of variable maps the reference orbit to the point $x = 0$, $z = 0$, and the corresponding dynamics

$$\dot{x} = z \quad \text{and} \quad \dot{z} = T^T \Omega [L(w(\phi^*))Tx + L(a)Tz]$$

describes the evolution of $\delta\phi$ and $\Omega\delta\gamma$ projected onto the subspace $\ker[\mathbf{1}_N^T]$. We now show the following theorem.

Theorem 2 (Orbits Instability): Given connected graphs G and G_a, positive definite Ω and positive semidefinite $L(a)$. Consider any orbit described by ω^* and ϕ^* as in (4). Whenever $L(w(\phi^*))$ has a negative eigenvalue, the orbit is *unstable*.

Proof: We prove this theorem by showing that if $L(w(\phi^*))$ has a negative eigenvalue, the equilibrium $(x^*, z^*) = (0, 0)$ is unstable. Thus, since x and y are projected version of $\delta\phi$ and $\Omega\delta\gamma$, this shows that in fact the orbit is unstable.

We will use Chetayev's instability theorem ([3] Th 4.3) to show our claim. That is, we will find a function $W(x, z)$ such that for any arbitrary neighborhood $\mathcal{B}$ of $(0, 0)$, there is a set $U = \{(x, z) \in \mathcal{B} | W(x, z) < 0\}$ with boundary $\partial U = \{(x, y) \in \mathcal{B} | W(x, y) = 0\}$ such that

(i) $U \neq \emptyset$,
(ii) $(0, 0) \in \partial U$,
(iii) and $\forall (x, y) \in U$, $\dot{W}(x, z) < 0$.

Let $W(x, z)$ be a slightly modified linearized version of (9), i.e.

$$W(x, z) = z^T (T^T \Omega T)^{-1} z + x^T T^T L(w(\phi^*))Tx.$$

Notice that $W(x, z)$ is well defined since Ω is positive definite and thus $T^T \Omega T$ is invertible.

Also, since $L(w(\phi^*))$ is symmetric and has at least one negative eigenvalue, there is some vector $v = T\hat{x} \in \ker[\mathbf{1}_N^T]$ such that $W(\hat{x}, 0) = -\varepsilon \|v\|^2 < 0$. Thus $U \neq \emptyset$ and (i) holds. Claim (ii) follows directly by definition of ∂U and $W(x, y)$.

Finally, a similar computation like the one in Theorem 1 for $\dot{W}(\phi, \gamma)$ shows that

$$\dot{W}(x, z) = -z^T T^T L(a)Tz < 0, \quad \forall z \neq 0,$$

where now $T^T L(a) T$ is positive definite since the range of T is the orthogonal complement of $\ker[L(a)] = \text{span}[\mathbf{1}_N^T]$. Therefore, (iii) follows and the orbit is unstable. ∎

Theorem 2 provides a connection between our clock synchronization algorithm and equal frequency coupled oscillators. It essentially shows that provided $\omega_i > 0\ \forall i$ and $L(a)$ is positive semidefinite with only one zero eigenvalue, both systems contain the same instability condition. This allows us to prove the main result of the paper.

Theorem 3 (Phase Consensus): Consider the clock system (11) running over connected undirected graphs G and G_a. Then, provided $f_{ij} \in \mathcal{F}_b$ with $b \in (0, \frac{2\pi}{N-1}]$, for almost every initial condition (ϕ^0, γ^0), (11) achieves phase and frequency consensus with ω^* as in (12).

Proof: Since G and G_a are connected and f_{ij} by definition is symmetric, odd and continuously differentiable, then by Theorem 1, (11) will always achieve frequency consensus. As mentioned before, since there are many possible synchronized orbits, this does not guarantee phase consensus.

However, since $f_{ij} \in \mathcal{F}_b$ with $b \in (0, \frac{2\pi}{N-1}]$, Corollary 5 of [7] guarantees that any other configuration ϕ^* of (4) will produce a negative eigenvalue in $L(w(\phi^*))$. Therefore, by Theorem 2, every limit cycle of (11) besides the phase consensus one is unstable.

So, unless the initial condition (ϕ, γ) belongs to the zero measure set that converges to these unstable orbits, (11) will always converge to the orbit with phase and frequency consensus. ∎

V. SIMULATIONS

We now present simulations to illustrate our results. In Figure 3 we simulate a network of three oscillator running the coupled oscillator algorithm (5) and the clock synchronization algorithm (11). Both graphs G and G_a are complete and the initial condition is

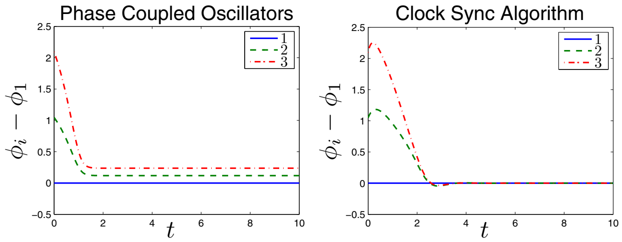
$$\phi^0 = (0, \frac{\pi}{3}, \frac{2\pi}{3})^T \text{ and } \gamma^0 = (1, 1, 1)^T,$$

where γ^0 is only used in (11). The frequency of each clock is $(\omega_1, \omega_2, \omega_3) = (1, 2, 3)$.

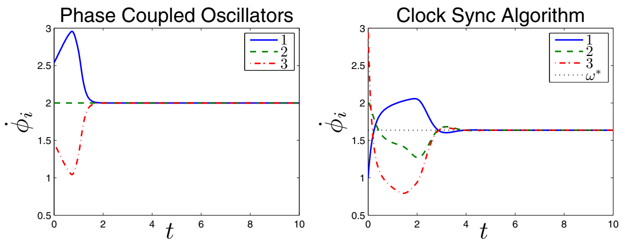
Figure 3(a) shows that while (11) can achieve phase consensus, (5) cannot achieve it due to the frequency difference. Figure 3(b) shows that both systems succeed in achieving frequency consensus. Since the initial γ^0 sums to $N = 3$, then (11) will have a ω^* as in (13), which in our case reduces to $\omega^* = 1.6364$.

We now show why a condition of $b \in (0, \frac{2\pi}{N-1}]$ is needed in order to guarantee phase consensus. We simulate (11) over a ring network of $N = 6$ nodes, set $\omega_i = 1\ \forall i \in V$ and initialize the state with values $\phi^0 = [\frac{2\pi k}{6}]_{k\in\{0,\ldots,5\}}$ and $\gamma^0 = \mathbf{1}_6$.

Figure 4 shows two simulations of the same ring network with exactly same initial conditions. The only difference is the choice of f_{ij}. Figure 4(a) shows that when we use a $b = \frac{\pi}{2} > \frac{\pi}{N-1}$ the system stays in the orbit defined by the initial condition. However, once $b = \frac{\pi}{6} < \frac{\pi}{N-1}$, Figure 4(b), the orbit is no longer stable and the system converges to the phase and frequency consensus.

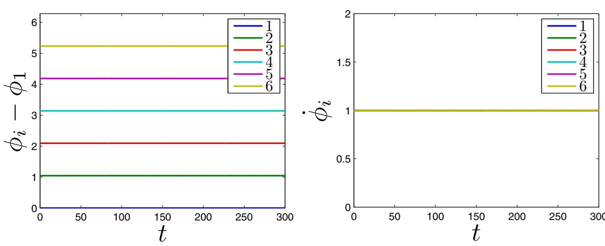


(a) Phase: Couple Oscillators have to compensate the frequency mismatch

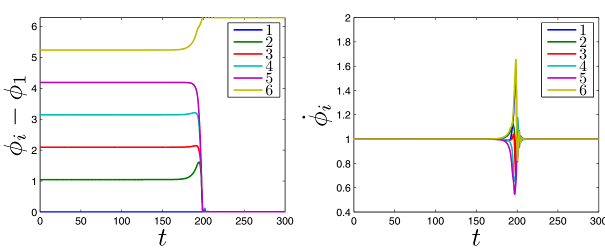


(b) Frequency: Both systems achieve a common frequency

Fig. 3. Different Frequency CO vs Clock Synchronization



(a) Stable Limit Cycle: $b = \frac{\pi}{2}$



(b) Stable Limit Cycle: $b = \frac{\pi}{6}$

Fig. 4. De-stablizing orbits by shrinking b below $\frac{\pi}{N-1}$

VI. Conclusion

This paper introduces a fully distributed synchronization algorithm that is able to achieve both frequency and phase consensus for heterogeneous oscillators. We provide a sufficient condition on the coupling function that guarantees almost global convergence for arbitrary connected topology. The synchronizing frequency is shown to be the harmonic mean of the natural oscillation frequencies.

For future directions, we are interested in studying the effect of communication delays as well as investigating discrete version of (11). It is also of great interest to quantify how much damping is needed to avoid the natural oscillation of the double integrator dynamics. In [2] it was shown that in the linear case on $\mathbb{R}^N$, G_a only needed to have one link for the dynamics to converge to the double consensus. However, here we require G_a to be connected. We also plan to further explore the relationship between coupled oscillators and our second order dynamics. Here, we showed that the instability of the orbits in both systems coincide but we believe there is a deeper connection.

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