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Hybrid Output Regulation with Unmeasured Clock

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***Abstract*— This paper addresses the problem of hybrid output regulation for linear systems in the case where the jumps of the system and exosystem state are triggered by a clock variable that is not measured. We extend the result presented in [13] by complementing the hybrid internal model-based regulator design proposed in that paper with a hybrid clock phase estimator yielding asymptotic regulation.**

I. Introduction

The problem of synchronizing dynamical systems has attracted a lot of interest in the control community. Studies in this field have focused on synchronization of chaotic systems [14], of linear networked systems [11], and of nonlinear systems by adaptive tools [1], just to mention a few. A particular research area related to synchronization is output regulation ([5], [6], [10]). In the latter the problem consists of "synchronizing" a controlled system with an exogenous system (the so-called exosystem) so that the effect of the latter on a regulation error is compensated.

In this paper we consider a problem of output regulation for hybrid linear systems [4] in which the jumps of the exogenous system (exosystem) are triggered by a clock variable that is not measured. In addressing the problem, we present a phase clock estimator that, joined to an internal model-based hybrid regulator, guarantees an asymptotically vanishing error. The paper complements the theory presented in [13] in which the problem at hand was addressed by designing a clock-dependent regulator. In that paper we developed a notion of steady state for hybrid linear systems and proposed a robust design solution that extended, to the hybrid setting, typical "continuous-time paradigms" based on the internal model principle ([2], [5]).

The solution presented in this paper starts from the solution in [13], and, in a kind of "certainty equivalence" paradigm, proposes a nonlinear controller obtained by replacing the unknown exogenous clock with a suitable estimate obtained by dynamically processing the regulation error. It is shown that, under a persistence of excitation condition and other technical assumptions, the steady-state control input converges, in appropriate sense, to the ideal input which guarantees that the error converges to zero asymptotically.

As in [13] we focus on the class of hybrid linear systems that are minimum-phase so that the high-gain stabilization results proposed in [15], [16] can be adopted. Interestingly enough, we show that the resulting "closed-loop system" fits in the framework of [16] in which asymptotic convergence of the error to zero is guaranteed in spite of the fact that term driving the error dynamics vanishes everywhere on the zero-error manifold except at a finite number of nonequilibrium points.

As the mathematical background regarding hybrid systems, this work rests upon the general framework and results of hybrid control systems presented, in an introductory way, in [4] from where also the hybrid formalism and notation are taken. We refer the reader to that work for details about the notion of solution of a hybrid system and the notion of asymptotic stability for hybrid systems that are extensively used in this paper.

Notation By $\mathbb{R}$, $\mathbb{Z}$ and $\mathbb{N}$ we denote the set of real, integer and nonnegative integers, respectively. With $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ we denote a set-valued mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$. $\text{Tr}(M)$ denotes the trace of a matrix M while $M^\dagger$ represents the Moore-Penrose pseudoinverse of M and $\text{Eig}(M)$ denotes the spectrum of M. By $\|M\|$ we denote the Frobenius matrix norm of M.

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II. The framework

We consider a hybrid linear system of the form

$$
\left.\begin{array}{rcl} \dot\tau &=& 1\\ \dot w &=& Sw\\ \dot x &=& Ax+Bu+Pw \end{array}\right\} (\tau,w,x)\in[0,\tau_{\max}]\times\mathcal{W}\times\mathbb{R}^n
$$
$$
\left.\begin{array}{rcl} \tau^+ &=& 0\\ w^+ &=& Jw\\ x^+ &=& Nw+Mx \end{array}\right\} (\tau,w,x)\in\{\tau_{\max}\}\times\mathcal{W}\times\mathbb{R}^n \qquad (1)
$$

with regulated output $e = Cx + Qw$ in which $u \in \mathbb{R}$ is the control input, $w \in \mathcal{W} \subset \mathbb{R}^s$ is the exogenous variable modeling disturbances to be rejected or references to be tracked, and τ is a clock variable triggering the jumps of the system that occur every $\tau_{\max}$ instances of time. In the paper we assume that $w(t, j) \in \mathcal{W}$ for all (t, j) in the solution domain and that $\mathcal{W}$ is a compact set. The value of $\tau_{\max}$ can be thus considered as a dwell time between two consecutive jumps. In the framework above our goal is to design a hybrid regulator of the form

$$
\begin{array}{l}
\left.\begin{array}{rcl} \dot\sigma &=& 1\\ \dot\zeta &=& \varphi(\sigma,\zeta,e) \end{array}\right\} (\sigma,\zeta)\in[0,\tau_{\max}]\times\mathbb{R}^m\\[2ex]
\left.\begin{array}{rcl} \sigma^+ &=& 0\\ \zeta^+ &\in& \psi(\sigma,\zeta,e) \end{array}\right\} (\sigma,\zeta)\in\{\tau_{\max}\}\times\mathbb{R}^m
\end{array} \qquad (2)
$$

with output $u=\gamma(\zeta,e)$, in which $\varphi:[0,\tau_{\max}]\times\mathbb{R}^m\times\mathbb{R}\to\mathbb{R}^m$ is a continuous function and $\psi:[0,\tau_{\max}]\times\mathbb{R}^m\times\mathbb{R}\rightrightarrows\mathbb{R}^m$ is an outer semicontinuous, locally bounded set-valued mapping, such that all complete trajectories of the resulting closed-loop system originating from a given compact set are uniformly bounded and satisfy

$$\lim_{t+j\to\infty} e(t+j)=0\,. \tag{3}$$

In (2) the variable σ represents an internal clock of the regulator not necessarily synchronized with the system clock τ. Indeed, this article places emphasis on the design of regulators that do not have access to the measure of τ by thus extending the design principles of [13] in which τ-dependent regulators was presented. The need of estimating the value of τ from the regulated error e will lead us to design nonlinear regulators obtained by combining internal-model based hybrid regulators of the form presented in [13] with dynamic estimators of the phase of τ. It is worth noting that the dwell time $\tau_{\max}$ is assumed to be known.

The problem at hand will be solved under some assumptions that are presented throughout the paper. As a first restriction, this paper focuses on systems (1) described in normal form with *unitary relative degree* between the input u and the output e. Specifically, we assume that the state x is partitioned as $x=\mathrm{col}(z,e)$ with $z\in\mathbb{R}^{n-1}$ and the matrices A,B,P,N,M are accordingly partitioned as

$$A=\begin{pmatrix} A_{11} & A_{12}\\ A_{21} & A_{22}\end{pmatrix}\quad B=\begin{pmatrix}0\\1\end{pmatrix}\quad P=\begin{pmatrix}P_1\\P_2\end{pmatrix}$$

$$N=\begin{pmatrix}N_1\\N_2\end{pmatrix}\quad M=\begin{pmatrix} M_{11} & M_{12}\\ M_{21} & M_{22}\end{pmatrix}.$$

In these coordinates, the system in question has a *hybrid zero dynamics* of the form

$$\left.\begin{array}{rcl}\dot\tau &=& 1 \quad \dot w = Sw\\ \dot z &=& A_{11}z+P_1 w\end{array}\right\}\ (\tau,w,z)\in[0,\tau_{\max}]\times\mathcal{W}\times\mathbb{R}^{n-1}$$

$$\left.\begin{array}{rcl}\tau^+ &=& 0 \quad w^+ = Jw\\ z^+ &=& N_1 w+M_{11}z\end{array}\right\}\ (\tau,w,z)\in\{\tau_{\max}\}\times\mathcal{W}\times\mathbb{R}^{n-1}\,. \tag{4}$$

On this system we formulate the following two assumptions.

Assumption 1 The eigenvalues of the matrix $M_{11}\exp(A_{11}\tau_{\max})$ are within the unitary disk. $\triangleleft$

Assumption 2 There exists a $T\in\mathbb{R}^{(n-1)\times s}$ solution of

$$0=A_{11}T-TS+P_1\,,\quad TJ=M_{11}T+N_1\,.\ \triangleleft$$

The first assumption represents a *minimum-phase assumption* for the system (1) and it is motivated by the fact that, as a stabilization tool, we shall use the high-gain design paradigm proposed in [15], [16]. The second assumption restricts the attention to zero dynamics (4) having a steady state response that is not dependent on τ. Specifically, by changing coordinates as $z\ \mapsto\ \tilde z:=z-Tw$ and using Assumption 2, system (4) transforms as

$$\left.\begin{array}{rcl}\dot\tau &=& 1 \quad \dot w = Sw\\ \dot{\tilde z} &=& A_{11}\tilde z\end{array}\right\}\ (\tau,w,\tilde z)\in[0,\tau_{\max}]\times\mathcal{W}\times\mathbb{R}^{n-1}$$

$$\left.\begin{array}{rcl}\tau^+ &=& 0 \quad w^+ = Jw\\ \tilde z^+ &=& M_{11}\tilde z\end{array}\right\}\ (\tau,w,\tilde z)\in\{\tau_{\max}\}\times\mathcal{W}\times\mathbb{R}^{n-1}\,. \tag{5}$$

This system, due to Assumption 1, has the set $\{(\tau,w,\tilde z)\in[0,\tau_{\max}]\times\mathcal{W}\times\mathbb{R}^{n-1}\ :\ \tilde z=0\}$ that is globally asymptotically stable, namely $z(t,j)$ converges asymptotically to $Tw(t,j)$ as $t+j\to\infty$.

As a further restriction, imposed by the stabilization tool of [15], [16], we assume that the value of the error e doesn't change during jumps and that $M_{11}=I$. Namely, we assume the following.

Assumption 3 $M_{21}=0\,,\ N_2=0\,,\ M_{22}=1\,, M_{11}=I\,.\ \triangleleft$

It is worth noting that the previous assumption is fulfilled in the special case where the plant is a continuous-time system and the jumping disturbance is matched with the input (in which case $M_{12}=0$, $P_1=0$ and $P_2=1$). In this special case Assumption 2 is automatically fulfilled with $T=0$ and Assumption 1 boils down to requiring that the matrix A_{11} is Hurwitz.

III. The Regulator and Its Tuning

The proposed regulator is a system that flows according to the dynamics

$$\begin{array}{rcl}\dot\sigma &=& 1\\ \dot\zeta &=& A_{11}\zeta+A_{12}e\\ \dot\eta &=& F(\eta-Ge)+Gu\\ \dot\xi &=& F(\xi-Ge)+Gu+GA_{22}e+GA_{21}\zeta\\ \dot W &=& 0\\ \dot p &=& 0\end{array} \tag{6}$$

whenever $(\sigma,\zeta,\eta,\xi,W,p,e)\in[0,\tau_{\max}]\times\mathbb{R}^{n-1}\times\mathbb{R}^\ell\times\mathbb{R}^\ell\times\mathcal{P}_0\times\mathbb{R}\times\mathbb{R}$, and jumps according to the rule

$$\begin{array}{rcl}\sigma^+ &=& 0\\ \zeta^+ &=& \zeta\\ \eta^+ &=& \eta\\ \xi^+ &=& \xi\\ W^+ &=& \lambda W+(\xi-Ge)(\xi-Ge)^T\\ p^+ &\in& \arg\min\limits_{\hat p\in[0,\tau_{max}]}\|\Gamma_{\rm f}(\sigma+\hat p)\,W\,\Gamma_{\rm f}(\sigma+\hat p)^T\|\end{array} \tag{7}$$

whenever $(\sigma,\zeta,\eta,\xi,W,p,e)\in\{\tau_{max}\}\times\mathbb{R}^{n-1}\times\mathbb{R}^\ell\times\mathbb{R}^\ell\times\mathcal{P}_0\times\mathbb{R}\times\mathbb{R}$, with control input that is chosen as

$$u=\Gamma(\sigma+p)(\eta-Ge)+v\,.$$

In the previous equations λ is a parameter fulfilling $\lambda\in(0,1)$, $\mathcal{P}_0$ is the set of symmetric positive semidefinite matrices in $\mathbb{R}^{\ell\times\ell}$, $(F,G)\in\mathbb{R}^{\ell\times\ell}\times\mathbb{R}^{\ell\times 1}$ is a controllable pair, $\Gamma:[0,2\tau_{\max}]\to\mathbb{R}^{1\times\ell}$ and $\Gamma_{\rm f}:[0,2\tau_{\max}]\to\mathbb{R}^{\nu\times\ell}$ are matrix-valued functions, v is a residual control input. The degree-of-freedom of the controller are ℓ, ν, (F,G), Γ, $\Gamma_{\rm f}$ and v, which will be fixed in a while.

The regulator state variables have the following meaning: σ is the internal clock of the regulator that is not necessarily synchronized with τ, p is the estimation of the clock phase (ideally, $\sigma+p$ converges, modulo $\tau_{\max}$, to τ), W is the matrix state of a "state-shared" estimator (by using the terminology of [3]) needed for tuning the phase variable p. The η and ξ are state variables of two *internal model units* (chosen exactly alike in the proposed solution) that are crucial to compute the steady state control input and to drive the W dynamics. Finally, the ζ dynamics introduces a partial replica of the zero dynamics of the system in the regulator. As clarified by the forthcoming analysis, the ζ-dynamics is introduced in order to make, in appropriate coordinates, the phase-estimation dynamics not affected by e.

In the following part of the section we complete the regulator design by fixing the degree-of-freedom ℓ, ν, (F, G), Γ, $\Gamma_{\rm f}$.

As far as ℓ and the pair (F, G) are concerned, we choose $\ell \geq s+1$, (F, G) controllable, with F fulfilling

$$\mathrm{Eig}(\exp(F\tau_{\max})) \cap \mathrm{Eig}(J\exp(S\tau_{\max})) = \emptyset\,,$$

and

$$\iota \in \mathrm{Eig}(\exp(F\tau_{\max})) \;\Rightarrow\; |\iota| < 1\,.$$

By Proposition 1 in [13], the F with the above properties guarantees that there exists a continuously differentiable function $\Pi : [0, \tau_{\max}] \to \mathbb{R}^{\ell\times s}$ solution of the following equations

$$\begin{aligned} \frac{d\Pi(\tau)}{d\tau} &= F\Pi(\tau) - \Pi(\tau)S - G\bar{P}_2 \\ 0 &= \Pi(\tau_{\max}) - \Pi(0)J \end{aligned} \tag{8}$$

with $\bar{P}_2 := A_{21}T + P_2$. In fact, the expression of $\Pi(\tau)$ is given by

$$\Pi(\tau) = (\exp(F\tau)\Pi(0) + D(\tau))\exp(-S\tau)$$

with $D(\tau)$ solution of

$$\frac{dD(\tau)}{d\tau} = FD(\tau) - G\bar{P}_2\mathrm{exp}(S\tau) \qquad D(0) = 0$$

and $\Pi(0)$ solution of the Sylvester equation

$$\exp(F\tau_{\max})\Pi(0) - \Pi(0)J\exp(S\tau_{\max}) + D(\tau_{\max}) = 0\,.$$

The tuning of the controller proceeds under a technical assumption, already present in [13], involving $\Pi(\tau)$.

Assumption 4 There exists $r \leq s$ such that $\mathrm{rank}\Pi(\tau) = r$ for all $\tau \in [0, \tau_{\max}]$. ◁

This assumption guarantees that $\Pi(\tau)^\dagger : [0, \tau_{\max}] \to \mathbb{R}^{s\times\ell}$ is a continuous function (see [12]). With the function $\Pi(\tau)$ and the number r in hand, the matrix-valued function $\Gamma_{\rm f}(\varsigma) : [0, 2\tau_{\max}] \to \mathbb{R}^{\nu\times\ell}$ can be constructed as follow. Take $\nu = \ell - r$ and let $L : [0, \tau_{\max}] \to \mathbb{R}^{\ell\times\ell-r}$ be obtained by integrating

$$\dot{L}(\varsigma) = (-\Pi(\varsigma)^\dagger)^T(\dot{\Pi}(\varsigma))^T L(\varsigma)$$

with initial condition $L(0)$ satisfying

$$L(0)^T \begin{bmatrix} \Pi(0) & L(0) \end{bmatrix} = \begin{bmatrix} 0 & I_{\ell-r} \end{bmatrix}.$$

It turns out that $L(\tau)^T L(\tau) = I$ for all $\tau \in [0, \tau_{\max}]$. Furthermore, using the fact that $L(0)$ and $L(\tau_{\max})$ are unitary matrices, let $U_m \in \mathbb{R}^{\ell-r\times\ell-r}$ be such that $U_m U_m^T = I$ and $L(\tau_{\max})U_m = L(0)$ and construct $U : [0, \tau_{\max}] \to \mathbb{R}^{\ell-r\times\ell-r}$ in such a way that (see Example 8.2 of [8]) $U(\varsigma)U(\varsigma)^T = I_{\ell-r}$, $U(0) = I_{\ell-r}$ and $U(\tau_{\max}) = U_m$. Then, $\Gamma_{\rm f}$ can be chosen as

$$\Gamma_{\rm f}(\varsigma) = U(\varsigma)^T L(\varsigma)^T$$

for $\varsigma \in [0, \tau_{\max}]$ and $\Gamma_{\rm f}(\varsigma) = \Gamma_{\rm f}(\varsigma - \tau_{\max})$ for $\varsigma \in (\tau_{\max}, 2\tau_{\max}]$. Note that $\Gamma_{\rm f}(\tau_{\max}) = \Gamma_{\rm f}(0) = L(0)^T$, namely, $\Gamma_{\rm f} : [0, 2\tau_{\max}] \to \mathbb{R}^{\nu\times\ell}$ is continuous.

Finally, we design $\Gamma : [0, 2\tau_{\max}] \to \mathbb{R}^{1\times\ell}$ as $\Gamma(\varsigma) = -\bar{P}_2\,\Pi(\varsigma)^\dagger$ for $\varsigma \in [0, \tau_{\max}]$ and $\Gamma(\varsigma) = \Gamma(\varsigma - \tau_{\max})$ for $\varsigma \in (\tau_{\max}, 2\tau_{\max}]$. Note that, in general, $\Gamma(0) \neq \Gamma(\tau_{\max})$, namely, $\Gamma : [0, 2\tau_{\max}] \to \mathbb{R}^{1\times\ell}$ is discontinuous. Indeed, for the sake of robustness, a set-valued regularization of Γ will be considered in the following analysis.

IV. MAIN RESULT

The following result provides the choice of the residual input v that guarantees the fulfillment of the regulation objective (3). The result is formulated under two additional assumptions specified below. The first is motivated by the so-called "partial injectivity condition" introduced in the output regulation context in [9] (see also [10], [7]). The second is a persistence of excitation condition requiring a minimum energy value to the regularly sampled exosystem trajectory.

Assumption 5 There exists a $\mu > 0$ such that

$$|\bar{P}_2 w_1 - \bar{P}_2 w_2| \leq \mu|\Pi(\tau_1)w_1 - \Pi(\tau_2)w_2|$$

for all $(w_1, w_2) \in \mathcal{W}^2$ and $(\tau_1, \tau_2) \in [0, \tau_{\max}]^2$. ◁

Assumption 6 (*Persistence of excitation*) There exists an $\alpha > 0$ such that for each increasing sequence $\{t_k\}_{-\infty}^{0}$, with the property that (t_k, k) belongs to the solution domain of the exosystem and $t_k - t_{k-1} = \tau_{\max}$, the following holds

$$\sum_{k=-\infty}^{0} \lambda^{-k} w(t_k, k)w(t_k, k)^T > \alpha I\,. \;◁$$

Proposition 1: Let Assumptions 1-6 be fulfilled. Let $\mathcal{K}_1 \subset \mathbb{R}^{n-1} \times \mathbb{R}$ and $\mathcal{K}_2 \subset \mathbb{R}^{n-1}\times\mathbb{R}^\ell \times \mathbb{R}^\ell \times \mathcal{P}_0 \times \mathbb{R}$ be given compact sets. There exists a continuous function $\kappa : \mathbb{R} \to \mathbb{R}$ such that for all $(\tau(0,0), \sigma(0,0)) \in [0, \tau_{\max}] \times [0, \tau_{\max}]$, $(z(0), e(0)) \in \mathcal{K}_1$ and $(\zeta(0,0), \xi(0,0), \eta(0,0), W(0,0), p(0,0)) \in \mathcal{K}_2$ the trajectories of (1) in closed-loop with (6)-(7) and

$$v = -\mathrm{sgn}(e)\,\kappa(|e|) \tag{9}$$

are bounded and (3) holds true.

In the following part of the section we present the proof of the proposition. The proof rests upon the following ideas. First, it is shown that the closed-loop system preserves, in appropriate coordinates, the property of being a hybrid nonlinear system in normal form with relative degree-one between the input v and the error e. The body of the proof is then the study of the associated hybrid zero dynamics that present a cascade structure, with (5) driving a system in cascade dependent on the regulator dynamics. It is shown that, due to the design choice in Section III, the hybrid zero dynamics presents an asymptotically stable compact set, denoted by $\mathcal{M}$, on which the interconnection term with the error dynamics is everywhere zero except a finite number of nonequilibrium points. This fact allows us to frame the stabilization problem in the framework of [16] and to conclude semiglobal asymptotic stability of the set $\mathcal{M}\times\{0\}$ with an high-gain law of the form (9). Due to space constraint, we omit all the proofs of the forthcoming technical results that can be found in the journal version of this paper under preparation.

Consider the change of variables $\eta \mapsto \chi_\eta = \eta - Ge$, $\xi \mapsto \chi_\xi = \xi - Ge$ and $\zeta \mapsto e_z = \zeta - \tilde{z}$. In the new coordinates, the closed-loop system is a hybrid system that flows according to the dynamics

$$\begin{array}{rcl} \dot\tau &=& 1\,, \qquad \dot\sigma = 1\,, \qquad \dot w = Sw \\ \dot{\tilde z} &=& A_{11}\tilde z + A_{12} e \\ \dot e_z &=& A_{11} e_z \\ \dot\chi_\eta &=& F\chi_\eta - G\bar P_2 w - GA_{21}\tilde z - GA_{22} e \\ \dot\chi_\xi &=& F\chi_\xi - G\bar P_2 w + GA_{21} e_z \\ \dot W &=& 0 \\ \dot p &=& 0 \\ \dot e &=& A_{21}\tilde z + A_{22} e + \bar P_2 w + \Gamma(\sigma+p)\chi_\eta + v \end{array} \tag{10}$$

when $((\tau,\sigma),w,\tilde z,e_z,\chi_\eta,\chi_\xi,W,p,e) \in [0,\tau_{\max}]^2\times\mathcal{W}\times\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}\times\mathbb{R}^\ell\times\mathbb{R}^\ell\times\mathcal{P}_0\times\mathbb{R}\times\mathbb{R}$, and that jumps according to the rules

$$\begin{array}{rcl} \tau^+ &=& 0\,, \qquad \sigma^+ = \sigma\,, \quad w^+ = Jw \\ \tilde z^+ &=& \tilde z\,, \quad e_z^+ = e_z \\ {\chi_\eta}^+ &=& \chi_\eta\,, \quad {\chi_\xi}^+ = \chi_\xi\,, \quad W^+ = W\,, \quad p^+ = p \\ e^+ &=& e \end{array} \tag{11}$$

when $(\tau,\sigma,w,\tilde z,e_z,\chi_\eta,\chi_\xi,W,p,e) \in \{\tau_{\max}\}\times[0,\tau_{\max}]\times\mathcal{W}\times\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}\times\mathbb{R}^\ell\times\mathbb{R}^\ell\times\mathcal{P}_0\times\mathbb{R}\times\mathbb{R}$ and

$$\begin{array}{rcl} \tau^+ &=& \tau\,, \quad \sigma^+ = 0\,, \quad w^+ = w \\ \tilde z^+ &=& \tilde z\,, \quad e_z^+ = e_z \\ {\chi_\eta}^+ &=& \chi_\eta\,, \quad {\chi_\xi}^+ = \chi_\xi \\ W^+ &=& \lambda W + \chi_\xi\,\chi_\xi^T \\ p^+ &\in& \arg\min_{\hat p\in[0,\tau_{\max}]} \|\Gamma_{\rm f}(\sigma+\hat p)\,W\,\Gamma_{\rm f}(\sigma+\hat p)^T\| \\ e^+ &=& e \end{array} \tag{12}$$

$(\tau,\sigma,w,\tilde z,e_z,\chi_\eta,\chi_\xi,W,p,e) \in [0,\tau_{\max}]\times\{\tau_{\max}\}\times\mathcal{W}\times\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}\times\mathbb{R}^\ell\times\mathbb{R}^\ell\times\mathcal{P}_0\times\mathbb{R}\times\mathbb{R}$. This hybrid system is in normal form with unitary relative degree between the input v and the output e. Motivated by the results in [15], [16], we start studying the *zero dynamics* of the system that are given by a hybrid system flowing according to the dynamics

$$\begin{array}{rcl} \dot\tau &=& 1\,, \quad \dot\sigma = 1\,, \quad \dot w = Sw \\ \dot{\tilde z} &=& A_{11}\tilde z\,, \dot e_z = A_{11} e_z \\ \dot\chi_\eta &=& F\chi_\eta - G\bar P_2 w - GA_{21}\tilde z \\ \dot\chi_\xi &=& F\chi_\xi - G\bar P_2 w + GA_{21} e_z \\ \dot W &=& 0\,, \qquad \dot p = 0 \end{array} \tag{13}$$

when $((\tau,\sigma),w,\tilde z,e_z,\chi_\eta,\chi_\xi,W,p) \in [0,\tau_{\max}]^2\times\mathcal{W}\times\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}\times\mathbb{R}^\ell\times\mathbb{R}^\ell\times\mathcal{P}_0\times\mathbb{R}$, and jumping according to the rules

$$\begin{array}{rcl} \tau^+ &=& 0\,, \quad \sigma^+ = \sigma\,, \quad w^+ = Jw \\ \tilde z^+ &=& \tilde z\,, \quad e_z^+ = e_z \\ {\chi_\eta}^+ &=& \chi_\eta\,, \quad {\chi_\xi}^+ = \chi_\xi\,, \quad W^+ = W\,, \quad p^+ = p \end{array} \tag{14}$$

when $(\tau,\sigma,w,\tilde z,e_z,\chi_\eta,\chi_\xi,W,p) \in \{\tau_{\max}\}\times[0,\tau_{\max}]\times\mathcal{W}\times\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}\times\mathbb{R}^\ell\times\mathbb{R}^\ell\times\mathcal{P}_0\times\mathbb{R}$ and

$$\begin{array}{rcl} \tau^+ &=& \tau\,, \quad \sigma^+ = 0\,, \quad w^+ = w \\ \tilde z^+ &=& \tilde z\,, \quad e_z^+ = e_z\,, \quad {\chi_\eta}^+ = \chi_\eta\,, \quad {\chi_\xi}^+ = \chi_\xi \\ W^+ &=& \lambda W + \chi_\xi\,\chi_\xi^T \\ p^+ &\in& \arg\min_{\hat p\in[0,\tau_{\max}]} \|\Gamma_{\rm f}(\sigma+\hat p)\,W\,\Gamma_{\rm f}(\sigma+\hat p)^T\| \end{array} \tag{15}$$

$(\tau,\sigma,w,\tilde z,e_z,\chi_\eta,\chi_\xi,W,p) \in [0,\tau_{\max}]\times\{\tau_{\max}\}\times\mathcal{W}\times\mathbb{R}^{n-1}\times\mathbb{R}^{n-1}\times\mathbb{R}^\ell\times\mathbb{R}^\ell\times\mathcal{P}_0\times\mathbb{R}$.

The zero dynamics has a nice cascade structure that is employed in the next part of the section in which the asymptotic properties of (13)-(15) are analyzed. With the theoretical support of Corollary 19 in [4], the asymptotic properties of a specific system of the cascade are studied by considering the driving system in steady-state. According to this, by bearing in mind Assumption 1, we consider $\tilde z = 0$ and $e_z = 0$ we start studying the hybrid system with flow dynamics

$$\begin{array}{rcl} \dot\tau &=& 1\,, \quad \dot\sigma = 1\,, \quad \dot w = Sw \\ \dot\chi_\eta &=& F\chi_\eta - G\bar P_2 w\,, \quad \dot\chi_\xi = F\chi_\xi - G\bar P_2 w \end{array} \tag{16}$$

governing the system when $((\tau,\sigma),w,\chi_\eta,\chi_\xi) \in [0,\tau_{\max}]^2\times\mathcal{W}\times\mathbb{R}^\ell\times\mathbb{R}^\ell$, and jump rule given by

$$\begin{array}{rcl} \tau^+ &=& 0\,, \quad \sigma^+ = \sigma\,, \quad w^+ = Jw \\ {\chi_\eta}^+ &=& \chi_\eta\,, \quad {\chi_\xi}^+ = \chi_\xi\,, \end{array} \tag{17}$$

when $(\tau,\sigma,w,\chi_\eta,\chi_\xi) \in \{\tau_{\max}\}\times[0,\tau_{\max}]\times\mathcal{W}\times\mathbb{R}^{n-1}\times\mathbb{R}^\ell\times\mathbb{R}^\ell$ and

$$\begin{array}{rcl} \tau^+ &=& \tau\,, \quad \sigma^+ = 0\,, \quad w^+ = w \\ {\chi_\eta}^+ &=& \chi_\eta\,, \quad {\chi_\xi}^+ = \chi_\xi \end{array} \tag{18}$$

when $(\tau,\sigma,w,\chi_\eta,\chi_\xi) \in [0,\tau_{\max}]\times\{\tau_{\max}\}\times\mathcal{W}\times\mathbb{R}^\ell\times\mathbb{R}^\ell$. The asymptotic properties of this system are detailed in the next proposition that follows immediately from (8) and from the definition of F.

Proposition 2: The set

$$\{((\tau,\sigma),w,\chi_\eta,\chi_\xi) \in [0,\tau_{\max}]^2\times\mathcal{W}\times\mathbb{R}^\ell\times\mathbb{R}^\ell \;:\; \chi_\eta = \Pi(\tau)w\,,\ \chi_\xi = \Pi(\tau)w\}$$

is globally asymptotically stable for (16)-(18).

We proceed by studying the W dynamics specialized with $\tilde z = 0$, $\chi_\xi = \Pi(\tau)w$, namely the hybrid system governed by the flow dynamics

$$\dot\tau = 1\,, \quad \dot\sigma = 1\,, \quad \dot w = Sw\,, \quad \dot W = 0 \tag{19}$$

when $((\tau,\sigma),w,W) \in [0,\tau_{\max}]^2\times\mathcal{W}\times\mathcal{P}_0$, and jump rules given by

$$\tau^+ = 0\,, \quad \sigma^+ = \sigma\,, \quad w^+ = Jw\,, \quad W^+ = W \tag{20}$$

when $(\tau, \sigma, w, W) \in \{\tau_{\max}\} \times [0, \tau_{\max}] \times \mathcal{W} \times \mathcal{P}_0$ and

$$\begin{array}{l} \tau^+ = \tau\,, \quad \sigma^+ = 0\,, \quad w^+ = w \\ W^+ = \lambda W + \Pi(\tau)\, w\, (\Pi(\tau)\, w)^T \end{array} \tag{21}$$

when $(\tau, \sigma, w, W) \in [0, \tau_{\max}] \times \{\tau_{\max}\} \times \mathcal{W} \times \mathcal{P}_0$. In the study of this system it turns out convenient to introduce an additional hybrid dynamics obtained by "sampling and holding" the exogenous dynamics at the times at which σ switches. Specifically, let (t_j, j) be a pair in the solution domain such that $\sigma(t_j, j) = \tau_{\max}$ and $\tau(t_j, j) \in (0, \tau_{\max}]$ and let[1] $\tau_s := \tau(t_j, j)$. Furthermore, let

$$J_s = \exp(S\, \tau_s)\; J\, \exp(S\, (\tau_{\max} - \tau_s)) \tag{22}$$

and consider the system

$$\begin{array}{lll} \dot{\sigma} = 1\,, \;\; \dot{w}_s = 0 & & (\sigma, w_s) \in [0, \tau_{\max}] \times \mathcal{W} \\ \sigma^+ = 0\,, \;\; w_s^+ = J_s w_s & & (\sigma, w_s) \in \{\tau_{\max}\} \times \mathcal{W} \end{array} \tag{23}$$

with initial condition that is set to

$$w_s(0,0) = \exp(S(\tau_{\max} - \sigma(0,0)))w(0,0)$$

if $\sigma(0,0) \geq \tau(0,0)$ and

$$w_s(0,0) = \exp(S(\tau(0,0) - \sigma(0,0)))\; J \cdot \\ \cdot \exp(S(\tau_{\max} - \tau(0,0)))w(0,0)$$

otherwise. By letting (t_j, j) such that $\sigma(t_j, j) = \tau_{\max}$ and $\tau(t_j, j) \in (0, \tau_{\max}]$, namely a time at which a switch of σ, and possibly of τ, are going to occur, the previous initialization of w_s guarantees that $w_s(t_j, j) = w(t_j, j)$.

Then the following proposition holds.

Proposition 3: Let

$$H(w_s) = \sum_{k=-\infty}^{0} \lambda^{-k} (J_s^{k-1}\, w_s)\, (J_s^{k-1}\, w_s)^T\,.$$

The set

$$\{((\tau, \sigma), (w, w_s), W) \in [0, \tau_{\max}]^2 \times \mathcal{W}^2 \times \mathcal{P}_0 \quad : \\ W = \Pi(\tau_s) H(w_s) \Pi(\tau_s)^T\}$$

is globally asymptotically stable for (19)-(21), (23).

As last step in the study of the zero dynamics, we analyze the asymptotic behavior of p by setting $W = \Pi(\tau_s) H(w_s) \Pi(\tau_s)^T$. Specifically, we focus on the hybrid system having flow dynamics

$$\dot{\sigma} = 1\,, \quad \dot{w}_s = 0\,, \quad \dot{p} = 0 \tag{24}$$

taking place when $(\sigma, w_s, p) \in [0, \tau_{\max}] \times \mathcal{W} \times \mathbb{R}$ and jumping according to the rule

$$\begin{array}{rcl} \sigma^+ & = & 0\,, \quad w_s^+ = J_s w_s \\ p^+ & \in & \arg \min\limits_{\hat{p} \in [0, \tau_{\max}]} \\ & & \|\Gamma_f(\sigma + \hat{p})\, \Pi(\tau_s) H(w_s) \Pi(\tau_s)^T\, \Gamma_f(\sigma + \hat{p})^T\| \end{array} \tag{25}$$

when $(\sigma, w_s, p) \in \{\tau_{\max}\} \times \mathcal{W} \times \mathbb{R}$.

[1] Namely, τ_s coincides with the value of τ at the time in which σ switches if σ and τ are not synchronized, otherwise it is set equal to $\tau_{\max}$.

For this system the following proposition holds.

Proposition 4: The set

$$\mathcal{P} = \{p \in \mathbb{R} \;:\; \Gamma_f(\tau_{\max} + p)\Pi(\tau_s) = 0\}$$

is not empty. As a consequence, under Assumption 6, the set $[0, \tau_{\max}] \times \mathcal{W} \times \mathcal{P}$ is globally asymptotically stable for (24)-(25).

Overall, the previous arguments have shown that the set

$$\mathcal{A} := \{((\tau, \sigma), (w, w_s), (\tilde{z}, e_z), (\chi_\eta, \chi_\xi), W, p) \in \\ [0, \tau_{\max}]^2 \times \mathcal{W}^2 \times \mathbb{R}^{2(n-1)} \times \mathbb{R}^{2\ell} \times \mathcal{P}_0 \times \mathcal{P} \qquad \text{such that} \\ (\tilde{z}, e_z) = 0\,, \chi_\eta = \chi_\xi = \Pi(\tau)w\,, W = \Pi(\tau_s)H(w_s)\Pi(\tau_s)^T\}$$

is globally asymptotically stable for the (13)-(15), (23).

We study now the interconnection term between the zero dynamics and the error dynamics in (10), namely the function

$$\gamma(\sigma, w, \tilde{z}, \chi_\eta, p) = A_{21}\tilde{z} + A_{22}e + \bar{P}_2 w + \Gamma(\sigma + p)\chi_\eta\,.$$

Our goal is to show that, along trajectories taking place in $\mathcal{A}$, the function γ is identically zero, possibly with the exception of a finite number of nonequilibrium points in which it is not vanishing.

To this end, pick an initial condition of (13)-(15), (23) in $\mathcal{A}$ and note that, on the resulting trajectory,

$$\gamma(\sigma(t,j), w(t,j), \tilde{z}(t,j), \chi_\eta(t,j), p(t,j)) = \\ \bar{P}_2 w(t,j) + \Gamma(\sigma(t,j) + p(t,j))\Pi(\tau(t,j))w(t,j)$$

with $p(t,j) \in \mathcal{P}$ for all (t,j).

In the following proposition we denote by $\mathcal{H} \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ the hybrid time domain of system (13)-(15), (23) associated to the selected initial condition.

Proposition 5: There exists a set $\mathcal{S} \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$ composed of isolated points, such that

$$\Gamma(\sigma(t,j) + p(t,j))\Pi(\tau(t,j))w(t,j) = \\ \Gamma(\tau(t,j))\Pi(\tau(t,j))w(t,j) \qquad \forall\, (t,j) \in \mathcal{H} \setminus \mathcal{S}\,. \tag{26}$$

By taking advantage from the previous proposition and from the fact that, under Assumption 4 and $\ell \geq s$, $\bar{P}_2 w = \bar{P}_2 \Pi(\tau)^\dagger \Pi(\tau) w$ for all $\tau \in [0, \tau_{\max}]$ (see Proposition 4 in [13]), it turns out that

$$\begin{array}{l} \bar{P}_2 w + \Gamma(\sigma(t,j) + p(t,j))\Pi(\tau(t,j))w(t,j) \\ = \bar{P}_2 w + \Gamma(\tau(t,j))\Pi(\tau(t,j))w(t,j) \\ = \bar{P}_2 w - \bar{P}_2 \Pi(\tau(t,j))^\dagger \Pi(\tau(t,j))w(t,j) = 0 \end{array}$$

for all $(t,j) \in \mathcal{H} \setminus \mathcal{S}$.

This fact allows us to cast the stabilization problem in the framework of [16].

Specifically, we let $x_0 = \mathrm{col}(w, w_s, e_z, \chi_\xi, p, W)$, $x_1 = \mathrm{col}(\tilde{z}, \chi_\eta)$, and rewrite the error dynamics in (10) as

$$\begin{array}{rcl} \dot{e} & = & A_{21}\tilde{z} + A_{22}e + \bar{P}_2 w + \Gamma(\sigma + p)\chi_\eta + v \\ & = & F_2(\tau, \sigma, x_0, x_1, e) + F_3(\tau, \sigma, x_0) + v \end{array}$$

with

$$F_2 \;=\; A_{21}\tilde{z} + A_{22}e + \bar{P}_2 w + \Gamma(\sigma+p)\chi_\eta - \big(\bar{P}_2 w + \Gamma(\sigma+p)\Pi(\tau)w\big)$$

and $F_3 = \bar{P}_2 w + \Gamma(\sigma+p)\Pi(\tau)w$.

In this way, system (10)-(12) and (23) fits in the framework of [16] by which the claim of Proposition 1 follows.

V. Simulation results

We consider the linear hybrid system in $\mathbb{R}^3$ that flows according to the dynamics

$$\dot{\tau} = 1\,, \qquad \dot{z} = -z + e\,, \qquad \dot{e} = z + e + u + w_1$$

and regularly jumps according to the jump rule $\tau^+ = 0$, $z^+ = z + e$ and $e^+ = e$ whenever τ gets the value $\tau_{\max} = 2$. The variable w_1 denotes a matched exogenous disturbance generated by a third order exosystem that is constant during flows, i.e. $S = 0$, and jumps according to the "shift rule"

$$w_1^+ = w_2\,, \quad w_2^+ = w_3\,, \quad w_3^+ = w_1$$

whenever the system jumps. It is easy to check that the system fulfills the minimum-phase Assumption 1 and Assumptions 2 and 3 with $T = 0$. The regulator in Section III has been tuned with

$$F = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -8 & -20 & -16 & 1 \\ 0 & 0 & 0 & -10 \end{pmatrix} \quad G = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix},$$

$\lambda = 0.001$ and $\kappa = 80$. By simulation, it is possible to verify that Assumptions 4 and 5 are also fulfilled. The initial condition of the exosystem have been taken as $w(0) = (1, -0.5, 2)$ so that the persistence of excitation condition in Assumption 6 is also fulfilled for some positive α. Simulation results are shown in Figures 1-2. In particular Figure 1 plots the disturbance w_1 against the control input u (with negative sign) while Figure 2 plots the regulation error e and the phase offset estimation parameter p against the initial condition of the clock variable, τ.

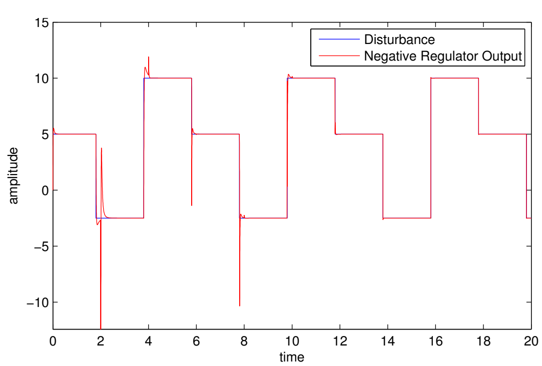

Fig. 1. Plot of the matched disturbance w_1 and of $-u$.

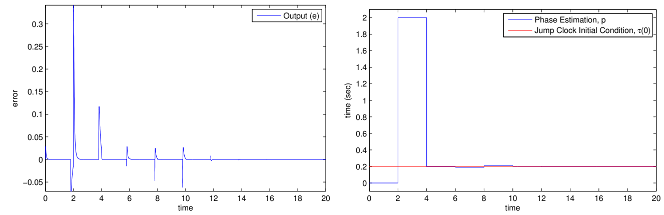

Fig. 2. Plot of the regulation error e (left), and of the phase estimation p (right).

VI. Conclusions

We considered the problem of hybrid output regulation for a class of linear systems. With respect to our earlier work on the subject, here we addressed the case in which the jump of the exosystem and of the plant are triggered by a clock that is not accessible for feedback. In this context we proposed a phase clock estimator that, joined to an internal model based regulator of the form proposed in [13], guarantee the fulfillment of the regulation objective. Future works on the subject are mainly directed to improve the solution in presence of parametric uncertainties characterizing the system and exosystem.

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