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Framework for Estimating System Reliability from Full System and Subsystem Tests with Dependence on Dynamic Inputs

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Abstract— **This paper develops a framework for estimating the reliability—with confidence regions—of a complex system based on a combination of full system and subsystem (and/or component or other) tests where some of the subsystems are dependent on dynamic inputs (independent predictor variables). It is assumed that the system is composed of multiple processes (e.g., the subsystems and/or components within subsystems), where the subsystems may be arranged in series, parallel (i.e., redundant), combination series/parallel, or other mode. The method of maximum likelihood estimation (MLE) is used to estimate subsystem and full system reliability. The MLE approach is well suited to providing asymptotic confidence bound through the Fisher information. As such, the Fisher information is derived for the general maximum likelihood estimator presented in the paper. A simple numerical study illustrates that the MLE recovers the reliability parameters of a system (plus some statistical uncertainty) when applied to a set of dynamic inputs and full system/subsystem output test data.**

I. Introduction

This paper considers the problem of estimating the reliability of a complex system based on a combination of information from tests on the subsystems, components, or other processes within the system, and, if available, tests on the full system. System, subsystem, component, interface, and other[1] tests are often carried out on complex systems to determine full system/subsystem reliability, evaluate aging or manufacturing process effects on reliability, and ensure that an operational performance requirements are satisfied. Fusing full system and subsystem test data to evaluate the reliability of a complex systems is desirable when full system testing is costly or dangerous or when it requires the destruction of the system itself. Additionally, it is desirable to include full system testing in an overall reliability assessment to help guard against possible mis-modeling of the relationship between the subsystems and full system in calculating overall system reliability [1]. One method of combining full system and subsystem reliability test data to form a full system estimate of reliability is the method of maximum likelihood [2]. This general maximum likelihood formulation for the combination of reliability test data applies across all system configurations (series, parallel, etc); only the optimization constraints change, leading to an appropriate maximum likelihood estimate (MLE). The method of maximum likelihood also provides a characterization of the estimation uncertainty—statistical uncertainty about the model parameters—through the Fisher Information on the parameters of the system reliability model.

The general maximum likelihood method of reliability estimation combines data from subsystem tests and full system tests via a model that reflects the constraints associated with the configuration of the full system. Test data for each subsystem and the full system are assumed to be independent and identically distributed (i.i.d.). However, reliability test data can be dependent on dynamic external performance predictors such as age, temperature, manufacturing lot, etc (for example see [3]). This leads to system and subsystem test data that are independent but not identically distributed. Herein, the general method of [2] and [4] is extended so that the model for the system also reflects dynamic subsystems reliabilities. The principles of maximum likelihood are applied to estimate overall system reliability from a combination of full system and subsystem tests that may be dependent on dynamic predictors. Such full system/subsystem reliability estimates (with quantified confidence) are valuable to decision makers for determining system operational limits and optimizing maintenance/upgrade schedules.

Certainly, other approaches exist for estimating system reliability when the subsystems are independent (see [2] for a review). However, these approaches do not allow for easy inclusion of dynamic external performance predictors. A Bayesian approach to including dynamic external performance predictors in full system/subsystem reliability estimates is developed in [5] and [6]. While prior information may be useful and appropriate in some situations, the MLE approach offers a prior free alternative to the Bayesian approach that is parsimonious in the model construction (no priors or hyperpriors) and in the subjective input (no prior parameters or hyperparameters). Also, the Bayesian estimation approach in [5] and [6] is ultimately one of numerical integration, where, maximum likelihood estimation is ultimately a problem of, the more simple, function maximization. Finally, the Bayesian estimator does not have the invariance property of maximum likelihood estimators, so, a Bayesian estimate of the system reliability model parameters does not necessarily give an optimal estimate of the system reliability.

Section 2 of this paper gives the formulation for the

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[1]To avoid the need to repeatedly refer to tests on subsystems, components, processes, and other aspects of the system as the key source of information other than full system tests, we will usually only refer to subsystem tests; subsystem tests in this context should be considered a proxy for all possible test information short of full system tests.

general MLE. Models for the full system and subsystems are specified, the input and output test data are described, and the likelihood function, score vector, and Fisher information are presented. Section 3 considers a specific case of the general MLE, where the dynamics of one subsystem is modeled with a logit function. A simulation study illustrates that the MLE recovers the fundamental reliability parameters of a system (plus some statistical uncertainty) when applied to simulated inputs and output test data. Section 4 offers some concluding remarks and a discussion of future work.

II. General Formulation

A. Modeling Approach

Consider a system composed of several subsystems that can be independently tested, and assume that the ultimate quantity of interest is the full system reliability. Suppose that reliability test data exists on both the full system and the subsystems. The subsystem reliabilities may be static or dynamic. The reliability of a static subsystem is assumed to be physically and stochastically independent of external performance factors when the subsystem is used as intended. In other words, the reliability of a static subsystem does not vary in its intended operating mode and environment. The reliability of a dynamic subsystem is assumed to be dependent on a set of external performance predictors. These inputs (or independent predictor variables) may be continuous or categorical. Typically, the inputs are a measure of the age of the subsystem (i.e. time since production, time since last service, or time in operational use), the condition of the subsystem, which may come from a condition-based maintenance sensor (i.e. amount of particulate in engine oil), or a discrete experience (i.e. manufacturing lot or mode of transport). Any system with a dynamic subsystem is also a dynamic system; the system reliability is stochastically dependent on all dynamic subsystem inputs.

Models for the full system and subsystems must be defined in order to estimate the reliability of a system. The models characterize the full system/subsystem dynamics and the system configuration. The models may take many different forms, subject to being able to write down a probabilistic characterization of the system that leads to a likelihood function. At the subsystem level, statistical or physical models must be developed that relate the dynamic inputs to the subsystem reliability. At the full system level, the physical configuration of the system (series, parallel, combination series/parallel) must be modeled. Further, since the system is dynamic, the full system model incorporates the subsystem models. It is the combination of subsystem and full system models (and any interaction between subsystem dynamics and system configuration) that enables the method of maximum likelihood to combine independent reliability test data observed at different input values.

Suppose a system is composed of q dynamic subsystems and r static subsystems. Let the dynamic subsystem response X_i be specified by a vector of unknown parameters $\boldsymbol{\beta}_i$, which are to be estimated from data for each dynamic subsystem, and a vector of inputs $\boldsymbol{\tau}_i$, for $i = 1,\ldots,q$. Then, the model for the i^{th} dynamic subsystem reliability (success probability) ρ_i is denoted

$$\rho_i = E(X_i) = \rho_i\left(\boldsymbol{\tau}_i, \boldsymbol{\beta}_i\right), \qquad i = 1,\ldots,q, \tag{1}$$

where $\rho_i(\cdot)$ is a differentiable function that links the parameters and input values to the expectation of X_i. Further, let the static subsystem response X_i be specified by ρ_i the static subsystem reliability, for $i = q+1,\ldots,q+r$. By definition, the static subsystem reliability is independent of any inputs, $\boldsymbol{\tau}_i$. The model for the i^{th} static subsystem reliability ρ_i is

$$\rho_i = E(X_i), \qquad i = q+1,\ldots,q+r. \tag{2}$$

The full system response Y is completely characterized by the models for the subsystems, the inputs to the dynamic subsystems, and an overall "system" model that specifies the system configuration. The overall "system" model maps the subsystem reliabilities into the system reliability according to

$$\rho = E(Y) = \rho\left(\boldsymbol{\tau}; \boldsymbol{\beta}_1,\ldots,\boldsymbol{\beta}_q; \rho_{q+1},\ldots,\rho_{q+r}\right), \tag{3}$$

$$= \rho\left(\rho_1\left(\boldsymbol{\tau}_1,\boldsymbol{\beta}_1\right),\ldots,\rho_q\left(\boldsymbol{\tau}_q,\boldsymbol{\beta}_q\right); \rho_{q+1},\ldots,\rho_{q+r}\right), \tag{4}$$

where $\boldsymbol{\tau} = \{\boldsymbol{\tau}_1^T,\ldots,\boldsymbol{\tau}_q^T\}^T$ is the stacked vector of inputs associated with each dynamic subsystem and $\rho(\cdot)$ is a differentiable function linking the parameters and the input values to the expectation of Y.

B. Inputs and Output Test Data

It is assumed that the full system and subsystem responses (output test data) used for estimating reliability are statistically independent. Subsystems may be tested by varying the inputs with each test or subsystems may be tested many times at relatively few values of the input. Full system tests (typically fewer in number than subsystem tests) may be conducted at relatively few input values, which need not be the same as the input values observed/used in the subsystem testing. Optimal experimental design for fuzing subsystem and full system output test data is outside the scope of this paper. However, a complete description of the subsystem and full system inputs and output test data is given.

For each dynamic subsystem, independent tests are performed at different values of the inputs. The i^{th} dynamic subsystem is tested at m_i values of the inputs. The input values are denoted $\boldsymbol{\tau}_{i1},\ldots,\boldsymbol{\tau}_{im_i}$, where $\boldsymbol{\tau}_{ij}$ is the j^{th} vector of input values at which the the i^{th} subsystem is tested. (So, $\boldsymbol{\tau}_{ij}$ is the vector of independent predictor variables associated with the X_{ij} response.) The m_i input values may include repeated (duplicate) quantities (so $\boldsymbol{\tau}_{ij} = \boldsymbol{\tau}_{ij'}$ for $j \neq j'$). For $j = 1,\ldots,m_i$, let X_{ij} represent the number of successes in n_{ij} i.i.d. tests of the i^{th} dynamic subsystem at the input value $\boldsymbol{\tau}_{ij}$. Thus, the total set of output test data on the i^{th} dynamic subsystem is $\boldsymbol{X}_i = \{X_{i1},\ldots,X_{im_i}\}$.

Static subsystems are assumed to be independent of external performance factors, so each test of a static subsystem is i.i.d. Let X_i represent the number of successes in n_i i.i.d. tests of the i^{th} static subsystem for $i = q+1,\ldots,q+r$.

The system response is dependent on the inputs of all q dynamic subsystems, and so, the input for a full system test is a set of vectors of all the subsystem inputs. For the full system, assume s independent tests are performed at different values of the subsystem inputs. Thus, the full system is tested at s levels of the subsystems' input; the levels are not necessarily distinct. The input values for the system tests are denoted $\boldsymbol{\tau}'_1,\ldots,\boldsymbol{\tau}'_s$, where $\boldsymbol{\tau}'_j = \left\{{\boldsymbol{\tau}'_{1j}}^T,\ldots,{\boldsymbol{\tau}'_{qj}}^T\right\}^T$, for $j = 1,\ldots,s$, is a stacked vector of the dynamic subsystem input levels at which the system tests are conducted. The prime is added to distinguish these inputs from the subsystem inputs. For $j = 1,\ldots,m_i$, let Y_j represent the number of successes in n'_j i.i.d. tests of the system at the input value $\boldsymbol{\tau}'_j$ (again, the prime is added to distinguish the sample size from the subsystem sample sizes).

For notational convenience, let $\boldsymbol{T} = \{\boldsymbol{\tau}_{i1},\ldots,\boldsymbol{\tau}_{im_i};\ldots;\boldsymbol{\tau}_{q1},\ldots,\boldsymbol{\tau}_{qm_q};\boldsymbol{\tau}'_1,\ldots,\boldsymbol{\tau}'_s\}$ represent the complete set of subsystem and full system input data, and let $\boldsymbol{Z} = \{\boldsymbol{X}_1,\ldots,\boldsymbol{X}_q,X_{q+1},\ldots,X_{q+r},Y_1,\ldots,Y_s\}$ represent the full set of subsystem and system output test data. The input and output test data notation is summarized in Table I.

C. MLE Formulation

The general MLE formulation involves a parameter vector $\boldsymbol{\theta}$ representing the parameters to be estimated, together with an associated log-likelihood criterion $\mathfrak{L}(\boldsymbol{\theta})$. The parameters that completely specify the system reliability include the dynamic subsystem parameters, $\boldsymbol{\beta}_1,\ldots,\boldsymbol{\beta}_q$, and the static subsystem reliabilities, $\rho_{q+1},\ldots,\rho_{q+r}$. The vector,

$$\boldsymbol{\theta} = \{\boldsymbol{\beta}_1^T,\ldots,\boldsymbol{\beta}_q^T,\rho_{q+1},\ldots,\rho_{q+r}\}^T,$$

is the vector of parameters to be estimated from data. Note, the full system reliability ρ is not included in $\boldsymbol{\theta}$ because it is uniquely determined by the models for the subsystems and the system configuration through the model (3). (For convenience throughout the paper, the parameter vector $\boldsymbol{\theta}$ is substituted for the complete set of parameters $\boldsymbol{\beta}_1,\ldots,\boldsymbol{\beta}_q;\rho_{q+1},\ldots,\rho_{q+r}$ in the argument of the full system reliability model (3).) Then, the estimate $\hat{\rho}$, as determined from applying (3) to the MLE of $\boldsymbol{\theta}$ (say $\hat{\boldsymbol{\theta}}$), is the MLE of ρ. The invariance property of the MLE applies even though the mapping from $\boldsymbol{\theta}$ to ρ is not generally one-to-one and may not be continuous (see, e.g., [7]).

Let $\boldsymbol{\Theta}$ denoted the feasible region for the elements of $\boldsymbol{\theta}$. To ensure that the relevent logarithms are defined and that the appropriate derivatives exist, it is assumed, at a minimum, that the feasible region $\boldsymbol{\Theta}$ is restricted such that $\rho_i \in (0,1)$, for $i = q+1,\ldots,q+r$ (the static reliabilities). The general MLE formulation for the parameter vector $\boldsymbol{\theta}$ is,

$$\hat{\boldsymbol{\theta}} = \hat{\boldsymbol{\theta}}(\boldsymbol{T},\boldsymbol{Z}) \equiv \arg\max_{\boldsymbol{\theta}\in\boldsymbol{\Theta}} \mathfrak{L}(\boldsymbol{\theta}) \tag{5}$$

subject to

$$\rho_i = \rho_i\left(\boldsymbol{\tau}_i,\boldsymbol{\beta}_i\right) \text{ for } i = 1,\ldots,q, \text{ and } \rho = \rho\left(\boldsymbol{\tau};\boldsymbol{\theta}\right).$$

The estimate of system reliability is derived from the MLE for $\boldsymbol{\theta}$ through the model for the system ρ, given the value of the inputs $\boldsymbol{T}$, and the responses, $\boldsymbol{Z}$, (system and subsystem output test data).

The MLE formulation in (5) is general, but not very practical for finding the MLE, which is typically found by finding the root of the score equation, $\partial\mathfrak{L}(\boldsymbol{\theta})/\partial\boldsymbol{\theta} = \boldsymbol{0}$. From the assumption of independence of all the test data, the probability mass function, say $p(\boldsymbol{Z}|\boldsymbol{\theta})$, is the product of binomial mass functions. Substituting the constraints from (5) into the probability mass function and taking the logarithm leads to the log-likelihood function $\mathfrak{L}(\boldsymbol{\theta}) \equiv \log p(\boldsymbol{Z}|\boldsymbol{\theta})$:

$$\begin{aligned}\mathfrak{L}(\boldsymbol{\theta}) = &\sum_{i=1}^{q}\sum_{j=1}^{m_i}\left[X_{ij}\log\left(\rho_i\left(\boldsymbol{\tau}_{ij},\boldsymbol{\beta}_i\right)\right) + \right.\\ &\left.\left(n_{ij}-X_{ij}\right)\log\left(1-\rho_i\left(\boldsymbol{\tau}_{ij},\boldsymbol{\beta}_i\right)\right)\right] + \\ &\sum_{i=q+1}^{r}\left[X_i\log\rho_i + (n_i - X_i)\log(1-\rho_i)\right] + \\ &\sum_{i=1}^{s}\left[Y_i\log\left(\rho\left(\boldsymbol{\tau}'_i,\boldsymbol{\theta}\right)\right) + \left(n'_i - Y_i\right)\log\left(1-\rho\left(\boldsymbol{\tau}'_i,\boldsymbol{\theta}\right)\right)\right]\\ &+ \text{ constant,}\end{aligned} \tag{6}$$

where the constant is not dependent on $\boldsymbol{\theta}$. Note that the expression in (6) for the log-likelihood applies regardless of the models for the dynamic subsystems and the model for the full system (the configuration of the system). Expression (6) is used below for derivation of the score function and Fisher information.

The elements of the score vector are a mixture of scalars and vectors,

$$\partial\mathfrak{L}/\partial\boldsymbol{\theta} = \begin{bmatrix}\partial\mathfrak{L}/\partial\boldsymbol{\beta}_1\\ \vdots \\ \partial\mathfrak{L}/\partial\boldsymbol{\beta}_q\\ \partial\mathfrak{L}/\partial\rho_{q+1}\\ \vdots\\ \partial\mathfrak{L}/\partial\rho_r\end{bmatrix}, \tag{7}$$

depending on the models for the dynamic subsystems. When ρ and ρ_i for $i = 1,\ldots,q$ are differentiable functions, (6) leads to the following elements of the score vector:

$$\begin{aligned}\frac{\partial\mathfrak{L}}{\partial\boldsymbol{\beta}_k} = &\sum_{i=1}^{s}\left[\frac{Y_i - n'_i\rho\left(\boldsymbol{\tau}'_i,\boldsymbol{\theta}\right)}{\rho\left(\boldsymbol{\tau}'_i,\boldsymbol{\theta}\right)\left(1-\rho\left(\boldsymbol{\tau}'_i,\boldsymbol{\theta}\right)\right)}\frac{\partial\rho\left(\boldsymbol{\tau}'_i,\boldsymbol{\theta}\right)}{\partial\rho_k\left(\boldsymbol{\tau}'_{ki},\boldsymbol{\beta}_k\right)}\frac{\partial\rho_k\left(\boldsymbol{\tau}'_{ki},\boldsymbol{\beta}_k\right)}{\partial\boldsymbol{\beta}_k}\right]\\ &+\sum_{i=1}^{m_k}\left[\frac{X_{ki}-n_{ki}\rho_k\left(\boldsymbol{\tau}_{ki},\boldsymbol{\beta}_k\right)}{\rho_k\left(\boldsymbol{\tau}_{ki},\boldsymbol{\beta}_k\right)\left(1-\rho_k\left(\boldsymbol{\tau}_{ki},\boldsymbol{\beta}_k\right)\right)}\frac{\partial\rho_k\left(\boldsymbol{\tau}_{ki},\boldsymbol{\beta}_k\right)}{\partial\boldsymbol{\beta}_k}\right],\end{aligned} \tag{8}$$

for $k = 1,\ldots,q$, and

$$\frac{\partial\mathfrak{L}}{\partial\rho_k} = \sum_{i=1}^{s}\left[\frac{Y_i - n'_i\rho\left(\boldsymbol{\tau}'_i,\boldsymbol{\theta}\right)}{\rho\left(\boldsymbol{\tau}'_i,\boldsymbol{\theta}\right)\left(1-\rho\left(\boldsymbol{\tau}'_i,\boldsymbol{\theta}\right)\right)}\frac{\partial\rho\left(\boldsymbol{\tau}'_i,\boldsymbol{\theta}\right)}{\partial\rho_k}\right] + \frac{X_k - n_k\rho_k}{\rho_k(1-\rho_k)}, \tag{9}$$

for $k = q+1,\ldots,r$.

Except in trivial cases, the analytical expression for the variance of the general MLE for system reliability is not easily found. However, the Fisher Information, given by

$$\boldsymbol{F}(\boldsymbol{\theta}) \equiv E\left[\frac{\partial\mathfrak{L}}{\partial\boldsymbol{\theta}}\left(\frac{\partial\mathfrak{L}}{\partial\boldsymbol{\theta}^T}\right)^T\right] = -E\left[\frac{\partial^2\mathfrak{L}}{\partial\boldsymbol{\theta}\partial\boldsymbol{\theta}^T}\right] \tag{10}$$

TABLE I

NOTATION FOR THE INPUTS, OUTPUT TEST DATA, AND MODEL PARAMETERS

	Inputs	Outputs	Sample Size	Parameters
i^{th} Dynamic Subsystem	$\boldsymbol{\tau}_{i1},\ldots,\boldsymbol{\tau}_{im_i}$	$X_{i1},\ldots,X_{im_i}$	$n_{i1},\ldots,n_{im_i}$	$\boldsymbol{\beta}_i$
i^{th} Static Subsystem	—	X_i	n_i	ρ_i
Full System	$\boldsymbol{\tau}'_1,\ldots,\boldsymbol{\tau}'_s$	$Y_1,\ldots,Y_s$	$n'_1,\ldots,n'_s$	—
Total Set $(q+r+1)$	$\boldsymbol{T}$	$\boldsymbol{Z}$	—	$\boldsymbol{\theta}$

is available for the general maximum likelihood estimator of the parameter vector $\boldsymbol{\theta}$. Invoking the Cramer-Rao inequality, the inverse of the Fisher information is a lower bound on the variance the MLE (for an unbiased estimator). The Fisher information matrix for $\boldsymbol{\theta}$, given the deterministic inputs $\boldsymbol{T}$, is specified in three parts. The Fisher information matrix for the dynamic subsystem parameters is specified by the following submatrix,

$$-E\left(\frac{\partial^2\mathfrak{L}}{\partial\boldsymbol{\beta}_k\partial\boldsymbol{\beta}_l}\right)=\sum_{i=1}^{s}\left[\frac{n'_i}{\rho\left(\boldsymbol{\tau}'_i,\boldsymbol{\theta}\right)\left(1-\rho\left(\boldsymbol{\tau}'_i,\boldsymbol{\theta}\right)\right)}\frac{\partial\rho\left(\boldsymbol{\tau}'_i,\boldsymbol{\theta}\right)}{\partial\rho_l}\cdot\frac{\partial\rho_l\left(\boldsymbol{\tau}'_{li},\boldsymbol{\beta}_l\right)}{\partial\boldsymbol{\beta}_l}\left(\frac{\partial\rho\left(\boldsymbol{\tau}'_i,\boldsymbol{\theta}\right)}{\partial\rho_k}\frac{\partial\rho_k\left(\boldsymbol{\tau}'_{ki},\boldsymbol{\beta}_k\right)}{\partial\boldsymbol{\beta}_k}\right)^T\right]+\sum_{i=1}^{m_k}\left[\frac{n_{ki}}{\rho_k\left(\boldsymbol{\tau}_{ki},\boldsymbol{\beta}_k\right)\left(1-\rho_k\left(\boldsymbol{\tau}_{ki},\boldsymbol{\beta}_k\right)\right)}\cdot\frac{\partial\rho_k\left(\boldsymbol{\tau}_{ki},\boldsymbol{\beta}_k\right)}{\partial\boldsymbol{\beta}_l}\left(\frac{\partial\rho_k\left(\boldsymbol{\tau}_{ki},\boldsymbol{\beta}_k\right)}{\partial\boldsymbol{\beta}_k}\right)^T\right],\tag{11}$$

for $k,l=1,\ldots,q$. Note, the function ρ is differentiated with respect to the scalar subsystem reliability ρ_l and ρ_k per application of the chain rule. The function ρ is explicitly dependent on the subsystem reliability through the model for the system (3). The elements of the Fisher information matrix for the static subsystem reliabilities are specified by the following scalar,

$$-E\left(\frac{\partial^2\mathfrak{L}}{\partial\rho_k\partial\rho_l}\right)=\sum_{i=1}^{s}\left[\frac{n'_i}{\rho\left(\boldsymbol{\tau}'_i,\boldsymbol{\theta}\right)\left(1-\rho\left(\boldsymbol{\tau}'_i,\boldsymbol{\theta}\right)\right)}\frac{\partial\rho\left(\boldsymbol{\tau}'_i,\boldsymbol{\theta}\right)}{\partial\rho_l}\cdot\frac{\partial\rho\left(\boldsymbol{\tau}'_i,\boldsymbol{\theta}\right)}{\partial\rho_k}\right]+\frac{n_k}{\rho_k(1-\rho_k)}\frac{\partial\rho_k}{\partial\rho_l},\tag{12}$$

for $k,l=q+1,\ldots,r$. The cross terms (between dynamic and static subsystem parameters) of the Fisher information matrix are specified by the following vector,

$$-E\left(\frac{\partial^2\mathfrak{L}}{\partial\boldsymbol{\beta}_k\partial\rho_l}\right)=\sum_{i=1}^{s}\left[\frac{n'_i}{\rho\left(\boldsymbol{\tau}'_i,\boldsymbol{\theta}\right)\left(1-\rho\left(\boldsymbol{\tau}'_i,\boldsymbol{\theta}\right)\right)}\frac{\partial\rho\left(\boldsymbol{\tau}'_i,\boldsymbol{\theta}\right)}{\partial\rho_l}\cdot\frac{\partial\rho\left(\boldsymbol{\tau}'_i,\boldsymbol{\theta}\right)}{\partial\rho_k}\frac{\partial\rho_k\left(\boldsymbol{\tau}'_{ki},\boldsymbol{\beta}_k\right)}{\partial\boldsymbol{\beta}_k}\right],\tag{13}$$

for $k=1,\ldots,q$ and $l=q+1,\ldots,r$ (again, the function ρ is differentiated with respect to the scalar subsystem reliability ρ_k per application of the chain rule).

III. ILLUSTRATION OF THE MLE FORMULATION

A. Series System with One Dynamic Subsystem

To illustrate the MLE formulation, a fully series system with one dynamic subsystem and an arbitrary number of static subsystems is modeled. (In this case, $q=1$, and so the subscript will be dropped from the vector $\boldsymbol{\beta}_1$.) The generalized linear model for a binary response is used to model the dynamic subsystem reliability. A generalized linear model relates the linear combination of parameters and inputs to the expected value of a response variable via a link function. Herein, the inverse logit function, $g(\eta)\equiv(e^{\eta}/(1+e^{\eta}))$, is selected to be the link function. The inverse logit function is chosen because i) it is the canonical link function for the generalized linear model for a binomial response (under certain conditions there exists a sufficient statistic equal in dimension to $\boldsymbol{\beta}$ for the linear predictor), ii) the linear combination $\boldsymbol{\tau}_{1,j}^T\boldsymbol{\beta}$ for any j can be interpreted as the log odds on the event of subsystem success, and iii) differences on the logistic scale can be estimated regardless whether the data are sampled prospectively or retrospectively [8, Chapter 4]. Further, the model is easily extended to capture more complex functional relation between the inputs and the response (the generalized linear model for a binary response using the logistic link function is a special case of a one node feed-forward neural network [9, Section 5.2]). Also, the logit function is chosen because it is commonly used in regression analysis for modeling a binary response, and so, statistical tools for assessing the appropriateness of the model (independence, additivity, and linearity of the inputs), the quality of the model fit, and tools for model identification are readily available. Nonetheless, care should be taken when using a linear logistic model. A modeler/statistician should always check to determine if its fixed functional form represents the physical or mathematical dependence structure present among the inputs [3].

The functions that comprise the model for the dynamic subsystem and the full system are as follows. The model for the dynamic subsystem reliabilty ρ_1 is

$$\rho_1\left(\boldsymbol{\tau}_1,\boldsymbol{\beta}\right)=\frac{e^{\boldsymbol{\tau}_1^T\boldsymbol{\beta}}}{\left(1+e^{\boldsymbol{\tau}_1^T\boldsymbol{\beta}}\right)},\tag{14}$$

and, based on the series system configuration, the functional form for the full system reliability ρ, is

$$\rho\left(\boldsymbol{\tau},\boldsymbol{\theta}\right)=\rho_1\left(\boldsymbol{\tau}_1,\boldsymbol{\beta}\right)\prod_{i=2}^{r}\rho_i.\tag{15}$$

The MLE formulation in (5) is now applied by inserting (14) and (15) into the log-likelihood function (6). The log-likelihood function for the new estimation problem is

$$\begin{aligned}\mathfrak{L}(\boldsymbol{\theta}) = &\sum_{i=1}^{m_1}\left[X_i\left(\boldsymbol{\tau}_{1i}^T\boldsymbol{\beta}\right) - n_i\log\left(1+\exp\left(\boldsymbol{\tau}_{1i}^T\boldsymbol{\beta}\right)\right)\right] + \\ &\sum_{i=2}^{r}\left[X_i\log\rho_i + (n_i - X_i)\log(1-\rho_i)\right] + \\ &\sum_{i=1}^{s}\left[Y_i\log\left(\rho\left(\boldsymbol{\tau}'_i,\boldsymbol{\theta}\right)\right) + (n'_i - Y_i)\log\left(1-\rho\left(\boldsymbol{\tau}'_i,\boldsymbol{\theta}\right)\right)\right] \\ &+ \text{ constant,}\end{aligned} \quad (16)$$

where $\boldsymbol{\theta} = \{\boldsymbol{\beta}^T, \rho_2, \ldots, \rho_r\}$. Note that the first line in (16) is the log-likelihood for the standard linear-logistic model, [8, Chapter 4]. The score vector can be written,

$$\begin{aligned}\frac{\partial\mathfrak{L}}{\partial\boldsymbol{\beta}} = &\sum_{i=1}^{m_1}\left[\left(X_{1i} - n_{1i}\rho_1\left(\boldsymbol{\tau}_{1i},\boldsymbol{\beta}\right)\right)\boldsymbol{\tau}_{1i}\right] + \\ &\sum_{i=1}^{s}\left[\frac{\left(Y_i - n'_i\rho\left(\boldsymbol{\tau}'_i,\boldsymbol{\theta}\right)\right)\left(1-\rho_1\left(\boldsymbol{\tau}'_{1i},\boldsymbol{\beta}\right)\right)}{1-\rho\left(\boldsymbol{\tau}'_i,\boldsymbol{\theta}\right)}\boldsymbol{\tau}'_{1i}\right], \end{aligned} \quad (17)$$

$$\frac{\partial\mathfrak{L}}{\partial\rho_k} = \sum_{i=1}^{s}\left[\frac{Y_i - n_{Y_i}\rho\left(\boldsymbol{\tau}'_i,\boldsymbol{\theta}\right)}{\rho_k\left(1-\rho\left(\boldsymbol{\tau}'_i,\boldsymbol{\theta}\right)\right)}\right] + \frac{X_k - n_k\rho_k}{\rho_k(1-\rho_k)}. \quad (18)$$

Assuming deterministic inputs, the Fisher Information matrix on $\boldsymbol{\beta}$ is

$$\begin{aligned}-E\left(\frac{\partial^2\mathfrak{L}}{\partial^2\boldsymbol{\beta}}\right) = &\sum_{i=1}^{s}\left[\frac{n'_i\rho\left(\boldsymbol{\tau}'_i,\boldsymbol{\theta}\right)\left(1-\rho_1\left(\boldsymbol{\tau}'_{1i},\boldsymbol{\beta}\right)\right)^2}{\left(1-\rho\left(\boldsymbol{\tau}'_i,\boldsymbol{\theta}\right)\right)}\boldsymbol{\tau}'_i\boldsymbol{\tau}'^{\,T}_i\right] + \\ &\sum_{i=1}^{m_1}\left[n_{1i}\rho_1\left(\boldsymbol{\tau}_{1i},\boldsymbol{\beta}\right)\left(1-\rho_1\left(\boldsymbol{\tau}_{1i},\boldsymbol{\beta}\right)\right)\boldsymbol{\tau}_{1i}\boldsymbol{\tau}_{1i}^T\right],\end{aligned} \quad (19)$$

and the elements of the Fisher Information matrix on $\rho_2, \ldots, \rho_r$ are

$$\begin{aligned}-E\left(\frac{\partial^2\mathfrak{L}}{\partial\rho_k\partial\rho_l}\right) = &\sum_{i=1}^{r}\left[\frac{n_i\rho\left(\boldsymbol{\tau}'_i,\boldsymbol{\theta}\right)}{\rho_k\rho_l\left(1-\rho\left(\boldsymbol{\tau}'_i,\boldsymbol{\theta}\right)\right)}\right] \\ &+\begin{cases}\frac{n_k}{\rho_k(1-\rho_k)} & \text{if } k=l \\ 0 & \text{if } k\neq l\end{cases},\end{aligned} \quad (20)$$

for $k, l = 2, \ldots, r$. The cross terms of the Fisher information matrix are specified by the following vector,

$$-E\left(\frac{\partial^2\mathfrak{L}}{\partial\boldsymbol{\beta}\partial\rho_l}\right) = \sum_{i=1}^{s}\left[\frac{n'_i\rho\left(\boldsymbol{\tau}'_i,\boldsymbol{\theta}\right)\left(1-\rho_1\left(\boldsymbol{\tau}'_{1i},\boldsymbol{\beta}\right)\right)}{\left(1-\rho\left(\boldsymbol{\tau}'_i,\boldsymbol{\theta}\right)\right)}\boldsymbol{\tau}'_i\right], \quad (21)$$

for $l = 2, \ldots, r$.

B. Numerical Example

This section describes a simple numerical study on the reliability estimation of a series system with four independent subsystems. The simulation study shows that the general MLE described in Section II recovers the fundamental reliability parameters of the system (plus some statistical uncertainty) when applied to inputs and output test data. The system is comprised of one dynamic subsystem and three static subsystems. The dynamic subsystem is assumed to be dependent on a constant and a single input. The example is analogous to assessing the reliability of a system that is degrading with the age of a particular subsystem (as such, parameter values for the dynamic subsystem are selected so that the dynamic subsystem reliability is monotonically decreasing as the input value increases). The model for the system reliability and dynamic subsystem reliability are given in (14) and (15), respectively, where $\boldsymbol{\beta} = \{\beta_0, \beta_1\}^T$ and $\boldsymbol{\tau} = \{\tau_0 = 1, \tau_1\}^T$.

The inputs producing the output test data for the system are generated as follows. First, a response from the dynamic subsystem is generated by sampling from a uniform distribution with lower and upper bound parameters 0 and 20, respectively, to obtain a value for the input τ_1. Then, the sampled value of τ_1 and the vector $\boldsymbol{\beta}$, specified in Table II, are used to compute the dynamic subsystem reliability, $\rho_1 = \rho_1(\{1,\tau_1\}^T, \{\beta_0,\beta_1\}^T)$. Finally, response is generated by randomly sampling from a Bernoulli distribution with mean specified by ρ_1. Inputs and responses are generated to create a dynamic subsystem sample of $m_1 = 200$. The static subsystem output test data are generated by randomly sampling from a binomial distribution with a means specified by the parameters in Table II and sample sizes, $n_2 = n_3 = n_4 = 200$. An input and response from the full system is generated in the same manner as the dynamic subsystem input/response, except before sampling from a Bernoulli distribution, the dynamic subsystem reliability is multiplied by the three static subsystem reliabilities listed in Table II. Again, inputs and responses are generated to create a full system sample of $s = 200$. The MLE, $\hat{\boldsymbol{\theta}} = \{\hat{\beta}_0, \hat{\beta}_1, \hat{\rho}_2, \hat{\rho}_3, \hat{\rho}_4\}^T$, is found by maximizing (16) at the simulated inputs and output test data.

Output test data from one replicate of the system/subsytem simulation described above are plotted in Figure 1. The full system and dynamic subsystem responses are generated from a Bernoulli distribution, and so, the responses are plotted as points with value one for success or zero for failure at the observed input value, τ_1 (recall that $\tau_0 = 1$). More full system failures occur at smaller values of the input than dynamic subsystem failures because the dynamic subsystem is in a series configuration with three other static subsystems. The MLE of the model parameters are $\hat{\beta}_0 = 6.314, \hat{\beta}_1 = -0.626, \hat{\rho}_2 = 0.968, \hat{\rho}_3 = 0.950, \hat{\rho}_4 = 0.936$. Through the invariance property of the MLE, these parameter estimates lead to the estimate of full system reliability plotted as a function of the input, τ_1, in Figure 1. The MLE of the full system reliability reflects *all* the test data (full system, static subsystem, and dynamic subsystem) given the models for the full system and dynamic subsystem. The statistical uncertainty in the estimate is also represented by plotting the 90% asymptotic confidence bounds, conditional on $\boldsymbol{\tau} = \{1, \tau_1\}^T$ (where τ_1 is plotted on the x-axis), computed from the Fisher information in (19)–(21).

To test that the general MLE described in Section II recovers the reliability parameters listed in Table II plus some statistical uncertainty, the simulation of inputs and output test data followed by maximum likelihood estimation are replicated 100 times. To account for the statistical uncertainty, the statistical z-test is used to produce probability

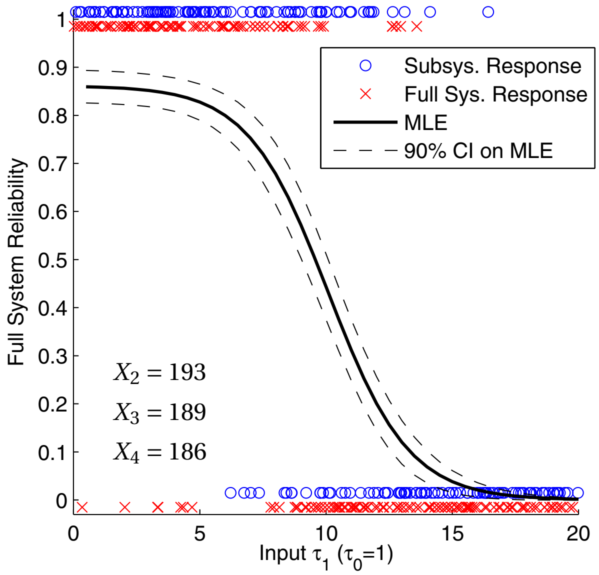

Fig. 1. Full system and subsystem test data from one replicate of the system/subsystem simulation and the corresponding MLE estimate of system reliability with its 90% asymptotic confidence interval (full system and dynamic subsystem responses are zero or one, they are offset from these values slightly to avoid clutter).

TABLE II

SUBSYSTEM PARAMETERS AND SIMULATION SUMMARY STATISTICS

Parameter	Value	Mean of Estimates	P-value
β_0	6.00	6.0968	0.2614
β_1	−0.60	−0.6100	0.2268
ρ_2	0.97	0.9699	0.9325
ρ_3	0.95	0.9515	0.2920
ρ_4	0.93	0.9249	0.7607

values (P-values) on the null hypothesis that the mean of the parameter MLEs are equal to the model parameter values listed in Table II. (The means of the MLEs and P-values are listed in Table II.) The large P-values for each parameter are consistent with the means of the parameter estimates being equal to the true model parameter values listed in Table II.

IV. CONCLUDING REMARKS

We have described above an MLE-based framework for estimating the reliability of a complex system by combining data from full system reliability tests and subsystem (or other) tests where some test data is dependent on dynamic external performance predictors (dynamic inputs) such as age, temperature, manufacturing lot, etc. The idea is a natural—but non-trivial—extension of [2] and [4], which did not consider dynamic inputs. By appropriately formulating constraints in an optimization problem, the approach accommodates general relationships between subsystem reliability and dynamic performance predictors and between the subsystems and full system. The method applies in general systems, where the subsystems may be arbitrarily arranged (in series, parallel, combination series/parallel, or other mode) and where any number of subsystem reliabilities may be dependent on external performance predictors. Interestingly, the MLE objective function (i.e., the likelihood or log-likelihood) has the same general form across all settings; only the constraints in the optimization problem change.

Significant work remains to move the framework described above to a robust methodology. The theoretical conditions on the subsystem and full system models under which the MLE will converge to the true system reliability and the asymptotic normality of the MLE need to be established. The results of such theoretical work will guide the process of establishing a class of models to be used for system/subsystem dynamics and possible extensions (such as use in a reliability-based controller for a linear stochastic system [10]). Future work also includes developing bootstrap methods to determine non-asymptotic confidence intervals on the vector $\hat{\boldsymbol{\theta}}$ and system reliability $\hat{\rho}$.

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