Motion Coordination of Planar Rigid Bodies

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Abstract— **Motion coordination of autonomous vehicles has
applications from target surveillance to climate monitoring.
Previous research has yielded stabilizing control laws for a self-
propelled-particle model with first-order rotational dynamics,
but this model may not adequately describe the rotational and
translational dynamics of vehicles in the atmosphere or ocean.
This paper describes the design of algorithms for decentralized
control of self-propelled vehicles with second-order rotational
and translational dynamics. We utilize a backstepping-based
controller to achieve the desired second-order rotational dy-
namics of each vehicle and feedback linearization to attain
the desired (constant) forward speed. We consider parallel and
circular formations. These controls extend prior results to a
more realistic vehicle model, while maintaining comparable
performance.**

I. INTRODUCTION

Dynamic models with ever increasing realism have been
used to describe the cooperative behavior of agents working
to accomplish a given task. In real-world systems the under-
lying dynamics unfold according to second-order differential
equations, however many cooperative control schemes have
been based on first-order models. In prior research, agents
have been modeled as particles for simplicity and ease of
control design. In some instances, the agents are modeled
as unicycles, meaning that they can only move forward and
turn. There are generally two control inputs in such models:
a speed controller and a steering controller. Sometimes
the problem is constrained further, as is the case when
the translational speed of the platform is assumed to be
constant. Examples of constant-speed models that rely solely
on steering control are described in [11], [4], [8], [12], [1].

Implementing cooperative control laws based on particle
dynamics enables only the location of the vehicle's center
of mass to be controlled. Additional control inputs must
be considered to ensure that not only the desired angular
orientation is achieved, but also that the forward and trans-
verse speeds achieve the desired values. Although there are
numerous dynamic models that focus on the development of
motion-planning algorithms for mobile robots, they tend to
consider only a single platform rather than a swarm. One
such example is [3], in which the motion-planning output of
a kinematic model is used with a dynamic rigid body model
in order to develop a trajectory for an autonomous vehicle
that can avoid obstacles. Another example is [2], in which
a Lagrangian formulation is used to derive the dynamic
equations of a single mobile robot with non-holonomic
constraints. To regulate movement of the robot, a velocity
controller is used to generate torque control by way of
integrator backstepping.

There is also a sizeable body of work that focuses on
using kinematic models for each agent in a swarm, however,
many such models typically do not incorporate rigid body
dynamics. An example of a kinematic control for a multi-
agent system is [6], in which integrator backstepping is used
for formation control of non-holonomic agents. A similar
approach has been used in [7], in which the authors intro-
duced a vehicle model based on [11] and [8] with second-
order rotational dynamics. We previously used integrator
backstepping to adapt controllers designed for first-order
rotational dynamics; however, forces and moments were not
considered before now.

In this paper, we consider a collection of vehicles with
second-order translational and rotational dynamics. We show
that under existing second-order steering control laws pro-
vided in [7] and a new speed control, the planar rigid
body model achieves comparable closed-loop performance
to a particle model. In prior work, the vehicles were each
considered to be particles with unit mass and unit speed and
the direction of travel was controlled by a force orthogonal
to the heading [8]. The vehicles we consider in the present
paper can rotate and translate with variable speed subject to
thrust, steering, and drag forces.

The contribution of this paper is to provide decentralized
controllers for a collection of planar rigid bodies. These
controllers enable the swarm to achieve parallel and circular
formations using thrust and steering forces. We use feedback
linearization to design the thrust controller and Lyapunov-
based control to design the steering controller. We control
the rigid-body dynamics to match those of a self-propelled
particle; that is, each platform moves with constant speed in
the desired formation.

This paper is organized as follows. Section II reviews the
second-order particle model with steering controllers devel-
oped via backstepping. Section III describes the dynamics
of a set of planar rigid bodies and Section IV derives the
controllers used to create parallel and circular formations.
Section V summarizes the paper and ongoing research.

II. BACKGROUND

An existing model for planar collective motion with second-order rotational dynamics is

$$\begin{aligned}
\dot{r}_k &= e^{i\theta_k} \\
\dot{\theta}_k &= \omega_k \\
\dot{\omega}_k &= a_k,
\end{aligned} \tag{1}$$

where $r_k \in \mathbb{C}$ is the position, θ_k is the heading, and a_k represents the second-order steering control of vehicle $k = 1,...,N$. In this model each vehicle moves at unit speed in the direction θ_k. Model (1) represents an extension of a self-propelled particle model with first-order rotational dynamics [7]. In the first-order case, phase and spacing potentials were used to derive control laws for parallel and circular formations in the absence of an external flowfield [11], and in the presence of a spatiotemporal flowfield [10]. Since first-order rotational dynamics may not adequately describe rigid-body motion, these first-order particle models were used as the first component of a second-order system for which we designed a controller using integrator backstepping.

The expression for a_k is found through the standard backstepping procedure to achieve $\omega_k = \phi_k$ as $t \to \infty$, where ϕ_k is the steering controller used to generate parallel and circular formations in the first-order system. We derive a_k via the composite Lyapunov function

$$V_c = V + \tfrac{1}{2}\sum_{k=1}^{N} z_k^2, \tag{2}$$

where V is the smooth potential that must be minimized in order to achieve collective parallel or circular motion in the first-order system. The term $z_k = \omega_k - \phi_k$ is the error between the actual and desired first-order rotational dynamics. Taking the derivative of (2) along solutions of (1) gives

$$\dot{V}_c = \sum_{k=1}^{N}\left[\frac{\partial V}{\partial \theta_k}\dot{\theta}_k + z_k \dot{z}_k\right], \tag{3}$$

where $\dot{\theta}_k = \phi_k + z_k$ and $\dot{z}_k = v_k$ is a controller that we design to achieve $\dot{V}_c \le 0$. The backstepping controller a_k is found by the transformation

$$a_k = v_k + \dot{\phi}_k. \tag{4}$$

There are two types of motion primitives for which we derive a_k. The first is parallel formations, which has the key control parameter p_θ defined as [11]

$$p_\theta = \frac{1}{N}\sum_{k=1}^{N} e^{i\theta_k}. \tag{5}$$

p_θ represents the average linear momentum of the particles (assuming they have unit mass). Collective parallel motion is achieved by maximizing the average linear momentum of the particle system via the phase potential [11]

$$V(\theta) = \tfrac{1}{2}\left(1 - |p_\theta|^2\right). \tag{6}$$

For the first-order system, the controller for parallel formations $\dot{\theta}_k = \phi_k(\theta)$ is [11]

$$\phi_k(\theta) = -K\langle p_\theta, ie^{i\theta_k}\rangle, \quad K < 0. \tag{7}$$

To achieve $\dot{V}_c \le 0$ we choose

$$v_k = \frac{1}{N}\langle p_\theta, ie^{i\theta_k}\rangle - \kappa z_k, \tag{8}$$

where κ is a positive gain. This gives us

$$\dot{V}_c = \sum_{k=1}^{N}\left[\frac{K}{N}\langle p_\theta, ie^{i\theta_k}\rangle^2 - \kappa z_k^2\right] \le 0. \tag{9}$$

The solutions θ_k and z_k converge to the largest invariant set for which $\dot{V}_c \equiv 0$, given by

$$\Lambda = \{\langle p_\theta, ie^{i\theta_k}\rangle \equiv 0, z_k \equiv 0 \ \forall \ k\}. \tag{10}$$

The condition that $\langle p_\theta, ie^{i\theta_k}\rangle \equiv 0$ implies that $\theta_j = \theta_k$ for all particle pairs j and k [11]. $z_k = 0$ implies $\omega_k = \phi_k(\theta)$; however, from the preceding condition we know that $\phi_k(\theta) = 0$ in Λ. This implies that θ_k is constant for all k. Thus, Λ contains the set of parallel formations.

Using transformation (4), a backstepping-based second-order controller for parallel formations in (1) is [7]

$$\begin{aligned}
a_k = &\frac{1}{N}\langle p_\theta, ie^{i\theta_k}\rangle - \kappa(\omega_k + K\langle p_\theta, ie^{i\theta_k}\rangle) \\
&- \frac{K}{N}\sum_{j=1}^{N}\left[\langle e^{i\theta_j}, e^{i\theta_k}\rangle(\omega_j - \omega_k)\right], \ K < 0, \kappa > 0.
\end{aligned} \tag{11}$$

The second motion primitive is circular formations. Collective circular motion is achieved by minimizing the spacing potential [11]

$$V(\mathbf{r},\theta) = \tfrac{1}{2}\langle \mathbf{c}, P\mathbf{c}\rangle. \tag{12}$$

The $N \times 1$ matrix $\mathbf{c}$ contains the centers c_k, $k = 1,...,N$, of the circular paths followed by each particle, where [11]

$$c_k = r_k + i\omega_0^{-1} e^{i\theta_k}. \tag{13}$$

$P = I_{N \times N} - \frac{1}{N}\mathbf{1}\mathbf{1}^T$ is an $N \times N$ matrix that projects onto the space orthogonal to $\mathbf{1} = [1,...,1]^T \in \mathbb{R}^N$ and P_k represents the kth row of P. For the first-order system, the controller for circular formations is [11]

$$\phi_k(\mathbf{r},\theta) = \omega_0(1 + K\langle P_k\mathbf{c}, e^{i\theta_k}\rangle), K > 0. \tag{14}$$

To achieve $\dot{V}_c \le 0$ we choose

$$v_k = -\kappa z_k + \omega_0^{-1}\langle P_k\mathbf{c}, e^{i\theta_k}\rangle, \tag{15}$$

which gives us

$$\dot{V}_c = \sum_{k=1}^{N}\left[-K\langle P_k\mathbf{c}, e^{i\theta_k}\rangle^2 - \kappa z_k^2\right] \le 0. \tag{16}$$

The solutions θ_k and z_k converge to the largest invariant set Λ for which $\dot{V}_c \equiv 0$, given by

$$\Lambda = \{\langle P_k\mathbf{c}, e^{i\theta_k}\rangle \equiv 0, z_k \equiv 0 \ \forall \ k\}. \tag{17}$$

$\langle P_k\mathbf{c}, e^{i\theta_k}\rangle = 0$ implies that $P_k\mathbf{c} = 0$, which is only true when all circular centers are the same; that is, $P_k\mathbf{c} = 0$ if and only if $\mathbf{c}$ is in the span of $\mathbf{1}$. $\langle P_k\mathbf{c}, e^{i\theta_k}\rangle = z_k = 0$; thus we have $\dot{\theta}_k = \phi_k(\theta) = \omega_0$. Thus, all N particles travel around the same circle of radius $1/|\omega_0|$.

Using transformation (4), a backstepping control to stabilize circular formations is [7]

$$\begin{aligned}
a_k = &-\kappa(\omega_k - \phi_k(\mathbf{r},\theta)) + \omega_0 K\omega_k\langle \tilde{r}_k, ie^{i\theta_k}\rangle \\
&+ K\left(\omega_0 - \frac{1}{N}\sum_{j=1}^{N}\left[\langle e^{i\theta_j}, e^{i\theta_k}\rangle(\omega_0 - (\omega_j - \omega_k))\right]\right) \\
&+ \omega_0^{-1}\left(\langle \tilde{r}_k, e^{i\theta_k}\rangle - \omega_0^{-1}\frac{1}{N}\sum_{j=1}^{N}\langle ie^{i\theta_j}, e^{i\theta_k}\rangle\right),
\end{aligned} \tag{18}$$

where $\tilde{r}_k = r_k - \frac{1}{N}\sum_{j=1}^{N} r_j$, and $K > 0$, $\kappa > 0$.

When a uniform, time-invariant flowfield is present, the second-order model (1) can be written as

$$
\begin{aligned}
\dot{r}_k &= e^{i\theta_k} + f_k \\
\dot{\theta}_k &= \omega_k \\
\dot{\omega}_k &= a_k,
\end{aligned} \tag{19}
$$

where $f_k \in \mathbb{C}$ is the flowfield. This model can also be expressed more succinctly using the magnitude $s_k = |e^{i\theta_k} + f_k|$ and orientation $\gamma_k = \arg\{e^{i\theta_k} + f_k\}$ of the inertial velocity [8]:

$$
\begin{aligned}
\dot{r}_k &= s_k e^{i\gamma_k} \\
\dot{\gamma}_k &= \Omega_k \\
\dot{\Omega}_k &= \lambda_k.
\end{aligned} \tag{20}
$$

Similarly to the flow-free case, we may define the average linear momentum as $p_\gamma = \frac{1}{N}\sum_{k=1}^{N} e^{i\gamma}$. Thus, the controllers can be re-expressed as [7]

$$
\begin{aligned}
\lambda_k &= \frac{1}{N}\langle p_\gamma, i e^{i\gamma_k}\rangle - \kappa(\Omega_k + K\langle p_\gamma, i e^{i\gamma_k}\rangle) \\
&\quad - \frac{K}{N}\sum_{j=1}^{N}\left[\langle e^{i\gamma_j}, e^{i\gamma_k}\rangle(\Omega_j - \Omega_k)\right]
\end{aligned} \tag{21}
$$

for parallel formations, and

$$
\begin{aligned}
\lambda_k &= -\kappa(\Omega_k - \phi_k(\gamma)) + \omega_0 K \Omega_k \langle \tilde{r}_k, i e^{i\gamma_k}\rangle + \omega_0 \dot{s}_k \\
&\quad + K\left(\omega_0 s_k - \frac{1}{N}\sum_{j=1}^{N}\left[\langle e^{i\gamma_j}, e^{i\gamma_k}\rangle(\omega_0 s_j - (\Omega_j - \Omega_k))\right]\right) \\
&\quad + \omega_0^{-1}\left(\langle \tilde{r}_k, e^{i\gamma_k}\rangle - \omega_0^{-1}\frac{1}{N}\sum_{j=1}^{N}\langle i e^{i\gamma_j}, e^{i\gamma_k}\rangle\right)
\end{aligned} \tag{22}
$$

for circular formations. When we consider, without loss of generality, a flowfield oriented along the real axis given by $f = \alpha \in \mathbb{R}$, the particle speed is

$$
\dot{s}_k = -\alpha \sin\gamma_k \left[1 + \frac{\alpha\cos\gamma_k}{\sqrt{1 - \alpha^2 \sin^2(\gamma_k)}}\right]\Omega_k. \tag{23}
$$

While models (1) and (20) are useful for understanding the movement of particles in the plane, they may not be sufficient for describing rigid-body dynamics. The presence of second-order rotational dynamics is only a part of the transformation process from particle model to rigid-body model. In order to complete the transformation, we consider second-order translational dynamics as well.

III. PLANAR RIGID-BODY DYNAMICS

Recall that in Eq. (1) each particle moves with unit forward speed and its direction of travel is controlled by θ_k. We can alternately use θ_k to describe the orientation of a planar rigid body. With this description, $e^{i\theta_k} = \bar{x}_k$ defines the body-fixed frame $\mathscr{B}_k = (k, \bar{x}_k, \bar{y}_k, \bar{z}_k)$, where $\bar{z}_k$ is out of the plane and $\bar{y}_k = \bar{z}_k \times \bar{x}_k$.

Let $u_k \in \mathbb{R}$ represent the forward speed of the kth vehicle in a body-fixed frame. The transverse speed in frame $\mathscr{B}_k$ is $v_k \in \mathbb{R}$. The kth velocity expressed as components in $\mathscr{B}_k$ is

$$
\dot{r}_k = u_k \bar{x}_k + v_k \bar{y}_k. \tag{24}
$$

The time-derivative of (24) with respect to inertial frame $\mathscr{I}$ yields the following rigid-body translational kinematics:

$$
\begin{aligned}
\ddot{r}_k &= \dot{u}_k \bar{x}_k + u_k \dot{\theta}_k \bar{y}_k + \dot{v}_k \bar{y}_k - v_k \dot{\theta}_k \bar{x}_k \\
&= (\dot{u}_k - v_k \dot{\theta}_k)\bar{x}_k + (u_k \dot{\theta}_k + \dot{v}_k)\bar{y}_k.
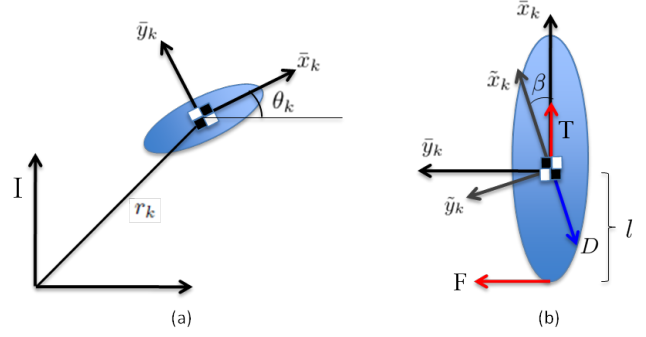\end{aligned} \tag{25}
$$

Fig. 1. (a) A single vehicle in the plane. (b) $\mathscr{B}_k$ and $\mathscr{C}_k$ reference frames; thrust T_k and steering force F_k.

We assume there are two control forces acting on the kth vehicle. The first is the thrust force T_k, which acts along the $\bar{x}_k$ axis. The steering control is F_k, which acts along the $\bar{y}_k$ axis a distance of l away from the center of mass. We also include a drag force $D_k = \frac{1}{2}\rho s_k^2 S C_D$. ($\rho$ is the density of the medium through which the vehicle is traveling; S is the vehicle reference area; and C_D is the appropriate drag coefficient.) The vehicle speed is $s_k = \sqrt{u_k^2 + v_k^2} \geq 0$.

We define a path frame for the kth vehicle as $\mathscr{C}_k = (k, \tilde{x}_k, \tilde{y}_k, \tilde{z}_k)$, where $\dot{r}_k = s_k \tilde{x}_k$ and $\tilde{z}_k = \bar{z}_k$. We denote the orientation of $\mathscr{C}_k$ relative to $\mathscr{B}_k$ as β_k. The drag force acts in the $-\tilde{x}_k$ direction [9]. Using Newton's second law with mass m_k we have

$$
\begin{aligned}
&(T_k - D_k\cos\beta_k)\bar{x}_k + (F_k - D_k\sin\beta_k)\bar{y}_k \\
&= m_k(\dot{u}_k - v_k\dot{\theta}_k)\bar{x}_k + m_k(u_k\dot{\theta}_k + \dot{v}_k)\bar{y}_k.
\end{aligned} \tag{26}
$$

Collecting the $\bar{x}_k$ terms, we have

$$
T_k - D_k\cos\beta_k = m_k(\dot{u}_k - v_k\dot{\theta}_k). \tag{27}
$$

We are able to obtain a dynamic expression for the forward speed u_k by solving (27) for $\dot{u}_k$:

$$
\dot{u}_k = -\frac{1}{m_k}D_k\cos\beta_k + \frac{1}{m_k}T_k + v_k\dot{\theta}_k. \tag{28}
$$

Following the same procedure for the terms in the $\bar{y}_k$-direction, we have

$$
\dot{v}_k = -\frac{1}{m_k}D_k\sin\beta_k + \frac{1}{m_k}F_k - u_k\dot{\theta}_k. \tag{29}
$$

In order to design the rotational dynamics, let M_k be the sum of the moments acting on the kth vehicle. Assuming there is no moment due to drag (i.e, the drag acts through the vehicle's center of mass), we have

$$
M_k = (-l\bar{x}_k) \times (F_k\bar{y}_k). \tag{30}
$$

The rotational dynamics are

$$
I_k\ddot{\theta}_k = -lF_k, \tag{31}
$$

where I_k is the moment of inertia about the center of mass of vehicle k. The equations of motion are

$$\dot{r}_k = u_k \bar{x}_k + v_k \bar{y}_k = s_k \tilde{x}_k$$

$$\dot{u}_k = -\frac{1}{m_k} D_k \cos \beta_k + \frac{1}{m_k} T_k + v_k \omega_k$$

$$\dot{v}_k = -\frac{1}{m_k} D_k \sin \beta_k + \frac{1}{m_k} F_k - u_k \omega_k \quad (32)$$

$$\dot{\omega}_k = -\frac{l}{I_k} F_k,$$

where $\omega_k = \dot{\theta}_k$, $D_k = \frac{1}{2}\rho s_k^2 S C_D$, and $\beta_k = \arccos\left(u_k/s_k\right)$.

IV. STABILIZATION OF MOVING FORMATIONS

A. Parallel Formation Control

In this section we design a decentralized feedback control to drive a collection of planar rigid bodies in the same direction at the same speed. In order to ensure that the steady-state forward speed of each vehicle is $u_k = u_0$, we use feedback linearization [5]. Choosing

$$T_k = m_k \left(\frac{1}{m_k} D_k \cos \beta_k - v_k \omega_k + K_f(u_0 - u_k)\right), \quad (33)$$

where $K_f > 0$, yields the following closed-loop dynamics

$$\dot{u}_k = K_f(u_0 - u_k). \quad (34)$$

The dynamics (34) ensure u_k exponentially converges to u_0.

Motivated by the backstepping procedure described in Section II, we choose

$$F_k = -\frac{I_k}{l} a_k, \quad (35)$$

with a_k given by (11).

Theorem 1: Consider the rigid body model (32) with thrust control (33) and steering control (35), where a_k is given by (11). Under these controllers, the set of parallel formations where $\beta_k = 0$ and $u_k = u_0$ for all k, and $\theta_k = \theta_j$ for all pairs j and k is asymptotically stable.

Proof: In order to analyze the closed-loop system, we propose the following candidate Lyapunov function

$$U_{par}(\mathbf{u}, \mathbf{v}, \theta, \mathbf{z}) = V_c + \frac{1}{2} \sum_{k=1}^{N}[(u_0 - u_k)^2 + v_k^2], \quad (36)$$

where V_c is given by (2) with phase potential (6). Taking the time-derivative along solutions of (32) gives

$$\dot{U}_{par} = \dot{V}_c + \sum_{k=1}^{N}[-(u_0 - u_k)\dot{u}_k + v_k \dot{v}_k]. \quad (37)$$

Observe that $\dot{V}_c \leq 0$ under the steering control (35). Note $a_k = a_k(\theta)$ is independent of the states u_k and v_k. If we plug in for $\dot{u}_k$ and $\dot{v}_k$ using (32) we obtain

$$\dot{U}_{par} = \sum_{k=1}^{N}\left[\frac{K}{N}\langle p_\theta, ie^{i\theta_k}\rangle^2 - \kappa z_k^2 - K_f(u_0 - u_k)^2 \right.$$

$$\left. -\frac{1}{m_k}v_k D_k \sin \beta_k + \frac{1}{m_k}v_k F_k - v_k u_k \omega_k\right].$$

The drag is $D_k = h s_k^2 \geq 0$, where $h = \frac{1}{2}\rho S C_D$. Furthermore, $\sin \beta_k = v_k/s_k$, and we consider $s_k \geq 0$. Consequently, $-v_k D_k \sin \beta_k = -h s_k v_k^2 \leq 0$.

According to the invariance principle, solutions converge to the largest invariant set Λ in which $\dot{V}_c = 0$, where Λ is
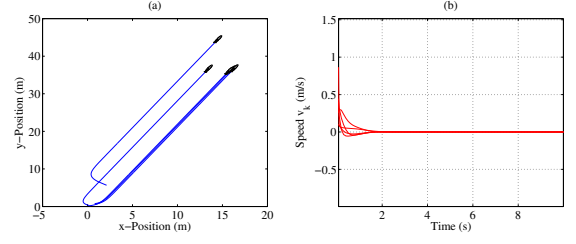

Fig. 2. (a) Rigid body parallel motion in the plane [$K = -1$, $\kappa = 5$, and $u_0 = 1$]; (b) velocity component of parallel motion with time.

given by (10). In Λ, $\omega_k \equiv 0$ because $z_k \equiv 0$ and $\phi_k \equiv 0$. Note that $\omega_k \equiv 0$ only if $\dot{\omega}_k = 0$ in Λ. Thus $F_k \equiv 0$ in Λ because $\dot{\omega}_k = a_k = -\frac{I_k}{l}F_k$. In Λ,

$$\dot{U}_{par} = \sum_{k=1}^{N}[-K_f(u_0 - u_k)^2 - \frac{h}{m_k}s_k v_k^2] \leq 0. \quad (38)$$

Application of the invariance principle in Λ shows that solutions starting in Λ converge to the largest set $M \subset \Lambda$ in which $\dot{U}_{par} = 0$. In M, $u_0 - u_k \equiv 0$ and $s_k v_k^2 \equiv 0$ which implies $u_k = u_0$ and $v_k = 0$. As a result, $\beta_k \equiv 0$ and M contains the set of parallel formations. The remainder of the proof follows from [11, Theorem 1]. ∎

Theorem 1 is illustrated in Fig. 2.

B. Circular Formation Control

To stabilize circular formations, we use the thrust control (33). However, instead of using the steering control (35) with a_k given by (18), we now consider an alternative backstepping control that takes into account the observation that for circular motion the steady-state crab angle β_k is not zero. (For parallel motion, the steady-state crab angle is zero.)

To find the steady-state crab angle and the corresponding steady-state speed, we differentiate $\tan \beta_k = v_k/u_k$ with respect to time and solve for $\dot{\beta}_k$ with $u_k = u_0 \neq 0$ to obtain

$$\dot{\beta}_k = \frac{\cos^2 \beta_k}{u_0}\left(-\frac{1}{m_k}D_k \sin \beta_k + \frac{1}{m_k}F_k - u_0 \omega_k\right). \quad (39)$$

For circular motion, $F_k = \dot{\omega}_k = 0$, which implies

$$\dot{\beta}_k = \frac{\cos^2 \beta_k}{u_0}\left(-\frac{1}{m_k}D_k \sin \beta_k - u_0 \omega_0 s_k\right). \quad (40)$$

The equilibrium points of (40) are

$$\beta_k = \pm \pi/2 \text{ and } \beta_k = \arcsin\left(-\frac{m_k u_0 \omega_0}{h s_k}\right), \quad (41)$$

where we used $D_k \sin \beta_k = h s_k v_k$.

The equilibrium points $\beta_k = \pm \pi/2$ are not possible for $u_k = u_0 \neq 0$ since $\cos \beta_k = u_k/s_k \neq 0$. For the second set of equilibrium points, $v_k = s_k \sin \beta_k$ and s_k are both constant. To see this, we solve

$$s_k^2 = u_0^2 + \left(-\frac{m_k u_0 \omega_0}{h}\right)^2, \quad (42)$$

obtaining

$$s_k = \left(\frac{u_0^2 + \sqrt{u_0^4 + 4(m_k u_0 \omega_0/h)^2}}{2}\right)^{1/2} \triangleq s_0 \quad (43)$$

and

$$v_k = -\frac{m_k u_0 \omega_0}{h} \triangleq v_0. \tag{44}$$

In light of this analysis, we modify the backstepping control (18) to allow for the vehicle speed $s_k \neq 1$. The first step is to recognize that, along solutions to (32), the circle center (13) becomes

$$c_k = r_k + i\omega_0^{-1} e^{i(\theta_k + \beta_k)}. \tag{45}$$

(We include the crab angle because the vehicle's total velocity heading must be tangent to the path.) The time derivative of (45) is $\dot{c}_k = (s_k - \omega_0^{-1}(\omega_k + \dot{\beta}_k))e^{i(\theta_k + \beta_k)}$. As a result, the desired ω_k dynamics (14) become

$$\phi_k(\mathbf{r}, \mathbf{u}, \mathbf{v}, \theta) = \omega_0(s_k + K\langle P_k \mathbf{c}, e^{i\psi_k}\rangle) - \dot{\beta}_k, \; K > 0, \tag{46}$$

where $\psi_k = \theta_k + \beta_k$. The backstepping control $\dot{\omega}_k = a_k$ that asymptotically stabilizes $\omega_k = \phi_k(\mathbf{r}, \mathbf{u}, \mathbf{v}, \theta)$ is

$$\begin{aligned}
a_k = & -\kappa(\omega_k - \phi_k(\mathbf{r}, \mathbf{u}, \mathbf{v}, \psi)) + \omega_0 K\dot{\psi}_k\langle \tilde{r}_k, ie^{i\psi_k}\rangle - \ddot{\beta}_k \\
& + K\Big(\omega_0 s_k - \frac{1}{N}\sum_{j=1}^{N}\big[\langle e^{i\psi_j}, e^{i\psi_k}\rangle\big(\omega_0 s_j - (\dot{\psi}_j - \dot{\psi}_k)\big)\big]\Big) \\
& \omega_0 \dot{s}_k + \omega_0^{-1}\Big(\langle \tilde{r}_k, e^{i\psi_k}\rangle - \omega_0^{-1}\frac{1}{N}\sum_{j=1}^{N}\langle ie^{i\psi_j}, e^{i\psi_k}\rangle\Big),
\end{aligned} \tag{47}$$

where $\tilde{r}_k = r_k - \frac{1}{N}\sum_{j=1}^{N} r_j$, $K > 0$, and $\kappa > 0$. (In practice the $\omega_0 \dot{s}_k$, $\dot{\beta}_k$, and $\ddot{\beta}_k$ terms are omitted, since they lead to a recursive formulation of the control and are zero in steady state.) We have the following result.

Theorem 2: Consider the rigid body model (32) with thrust control (33) and steering control (35), where a_k is given by (47). Under these controllers, the set of circular formations with radius $1/|\omega_0|$ and direction of rotation determined by the sign of ω_0 is asymptotically stable. In this set, $u_k = u_0$ and $v_k = v_0$ for all k and $c_j = c_k$ for all pairs j and k, where v_0 is given in (44).

Proof: To determine if the chosen controllers for T_k and F_k establish the desired closed loop behavior, we begin by defining a Lyapunov function for circular formations as

$$U_{circ}(\mathbf{r}, \mathbf{u}, \mathbf{v}, \psi, \mathbf{z}) = V_c + \frac{1}{2}\sum_{k=1}^{N}\big[(u_0 - u_k)^2 + (v_0 - v_k)^2\big], \tag{48}$$

where V_c is given by (2) with circular potential (12) and c_k given by (45). Taking the time-derivative along solutions of (32) gives

$$\begin{aligned}
\dot{U}_{circ} = & \sum_{k=1}^{N}[-K\langle P_k\mathbf{c}, e^{i\psi_k}\rangle^2 - \kappa z_k^2 \\
& -(u_0 - u_k)\big(-\frac{1}{m_k}D_k\cos\beta_k + \frac{1}{m_k}T_k + v_k\omega_k\big) \\
& +(v_0 - v_k)\big(\frac{1}{m_k}D_k\sin\beta_k - \frac{1}{m_k}F_k + u_k\omega_k\big)].
\end{aligned} \tag{49}$$

Choosing T_k to be the stabilizing control (33) ensures that u_k converges to u_0 according to the closed-loop dynamics (34). Furthermore, $\dot{V}_c \leq 0$ along solutions of (32). Therefore, solutions converge to the largest invariant set Λ in which $u_k - u_0 \equiv 0$ and $\dot{V}_c \equiv 0$, i.e.,

$$\Lambda = \{\langle P_k\mathbf{c}, ie^{i\psi_k}\rangle \equiv 0, z_k \equiv 0, u_k \equiv u_0, \forall k\}. \tag{50}$$
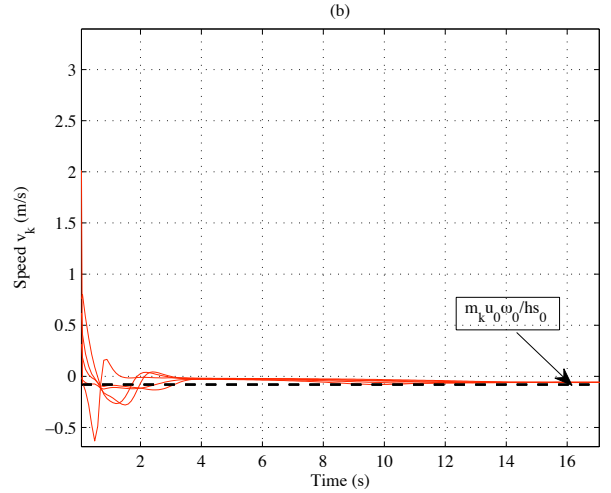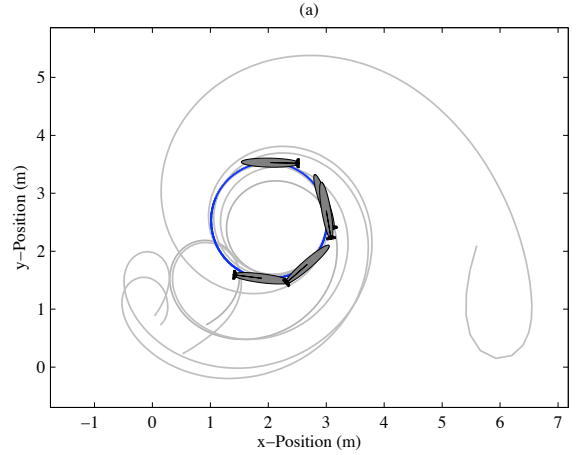

Fig. 3. (a) Rigid body circular motion in the plane [$K = 1$, $\kappa = 5$, $\omega_0 = 1$, and $u_0 = 1$]; (b) velocity components of circular motion with time.

In Λ, $\omega_k \equiv \omega_0 s_k$ and $F_k \equiv 0$, which implies that $s_k \equiv s_0$. For solutions starting in Λ, we have

$$\begin{aligned}
\dot{U}_{circ} &= \sum_{k=1}^{N}(v_0 - v_k)(hs_0 v_k/m_k + u_k\omega_0 s_0) \\
&= \sum_{k=1}^{N} -\frac{hs_0}{m_k}(v_0 - v_k)^2 \leq 0.
\end{aligned} \tag{51}$$

Application of the invariance principle in Λ shows that solutions starting in Λ converge to the largest set $M \subset \Lambda$ in which $\dot{U}_{circ} = 0$. M contains the set of circular formations with $u_k \equiv u_0$ and $v_k \equiv v_0$. ∎

Theorem 2 is illustrated in Fig. 3.

V. CONCLUSION

In this paper we describe a planar rigid body model for motion coordination. In addition to forward dynamics in

the body-fixed frame, second-order transverse dynamics are included in the model. The condition that the vehicle move at constant forward speed is relaxed; controllers are designed to ensure that the forward speed of each vehicle decays to the desired speed u_0. Decentralized controllers are provided to stabilize parallel and circular formations and illustrated in simulation. In ongoing work, we are extending these results to incorporate a steady, uniform flowfield.

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