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Optimal Navigation in a Planar Time-Varying Point-Symmetric Flow-Field

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***Abstract*— This paper is concerned with time-optimal navigation for flight vehicles in a planar, time-varying point-symmetric flow-field — such as inside vortices or regions of eddy-driven upwelling/downwelling — where the objective is to find the fastest trajectory between initial and final points. The primary contribution of the paper is the observation that for time-optimality the rate of the steering angle has to be equal to one-half of the instantaneous vertical vorticity. Consequently, if the vorticity is zero, then the steering angle has to be constant. The result can be applied to find the time-optimal trajectories in practical control problems, by reducing the infinite-dimensional continuous problem to numerical optimization involving at most two unknown scalar parameters.**

I. Introduction

This paper is concerned with optimal navigation for flight vehicles in a planar flow-field. In the present context flight vehicles should be understood as general self-propelled particles that are subject to advection with the ambient exogenous flow, and not necessarily winged flying machines. The planar particle model may be applied to marine vessels or submersibles subject to ocean currents, airplanes or airships subject to wind, or gliders exploring the atmosphere of distant celestial bodies [9]. Although most of these flight vehicles have the potential to move in the general three-dimensional space, studying the projection of their motion onto the horizontal plane is well justified given that their typical operation is planar, and because the ambient flow structures show relatively little variation in the vertical direction. The flow-relative speed of the flight vehicles will be assumed constant. This restriction simplifies the analysis, and also makes the results applicable to vehicles, whose speed cannot be changed due to the principle of operation or other mission objectives. Airplanes typically fly at the constant minimum-drag speed to maximize fuel economy [10]. Autonomous Underwater Vehicles (AUVs) travel at the optimum speed determined by the trade-off between the power required to overcome drag, and the power consumed by onboard computers and sensors (hotel load) [1]. Under this constraint, the objective is then to find the optimal steering angle of the constant-speed flight vehicle, whose inertial (ground-relative) speed is influenced by the ambient fluid motion. For high-powered flight vehicles, such as commercial airliners flying at transonic speed, the variation between the optimal trajectory and neighboring sub-optimal solutions may not significantly alter the course of motion. In the case of slowly moving vehicles, such as airships or underwater vehicles, the picture is significantly different. When the vehicle speed is comparable to the speed of the ambient fluid, then the flow can severely alter the vehicle's domain of maneuverability [14], and hence the optimal trajectory between the initial and final points may have a very different shape than just a straight line.

The optimal control problem discussed in this paper is known as *Zermelo's problem* after Ernst Zermelo, who first studied optimal navigation of airships in wind-fields using variational principles [15]. Zermelo's optimal navigation formula is given in the form of a differential equation that depends on the partial derivatives of the ambient flow with respect to position [2, Section 2.7]

$$\dot{\psi}(t) = (u_x - v_y)\cos\psi\sin\psi + u_y\cos^2\psi + v_x\sin^2\psi.$$

In this equation $\psi(t)$ is the heading angle (the control signal), u and v denote the North and East components of the ambient flow, respectively, and the subscripts x and y denote partial derivatives with respect the spatial North and East coordinates. The equation provides a necessary condition for the optimal steering angle. Although in practice the initial condition to this differential equation is also needed to find the optimal steering history, the equation gives important clues about the character of extremals in some special cases. (Extremals are feasible paths that satisfy the necessary conditions of optimality; hence they are candidate optimal paths.) An example is the case where the flow is uniform, but possibly temporally changing. In that case, the rate of the steering angle is zero, hence the steering angle is constant. Due to its intuitive interpretation and elegant closed-form solution, Zermelo's problem is a common example found in nearly every textbook on introductory optimal control (see e.g. [2], [7], [8]), however most of these discussions stop short at the uniform and time-invariant ambient flow model, for which a simple geometric solution exists. A more general case is explored in [2], by requiring the flow to be time-invariant, and also invariant in one inertial direction.

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In that special case, one of the adjoint states becomes an integral of motion [2, Section 2.7], and a closed-form solution is presented. In most other situations, the problem is transformed to a two-point boundary-value problem (TPBVP), and the solution is obtained using shooting methods (also known as variation of extremals [7]). Alternative approaches, based on direct trajectory optimization, have been explored recently in [4], [5].

In this paper we revisit Zermelo's problem employing the achievements of optimal control theory and modern systems theory [12], [13]. The flow model presented in Section II is able to represent a rich family of flow phenomena. Examples include vortices or eddy-driven upwelling/downwelling at the ocean surface. A similar model has been employed in [11] to identify the location and strength of flow singularities with the aim to improve the navigation precision of AUVs operating under the influence of such flow structures. The methods presented in [11] could provide the underlying flow model that can be utilized by the optimal control algorithms presented in this paper. The primary contribution of the paper, presented in Section III, is the observation that — near a point-symmetric flow structure — the rate of the steering angle has to be equal to one-half of the instantaneous vertical vorticity. In Section IV we present an example that demonstrates the usefulness of this result for finding the time-optimal trajectory in a vortical, time-varying flow-field.

II. Flow-field Model

Similarly as in [11], we focus our attention on a planar flow model that includes a flow singularity at some fixed location, $\boldsymbol{r}_s = [x_s, y_s]^T \in \mathbb{R}^2$, in the horizontal plane. The components of the fluid velocity, $\boldsymbol{V}_w = [u, v]^T \in \mathbb{R}^2$, in Cartesian coordinates are given by

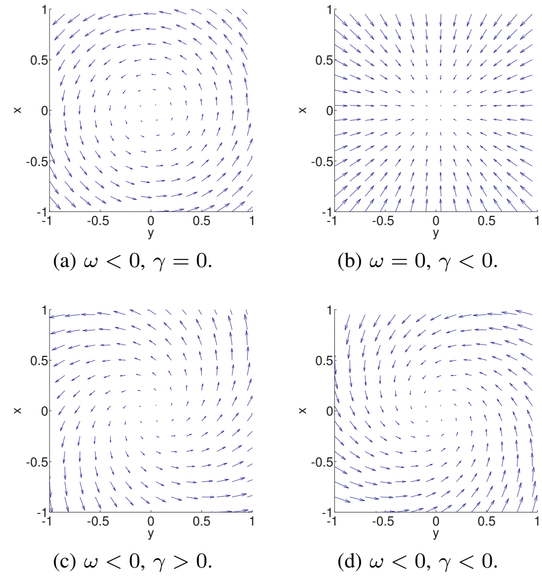
$$\boldsymbol{V}_w(x, y, t) = \boldsymbol{U}(t) + \boldsymbol{A}(t)(\boldsymbol{r} - \boldsymbol{r}_s), \qquad (1)$$

where $\boldsymbol{U}(t)$ is the uniform flow component, $\boldsymbol{r} = [x, y]^T$ is the planar position, and the matrix

$$\boldsymbol{A}(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) \\ a_{21}(t) & a_{22}(t) \end{bmatrix}$$

characterizes the flow singularity. The flow velocity components will be implicitly assumed to depend on the position and time, $[u, v]^T = [u(x, y, t), v(x, y, t)]^T$, and the parentheses will be omitted to simplify notation. Without loss of generality, the flow singularity will be assumed to be at the origin. Depending on the components of $\boldsymbol{A}$, the above model can describe a number of interesting flow phenomena. The divergence of the flow, given by

$$\boldsymbol{\nabla} \cdot \boldsymbol{V}_w = \frac{\partial}{\partial x} u + \frac{\partial}{\partial y} v = a_{11} + a_{22},$$



(a) $\omega < 0$, $\gamma = 0$. (b) $\omega = 0$, $\gamma < 0$.

(c) $\omega < 0$, $\gamma > 0$. (d) $\omega < 0$, $\gamma < 0$.

Fig. 1: Example flow fields that can be represented by equation (2).

is equal to the net inflow or outflow of fluid particles into the region containing the singularity [6]. The symbol $\boldsymbol{\nabla}$ denotes the gradient operator. If $\boldsymbol{\nabla} \cdot \boldsymbol{V}_w > 0$ the singularity is a source, if $\boldsymbol{\nabla} \cdot \boldsymbol{V}_w < 0$ the singularity is a sink, and if $\boldsymbol{\nabla} \cdot \boldsymbol{V}_w = 0$, then the flow is divergence free and purely rotational. The rotation, or vertical vorticity, given by

$$\boldsymbol{\nabla} \times \boldsymbol{V}_w = \frac{\partial}{\partial x} v - \frac{\partial}{\partial y} u = a_{21} - a_{12},$$

describes the rotational motion of the flow. Positive vertical vorticity describes clockwise rotation and vice versa. Note that in this paper the navigation axis convention is adopted, where the x-axis points North, the y-axis points East and the z-axis points down. Hence, in the plane — as seen from above — the x-axis is vertical and the y-axis is horizontal. Also, observe that in this model the instantaneous vorticity is related to the angular rate of the fluid particles by

$$\omega(t) = \frac{1}{2} \boldsymbol{\nabla} \times \boldsymbol{V}_w.$$

A. Point-Symmetric Singular Flow

Consider the following symmetric flow structure:

$$\boldsymbol{A}_p = \begin{bmatrix} \gamma(t) & -\omega(t) \\ \omega(t) & \gamma(t) \end{bmatrix}. \qquad (2)$$

This structure allows to treat fairly general flow phenomena, such as the ones shown in Figure 1. These flow patterns include the combination of sources/sinks and vortices, and can model interesting oceanographic phenomena such as eddy-driven upwelling or downwelling at the ocean surface. Flow patterns that can represented by the matrix $\boldsymbol{A}_p$ will be referred to as *point-symmetric* in this paper.

Remark 2.1: Note that if the fluid is rotating as a solid body, then $\omega(t)$ gives the instantaneous angular velocity of the fluid particles in the entire flow region. Also note that if $\gamma(t) \equiv 0$, then this flow model represents the flow-field inside the core region of a Rankine vortex.

The structure imposed by equation (2) allows the establishment of a simple geometric connection between the vertical vorticity and the optimal steering angle near a flow singularity.

III. Optimal Navigation

A. Model

A flight vehicle must travel through a region of $\mathbb{R}^2$ in the presence of spatially and temporally varying flow-field:

$$\dot{x} = V\cos\psi + u \tag{3}$$
$$\dot{y} = V\sin\psi + v, \tag{4}$$

where $[x, y]^T \in \mathbb{R}^2$ are the spatial position coordinates, V is the flow-relative velocity, which is assumed to be constant, and ψ is the heading angle due North. The control signal for the system is the steering angle, ψ, which can take values in the set

$$\psi \in \mathcal{U} = [-\pi, \pi], \qquad \forall t \in [0, t_f].$$

B. General Properties of Extremals

The Hamiltonian for the system is

$$\mathcal{H} = \lambda_x(V\cos\psi + u) + \lambda_y(V\sin\psi + v). \tag{5}$$

The adjoint equations are

$$\dot{\lambda}_x = -\lambda_x\frac{\partial u}{\partial x} - \lambda_y\frac{\partial v}{\partial x} \tag{6}$$
$$\dot{\lambda}_y = -\lambda_x\frac{\partial u}{\partial y} - \lambda_y\frac{\partial v}{\partial y}. \tag{7}$$

The necessary condition for optimality by Pontryagin's minimum principle [12, Theorem 5] is that the Hamiltonian attains its minimum at ψ for all values of the states. To carry out the minimization we write

$$\frac{\partial\mathcal{H}}{\partial\psi} = -V\lambda_x\sin\psi + V\lambda_y\cos\psi \equiv 0.$$

It follows that

$$\boldsymbol{\lambda} \perp \begin{bmatrix}-\sin\psi\\ \cos\psi\end{bmatrix} \quad \Longleftrightarrow \quad \boldsymbol{\lambda} \parallel \begin{bmatrix}\cos\psi\\ \sin\psi\end{bmatrix},$$

and since the adjoint vector cannot be zero, that is $\bar{\lambda} = \sqrt{\lambda_x^2 + \lambda_y^2} > 0$, we can write

$$\cos\psi = \delta\frac{\lambda_x}{\bar{\lambda}}, \quad \sin\psi = \delta\frac{\lambda_y}{\bar{\lambda}},$$

where $\delta = \pm 1$. Taking the second derivative of the Hamiltonian with respect to the control gives

$$\frac{\partial^2\mathcal{H}}{\partial\psi^2} = -V\lambda_x\cos\psi - V\lambda_y\sin\psi = -V\bar{\lambda}\delta.$$

In order for the extremal to provide a local minimum we need

$$\frac{\partial^2\mathcal{H}}{\partial\psi^2} = -V\bar{\lambda}\delta > 0,$$

hence $\delta = -1$. This choice guarantees that along the extremal path the solution will be at least locally optimal. With $\delta = -1$, the adjoint vector has to be anti-parallel to the flow-relative velocity vector at all times. The above findings can be summarized in the following

Lemma 3.1: (cf.[2, Section 2.7]) A necessary condition for time-optimality is that the flow-relative velocity vector is at all times anti-parallel to the current value of the adjoint vector, that is

$$\psi^*(t) = \tan^{-1}\left(\frac{-\lambda_y(t)}{-\lambda_x(t)}\right), \tag{8}$$

where the four-quadrant arctangent function is used.

In a flow region that can be described by equation (1), the adjoint equations (6)-(7) can be written as

$$\dot{\boldsymbol{\lambda}} = -\boldsymbol{A}_p^T\boldsymbol{\lambda}.$$

In this special case the adjoint system is linear and decoupled from the system states. The solution to this linear system is given by

$$\boldsymbol{\lambda}(t) = \boldsymbol{\Phi}_p(t, 0)\boldsymbol{\lambda}_0, \tag{9}$$

where $\boldsymbol{\Phi}_p(t, 0)$ is the state transition matrix corresponding to the matrix $-\boldsymbol{A}_p^T$. Lemma 3.1 states that the angle of the adjoint vector characterizes the optimal steering policy, hence it is instrumental to study the evolution of adjoint vector, or equivalently the state transition matrix $\boldsymbol{\Phi}_p(t, 0)$.

C. Point-symmetric Flow

The state matrix can be written as the sum of two matrices

$$-\boldsymbol{A}_p^T = \boldsymbol{\Gamma}(t) + \boldsymbol{\Omega}(t), \tag{10}$$

where

$$\boldsymbol{\Gamma}(t) = \begin{bmatrix}-\gamma(t) & 0\\ 0 & -\gamma(t)\end{bmatrix}, \quad \boldsymbol{\Omega}(t) = \begin{bmatrix}0 & -\omega(t)\\ \omega(t) & 0\end{bmatrix}.$$

The state transition matrices $\boldsymbol{\Phi}_\Gamma(t, 0)$ and $\boldsymbol{\Phi}_\Omega(t, 0)$, corresponding to the matrices $\boldsymbol{\Gamma}(t)$, and $\boldsymbol{\Omega}(t)$, respectively,

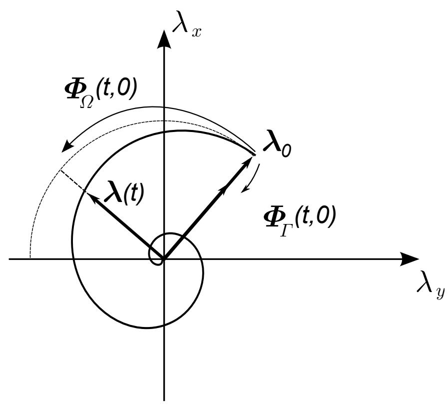


Fig. 2: Adjoint transformation.

can be written as

$$\Phi_\Gamma(t,0) = \begin{bmatrix} e^{-\Gamma(t)} & 0 \\ 0 & e^{-\Gamma(t)} \end{bmatrix}, \tag{11}$$

$$\Gamma(t) = \int_0^t \gamma(\sigma)d\sigma, \tag{12}$$

$$\Phi_\Omega(t,0) = \begin{bmatrix} \cos\Omega(t) & -\sin\Omega(t) \\ \sin\Omega(t) & \cos\Omega(t) \end{bmatrix}, \tag{13}$$

$$\Omega(t) = \int_0^t \omega(\sigma)d\sigma. \tag{14}$$

Notice that $\Phi_\Omega(t,0)$ is a proper rotation matrix about the z-axis with angle $\Omega(t)$:

$$\Phi_\Omega(t,0) = \boldsymbol{R}_3(-\Omega(t)).$$

Recall that for arbitrary matrix $\boldsymbol{X}(t)$, the state transition matrix $\Phi_X(t,0)$ satisfies

$$\dot{\Phi}_X(t,0) = \boldsymbol{X}(t)\Phi_X(t,0).$$

From this property, and from the fact that $\Phi_\Gamma(t,0) = e^{-\Gamma(t)}\mathbf{I}$, it follows that

$$\Phi_p(t,0) = \Phi_\Gamma(t,0)\Phi_\Omega(t,0).$$

The adjoint vector is obtained from its initial value by two consecutive linear operations: a rotation and a scaling. Since the matrix $\Phi_\Gamma(t,0)$ is diagonal and has identical values, the order of operations is interchangeable. This linear transformation — illustrated in Figure 2 — has the following properties:

1) The scaling leaves the ratio of the components of the adjoint vector invariant.
2) The rotation leaves the length of the adjoint vector invariant.

Consequently, the angle of the adjoint vector is given by $\Omega(t) + \phi_{\lambda_0}$, where ϕ_{λ_0} is the angle of the adjoint vector at $t = 0$. Since the adjoint vector $\boldsymbol{\lambda}(t)$ is always anti-parallel to the flow-relative velocity vector, we have established the following

Theorem 3.2: In a planar point-symmetric flow-field the optimal steering policy (if it exists) is given by

$$\psi^*(t) = \Omega(t) + \psi_0, \tag{15}$$

where $\Omega(t)$ — given in equation (14) — is the time integral of the fluid angular rate. Consequently, the rate of the steering angle is equal to the instantaneous fluid angular rate, or equivalently, to one-half of the instantaneous vertical vorticity:

$$\dot{\psi}^*(t) = \omega(t) \equiv \frac{1}{2}\boldsymbol{\nabla} \times \boldsymbol{V}_w. \tag{16}$$

Corollary 3.3: If the fluid is irrotational, then the optimal steering angle is constant.

Remark 3.4: It is important to emphasize again, that this result applies for the flow model given by (2). The necessity to impose this structure is clear from the geometric interpretation illustrated in Figure 2. If the diagonal elements of $\boldsymbol{A}_p$ were not equal, then the linear operation defined by $\Phi_\Gamma(t,0)$ would impose a rotation as well as a scaling, and the steering angle need not be constant even in irrotational flow.

D. *Special Case: Time-invariant, divergence free flow*

Suppose the flow takes the following, purely rotational form

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \tag{17}$$

where now the rotation rate is time-invariant (constant). Then the closed-form expression for the state transition matrix in (9) is given by

$$\Phi(t,0) = \begin{bmatrix} \cos(\omega t) & -\sin(\omega t) \\ \sin(\omega t) & \cos(\omega t) \end{bmatrix}.$$

(Note that the rotation leaves the length invariant, hence the length of the adjoint vector $\bar{\lambda} = \sqrt{\lambda_x^2 + \lambda_y^2}$ is a first integral.) By Theorem 3.2, the optimal steering control is a linear function of time:

$$\psi^*(t) = \omega t + \psi_0. \tag{18}$$

Since the optimal steering angle is linear, we can obtain closed-form solution for candidate extremal trajectories. Substituting equation (18) into equations (3)-(4), we get

$$\begin{bmatrix} \dot{x} \\ \dot{y} \end{bmatrix} = \begin{bmatrix} V\cos\psi^* - \omega y \\ V\sin\psi^* + \omega x \end{bmatrix} = \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} + \boldsymbol{B}(t),$$

where

$$\boldsymbol{B}(t) = \begin{bmatrix} V\cos(\omega t + \psi_0) \\ V\sin(\omega t + \psi_0) \end{bmatrix}.$$

This system can be thought of as a linear time-invariant system with sinusoidal forcing. The solution can be found using the variation of constants formula:

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \Phi(t,0) \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} + \int_0^t \Phi(t,\sigma)\boldsymbol{B}(\sigma)d\sigma,$$

which gives

$$\begin{bmatrix} x(t) \\ y(t) \end{bmatrix} = \mathbf{\Phi}(t,0) \begin{bmatrix} x(0) \\ y(0) \end{bmatrix} + Vt \begin{bmatrix} \cos(\psi_0 + \omega t) \\ \sin(\psi_0 + \omega t) \end{bmatrix}. \quad (19)$$

In order to solve for specific extremal trajectories, the boundary conditions have to be specified.

IV. NUMERICAL EXAMPLE

In this section a simple example is presented that illustrates the utility of methods established in Section III. The goal is to find the time-optimal trajectory between initial and final points $x_0 = R\cos(\pi/6)$, $y_0 = R\sin(\pi/6)$, $x_f = 0$, $y_f = R$, where $R = 1$. (All variables are assumed to be non-dimensional in this paper.) We take $V = 1$.

The approach we use in this example is based on numerical optimization involving two unknown parameters: the final time, t_f, and the initial steering angle, ψ_0. If both t_f and ψ_0 were known, then the trajectory would be uniquely determined given that the rate of the steering angle is known from equation (16), and numerical integration can be used to obtain $[x(t), y(t)]^T$ from equations (3)-(4). (Note that, if the flow takes the special form (17), then the closed-form expression (19) can be used to find the solution to the initial value problem.) Since the two parameters $(t_f; \psi_0)$ are unknown, let $[x(t_f;\psi_0), y(t_f;\psi_0)]^T$ denote the solution to the initial value problem. Let us define the following function

$$\boldsymbol{F}(t_f, \psi_0) = \begin{bmatrix} x(t_f;\psi_0) - x_f \\ y(t_f;\psi_0) - y_f \end{bmatrix}. \quad (20)$$

All the solutions to the equation $\boldsymbol{F}(t_f, \psi_0) \equiv \mathbf{0}$, where $t_f > 0$ define feasible trajectories that transfer the initial point to the desired final point. Furthermore, the trajectories satisfy the necessary conditions for optimality as established by Theorem 3.2, and are guaranteed to be at least locally optimal. The roots of equation (20) were found using the Levenberg-Marquardt algorithm (see e.g. [3]).

In this example the flow is described by equation (2), with $\gamma = -0.3$, and $\omega(t) = t - 1$. In this case the flow model represents a constant sink plus a vortex that changes vorticity linearly during the maneuver. In this simple linear case, the fluid angular rate is a linear function of time. Consequently, by Theorem 3.2, the optimal steering policy has to be parabolic, and is given by $\psi^*(t) = t^2/2 - t + 2.605$. The planar position plots can be seen in Figure 3. From the minimum principle [12, Theorem 5], the variable

$$\mathcal{H}_1 = \mathcal{H} - \int_{t_f}^{t} \frac{\partial \mathcal{H}}{\partial t} d\tau.$$

has to be a non-positive constant along the optimal trajectory. The condition is satisfied for the example presented in this paper. The evolution of the steering angle, the variable $\mathcal{H}_1$, and the adjoint states are shown in Figure 4. Notice that at $t = 1$, when the vertical vorticity vanishes, the derivative of the steering angle is zero.

V. CONCLUSIONS

In this paper we revisited Zermelo's time-optimal navigation problem. Recalling the geometric connection between the adjoint vector's angle and the optimal steering policy, and exploiting the linear structure of the imposed flow model, we established a necessary condition for candidate extremal trajectories. The contribution is the observation that — in a point-symmetric, time-varying flow-field — the optimal steering policy necessarily has to be such that the rate of the steering angle equals the angular rotation rate of the fluid particles, or equivalently one-half of the instantaneous vertical vorticity. The point-symmetric flow model may represent a variety of flow phenomena as they are observed in nature, such as large-scale vortical motions and eddy-driven upwelling or downwelling at the ocean surface. The result can be used to set up efficient numerical routines to find extremal trajectories in time-varying flow structures. In addition to finding extremal trajectories in specific scenarios, the result also gives an intuitive rule of thumb as of how to helm a ship or aircraft in a point-symmetric flow-field. This rule could be potentially exploited in present day human piloted marine vehicle operations. One such area is underwater gliders, which are deployed at an increasing rate to collect oceanographic data. Although there has been considerable effort recently to equip them with optimal trajectory planning methods, presently these vehicles are almost exclusively piloted by human operators, who provide way-point commands between consecutive dives.

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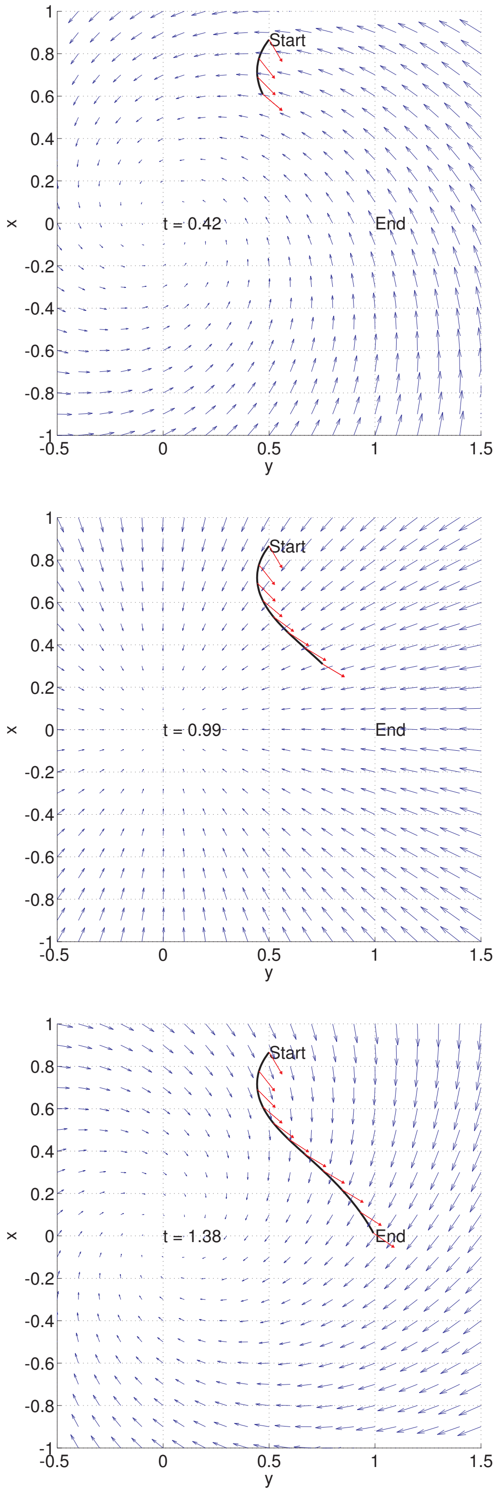

Fig. 3: Example 3: Planar position plot of the time-optimal trajectory in time-varying flow that includes a sink $\gamma < 0$, and a vortex that changes vorticity linearly during the maneuver.

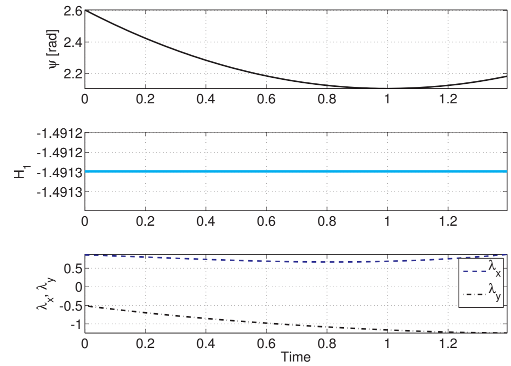

Fig. 4: Example 3: Evolution of the optimal steering policy, the Hamiltonian, and the adjoint states.

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