

Decentralised minimal-time consensus

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Abstract—This study considers the discrete-time dynamics of a network of agents that exchange information according to the nearest-neighbour protocol under which all agents are guaranteed to reach consensus asymptotically. We present a fully decentralised algorithm that allows any agent to compute the consensus value of the whole network in finite time using only the minimal number of successive values of its own history. We show that this minimal number of steps is related to a Jordan block decomposition of the network dynamics and present an algorithm to obtain the minimal number of steps in question by checking a rank condition on a Hankel matrix of the local observations. Furthermore, we prove that the minimal number of steps is related to other algebraic and graph theoretical notions that can be directly computed from the Laplacian matrix of the graph and from the underlying graph topology.

I. INTRODUCTION

Fuelled by applications in a variety of fields, there has been a recent surge of interest in consensus dynamics. In its most basic formulation, the consensus problem studies the linear discrete-time dynamics of a network of agents that exchange information according to the nearest-neighbour averaging rule. The consensus problem has broad implications beyond the analysis and design of collective behaviour in multi-agent systems. Various applications can be cast in this framework, including swarming and flocking [1], [2], distributed computing [3], agreement in social networks [4], [5] or synchronisation of coupled oscillators [6], [7], [8].

The design of efficient distributed consensus algorithms is a current focus of active research in the Control literature. Under broad assumptions, well-known results [9], [10], [11] give conditions to ensure that the state of each agent reaches the consensus value *asymptotically*. From a practical point of view, however, requiring an infinite or arbitrarily long time to obtain the final consensus value of the system is unsatisfactory. A finite-time protocol is designed in [12]. The principles for the computation of the asymptotic final value of the network in finite time were introduced in [13]. In [14], we extended those results and studied the *minimal* number of discrete-time steps required by an arbitrarily chosen agent to compute the asymptotic final value of the network without any prior knowledge of the system dynamics. Importantly, the information used for that purpose was solely based on the accumulation of the successive state values of the agent under consideration, and, consequently, the corresponding algorithm was truly decentralised.

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This paper presents a characterisation of our results for decentralised minimal-time consensus. Firstly, we introduce an algorithm that allows any agent in a consensus-guaranteed network to compute the consensus value using one less step than in [14]. This algorithm relies on the analysis of the rank of a Hankel matrix constructed from local observations at any chosen node. Furthermore, we show that the minimal number of steps is linked to a global property of the network: the degree of a specific matrix polynomial. This provides us with an algebraic characterisation of the local convergence to consensus in terms of properties of the Laplacian matrix of the graph. Finally, we show that the minimal number of steps required to compute the consensus value from local observations of any chosen node can also be characterised in terms of a combinatorial graph theoretical property: the minimal external equitable partition of the graph with respect to that node. Throughout the paper we illustrate our results with relevant examples to highlight how our framework can establish a link between the spectral and graph theoretical properties of a network of interacting agents and the minimal-time solution of distributed decision-making problems.

Notation: The notation in this paper is standard. For a matrix $A \in \mathbb{R}^{M \times N}$, $A[i, j] \in \mathbb{R}$ denotes the element in the i^{th} row and j^{th} column, $A[i, :] \in \mathbb{R}^{1 \times N}$ denotes its i^{th} row, $A[:, j] \in \mathbb{R}^{M \times 1}$ denotes its j^{th} column, and $A[i_1 : i_2, j_1 : j_2]$ denotes the submatrix of A defined by the rows i_1 to i_2 and the columns j_1 to j_2 . For a column vector $\alpha \in \mathbb{R}^{N \times 1}$, $\alpha[i]$ denotes its i^{th} element. We denote by $e_r^T = [0, \dots, 0, 1, \dots, 0] \in \mathbb{R}^{1 \times N}$. Furthermore, I_N denotes the identity matrix of dimension N .

II. CONSENSUS DYNAMICS: FORMULATION AND PREVIOUS RESULTS

A. Formulation of the problem

Consider a directed unweighted graph denoted by $\mathcal{G} = (\mathcal{V}, \mathcal{E}, W)$, where $\mathcal{V} = \{\nu_1, \dots, \nu_n\}$ is the set of nodes, $\mathcal{E} \subset \mathcal{V} \times \mathcal{V}$ is the set of edges, and $W = \{W[i, j]\}_{i, j=1, \dots, n}$ is the corresponding n by n adjacency matrix, with $W[i, j] = 1$ when there is a link from j to i , and $W[i, j] = 0$ when there is no link from j to i . Let $x[i] \in \mathbb{R}$ denote the state of node i , which might represent a physical quantity such as altitude, position, temperature, voltage, etc. The classical consensus problem on a network of continuous-time integrator agents is defined by the following dynamics [10]:

$$\dot{x}(t) = -Lx(t),$$

where $L \in \mathbb{R}^{n \times n}$ is the Laplacian matrix induced by the topology \mathcal{G} . L is defined as $L[i, i] = \sum_{l \neq i} W[i, l]$, $\forall i = 1, \dots, n$ and $L[i, j] = -W[i, j]$, $\forall i \neq j$.

Here, we consider the associated discrete-time consensus dynamics on a network:

$$\begin{aligned} x_{k+1} &= (I_n - \epsilon L) x_k \triangleq A x_k \\ y_k &= e_r^T x_k = x_k[r], \end{aligned} \quad (1)$$

where $x_k \in \mathbb{R}^n$ and ϵ is the sampling time. Without loss of generality, we concentrate on the case where the measurable output $y_k \in \mathbb{R}$ corresponds to the local state of an arbitrarily chosen agent labelled r .

B. Global asymptotic convergence to distributed consensus (see [9], [10]):

Let $d_{\max} = \max_i L[i, i]$ denote the maximal node out-degree of the graph \mathcal{G} . If the network has a rooted directed spanning tree (or is connected in the case of an undirected graph) over time, and the sampling time ϵ is such that $0 < \epsilon < 1/d_{\max}$, then the discrete-time version of the classical consensus protocol given in (1) ensures global asymptotic convergence to consensus in the sense that

$$\lim_{k \rightarrow \infty} x_k = (c^T x_0) \mathbf{1}_{n \times 1}$$

where $\mathbf{1}_{n \times 1}$ is a column vector with all components equal to 1, and c^T is a constant row vector. In other words, the values of all nodes converge asymptotically to the same linear combination of the initial node values x_0 .

Algebraic characterisation of distributed asymptotic consensus [15]: When $c^T \mathbf{1} = 1$, the iteration given by (1) achieves distributed consensus if and only if:

- A.1 A has a simple eigenvalue at 1, and all other eigenvalues have a magnitude strictly less than 1.
- A.2 The left and right eigenvectors of A corresponding to the eigenvalue 1 are c^T and $\mathbf{1}$, respectively.

C. Finite-time computation of the final consensus value [13]

Recent work by Sundaram and Hadjicostis [13] showed that it is possible to obtain the final value of the consensus dynamics in a finite number of steps. Their result hinges on the use of the minimal polynomial associated with the consensus dynamics (1) in conjunction with the final value theorem.

Definition 1 (Minimal polynomial of a matrix): The minimal polynomial of matrix $A \in \mathbb{R}^{n \times n}$ is the unique monic polynomial $q(t) \triangleq t^{D+1} + \sum_{i=0}^D \alpha_i t^i$ with minimal degree $D+1$ that satisfies $q(A) = 0$.

Given the explicit solution of the linear system in (1) with initial state x_0 , it follows from the definition of the minimal polynomial that the dynamics in (1) satisfies the linear regression equation:

$$x_{k+D+1} + \alpha_D x_{k+D} + \dots + \alpha_1 x_{k+1} + \alpha_0 x_k = 0, \quad \forall k \in \mathbb{N}. \quad (2)$$

Similarly, the regression equation for $y_k = x_k[r]$, the measurable output at node r , is determined by the minimal polynomial of the corresponding matrix observability pair $[A, e_r^T]$.

Definition 2 (Minimal polynomial of a matrix pair): The minimal polynomial associated with the matrix pair $[A, e_r^T]$ denoted by $q_r(t) \triangleq t^{D_r+1} + \sum_{i=0}^{D_r} \alpha_i^{(r)} t^i$ is the unique

monic polynomial of minimal degree $D_r + 1 \leq D + 1$ that satisfies $e_r^T q_r(A) = 0$.

Again, it is straightforward to show that:

$$y_{k+D_r+1} + \alpha_{D_r}^{(r)} y_{k+D_r} + \dots + \alpha_1^{(r)} y_{k+1} + \alpha_0^{(r)} y_k = 0, \quad \forall k \in \mathbb{N}. \quad (3)$$

Therefore each node r will be associated with a particular length of the regression ($D_r + 1$) which is upper bounded by the degree of the minimal polynomial of the dynamical matrix A .

Consider now the Z-transform of y_k ¹:

$$Y(z) = \frac{\sum_{i=1}^{D_r+1} \alpha_i^{(r)} \sum_{\ell=0}^{i-1} y_\ell z^{i-\ell}}{q_r(z)} \triangleq \frac{H(z)}{q_r(z)}. \quad (4)$$

Under the assumptions specified in Section II-B, the minimal polynomial $q_r(t)$ does not possess any unstable root except for one single root located at 1. We can then define the following polynomial:

$$p_r(z) \triangleq \frac{q_r(z)}{z-1} \triangleq \sum_{i=0}^{D_r} \beta_i z^i. \quad (5)$$

The application of the final value theorem [16] then gives the consensus value

$$\phi = \lim_{z \rightarrow 1} (z-1)Y(z) = \frac{H(1)}{p_r(1)} = \frac{y_{D_r}^T \beta}{\mathbf{1}^T \beta} \quad (6)$$

where $y_{D_r}^T = [y_0 \ y_1 \ \dots \ y_{D_r}]$ and $\beta_{(D_r+1) \times 1}$ is the vector of coefficients of the polynomial $p_r(z)$ defined in eq. (5).

Based on these results, an algorithm to obtain the consensus value was proposed in [13]. The proposed algorithm was distributed but not entirely local, in the sense that a local calculation is repeated over n independent iterations (where n is the total number of nodes of the network) and at each iteration, it requires each node to store its own values for $n+1$ steps. Hence, a total of $n(n+1)$ successive values of $x[r]$ are required for the calculation of ϕ .

D. Minimal-time, decentralised computation of the final consensus value

The main purpose of this paper is to characterise the computation in *minimal time* of the final consensus value ϕ using only the output observations $y_k = x_k[r]$ of the node r alone. We formalise and improve here our previous results [14] and show that, for a general arbitrary initial condition, except for a set of initial conditions with Lebesgue measure zero [18], the consensus value can be obtained from local observations in a minimal number of steps that does not depend explicitly on the total size of the graph. In our framework, the minimal number of steps is computed in a truly decentralised manner by checking a rank condition of a Hankel matrix constructed exclusively from local output observations. We also provide a graph theoretical characterisation of this local property in terms of the minimal external equitable partition of the graph. This characterisation provides insight into which properties

¹This follows from the time-shift property of the Z-transform: $\mathcal{Z}(x_{k+n}) = z^n X(z) - \sum_{l=0}^{n-1} z^{n-l} x_l$ where $X(z) = \mathcal{Z}(x_k)$.

of the graph contribute to the disparity in the ability of the different nodes to compute the global consensus value from local information.

III. MINIMAL TIME CONSENSUS AND THE JORDAN BLOCK DECOMPOSITION OF THE CONSENSUS DYNAMICS

Given the linear system in (1) and an initial state x_0 , it follows from above that there always exist scalars $d \triangleq d(r, x_0) \in \mathbb{N}$ and $a_0, \dots, a_d \in \mathbb{R}$ such that the following linear regression equation is satisfied $\forall k \in \mathbb{N}$

$$x_{k+d+1}[r] + a_d x_{k+d}[r] + \dots + a_1 x_{k+1}[r] + a_0 x_k[r] = 0. \quad (7)$$

From the definitions above, it is clear that $D_r + 1$ is the *minimal length of recursion*:

$$D_r + 1 = \min_{d \in \mathbb{N}} \max_{x_0 \in \mathbb{R}^n} \{d(r, x_0) + 1: \text{eq. (7) holds } \forall k\}.$$

Remark 1: Among the many recursions of length d that are not necessarily minimal, $(D_r + 1)$ appears as a min-max over the space of (d, x_0) . When $d + 1 = D_r + 1$, the coefficients a_i in (7) correspond to $\alpha_i^{(r)}$, the coefficients of the minimal polynomial of the matrix pair $[A, e_r^T]$ in (3).

In this section, we give an algebraic characterisation of the minimal number of steps $D_r + 1$ based on the projection of the Jordan block decomposition of A^k on e_r^T . Our aim is to obtain the coefficients $\alpha_i^{(r)}$ in (3) from data, so that we can compute future outputs recursively. Consider the standard Jordan decomposition:

$$A = SJS^{-1} \quad \text{where} \quad (8)$$

$$S = [s_1 \ s_2 \ \dots \ s_n] \quad (9)$$

$$J = \text{diag} \{J_1(\lambda_1), J_2(\lambda_2), \dots, J_l(\lambda_l)\} \quad (10)$$

where

$$J_i(\lambda_i) = \begin{bmatrix} \lambda_i & 1 & & & \\ & \lambda_i & 1 & & \\ & & \ddots & \ddots & \\ & & & \lambda_i & 1 \\ & & & & \lambda_i \end{bmatrix}_{n_i \times n_i}, \quad (11)$$

and s_i , the columns of the non singular matrix S , are the generalised eigenvectors of A [22]. The matrix A has l (possibly degenerate) eigenvalues λ_i , each of them associated with a Jordan block of size n_i , such that $\sum_{i=1}^l n_i = n$. Without loss of generality, we assume that the blocks are ordered according to decreasing size: $n_1 \geq n_2 \geq \dots \geq n_l$.

Using eq. (8), the linear dynamics (1) can be rewritten as follows:

$$x_k[r] = e_r^T A^k x_0 = (e_r^T S) J^k (S^{-1} x_0) \triangleq \sigma^T J^k \chi, \quad (12)$$

where the vectors

$$\sigma^T = [\sigma_1^T \ \sigma_2^T \ \dots \ \sigma_l^T]_{1 \times n} \quad (13)$$

$$\chi^T = [\chi_1^T \ \chi_2^T \ \dots \ \chi_l^T]_{1 \times n} \quad (14)$$

are partitioned according to the Jordan blocks in (8), e.g., $\sigma_1^T = [\sigma_{11} \ \dots \ \sigma_{1n_1}]$ and $\chi_1^T = [\chi_{11} \ \dots \ \chi_{1n_1}]$. Here,

$$J^k = \text{diag} \{J_1^k(\lambda_1), J_2^k(\lambda_2), \dots, J_l^k(\lambda_l)\}$$

has the well known structure [17]:

$$J_i^k(\lambda_i) = \sum_{m=0}^{k-1} \binom{k}{m} \lambda_i^{k-m} J_i^m(0), \quad (15)$$

where $J_i^m(0)$ is the m -th power of a Jordan block, as defined in (11).

The output dynamics (12) then becomes:

$$x_k[r] = \sum_{i=1}^l \sum_{m=0}^{k-1} \binom{k}{m} \lambda_i^{k-m} [\sigma_i^T J_i^m(0) \chi_i]. \quad (16)$$

Note that, because of its Jordan block structure, the matrix $J_i^m(0)$ induces a strict m -shift on the vector χ_i for $m \leq n_i$. Therefore, if $k \geq \max_i \{n_i\}$, we have:

$$x_k[r] = \sum_{i=1}^l \sum_{m=0}^{n_i-1} \binom{k}{m} \lambda_i^{k-m} \left[\sum_{j=1}^{n_i-m} \sigma_{ij} \chi_{ij+m} \right] \quad (17)$$

$$\triangleq \sum_{i=1}^l \sum_{m=0}^{n_i-1} \binom{k}{m} \lambda_i^{k-m} g_{im} \quad (18)$$

However, some of the g_{im} might be zero (we might even have situations where all the coefficients associated with a particular eigenvalue are zero) so that the dynamics of node r can be written as:

$$x_k[r] = \sum_{i=1}^{l_r} \sum_{m=0}^{n_i^r-1} \binom{k}{m} \lambda_i^{k-m} g_{im} \quad (19)$$

where $n_i^r \leq n_i$ and $l_r \leq l$. Here, $\{\lambda_1, \dots, \lambda_{l_r}\}$ is an ordered subset of *distinct* eigenvalues from the original Jordan block decomposition. As a consequence, the degree of the characteristic polynomial that underlies the length of the recursion for node r is:

$$\sum_{i=1}^{l_r} n_i^r = D_r + 1.$$

Eq. (19) can be rewritten as a dot product:

$$x_k[r] = \mathbf{v}_r(k)^T \mathbf{g}_r \triangleq [v_1^T(k) \ v_2^T(k) \ \dots \ v_{l_r}^T(k)] \begin{bmatrix} g_1 \\ g_2 \\ \vdots \\ g_{l_r} \end{bmatrix}$$

where

$$v_i^T(k) \triangleq \left[\binom{k}{0} \lambda_i^k \ \binom{k}{1} \lambda_i^{k-1} \ \dots \ \binom{k}{n_i^r-1} \lambda_i^{k-n_i^r+1} \right]_{1 \times n_i^r}$$

$$g_i^T \triangleq [g_{i0} \ \dots \ g_{i(n_i^r-1)}].$$

Based upon the decomposition of confluent Vandermonde matrices introduced in [19], it is easy to see that

$$v_i^T(k) = e_i^T \mathcal{J}_i^k(\lambda_i)$$

where $\mathcal{J}_i(\lambda_i)$ is a Jordan block of size n_i^r as defined in (11) and $e_i^T = [1 \ 0 \ \dots \ 0]_{1 \times n_i^r}$ is the unit vector of the same length. The dynamics (12) can thus be rewritten in terms of a Jordan decomposition of reduced dimensionality as follows:

$$x_k[r] = \mathbf{E}_r^T \mathbf{J}_r^k \mathbf{g}_r, \quad \forall k, \quad (20)$$

where

$$\begin{aligned} \mathbf{E}_r^T &\triangleq [e_1^T \ \dots \ e_{l_r}^T]_{1 \times (D_r+1)} \quad \text{and} \\ \mathbf{J}_r &\triangleq \text{diag} \{ \mathcal{J}_1(\lambda_1), \mathcal{J}_2(\lambda_2), \dots, \mathcal{J}_{l_r}(\lambda_{l_r}) \} \end{aligned} \quad (21)$$

are partitioned according to the l_r blocks.

From the analysis above, we have the following lemma.

Lemma 1: Consider the discrete-time LTI system (1). The minimal polynomial associated with $x[r]$, as given in Definition 2, is the characteristic polynomial of the matrix \mathbf{J}_r in eq. (20) which has order $D_r + 1 = \sum_{i=1}^{l_r} n_i^r$. The final consensus value ϕ can be computed from eq. (6) based on the coefficients of the minimal polynomial of the pair $[A, e_r^T]$ and the successive values of $x[r]$.

Proof: The Jordan matrix \mathbf{J}_r in eq. (20) has the property that each of its Jordan block has distinct eigenvalues. Hence, the minimal polynomial of $[A, e_r^T]$ is the same as the characteristic polynomial of $[\mathbf{J}_r, e_r^T]$ (see [17]): $e_r^T q_r(A) = e_r^T q_r(\mathbf{J}_r)$. Therefore, the minimal polynomial possesses the following explicit form: $\det(\mathbf{J}_r - tI) = \prod_{i=1}^{l_r} (t - \lambda_i)^{n_i^r} = t^{D_r+1} + \alpha_{D_r} t^{D_r} + \dots + \alpha_1 t + \alpha_0$, and has degree $D_r + 1$. This latter relationship also shows that $D_r + 1 = \sum_{i=1}^{l_r} n_i^r$. ■

Remark 2: Lemma 1 states that instead of an n -dimensional Jordan block form J of $x_k[r]$, as in eq. (12), the general expression of $x_k[r]$ can be written in terms of a smaller $D_r + 1$ -dimensional Jordan matrix \mathbf{J}_r , as in eq. (20).

Remark 3: The minimal integer value $D_r + 1$ necessary for the recursion (7) to hold for *almost any* initial condition x_0 is given by the degree of the minimal polynomial of the observability pair $[A, e_r^T]$ (see [14]). In other words, eq. (7) holds for a randomly chosen initial state x_0 , except for a set of initial conditions of Lebesgue measure zero [18].

IV. DECENTRALISED MINIMAL-TIME CONSENSUS COMPUTATION ALGORITHM

In the decentralised problem, we assume that node r does not have access to any external information such as the total number of agents n in the network, the local communication links around node r or the state values or number of its neighbours. In [14], we showed that for the general discrete-time LTI system (1), $2D_r + 3$ successive discrete-time steps are needed by agent r to compute the final value in a fully decentralised manner. If the communication network is well-designed for consensus (i.e., Assumptions A.1 and A.2 are satisfied and asymptotic convergence to consensus is guaranteed), we hereby propose an algorithm that computes the final value using $2D_r + 2$ successive discrete-time steps, i.e., one fewer step than [14].

Problem 1 (Decentralised problem): Consider the discrete-time LTI dynamics in eq. (1) where an arbitrarily chosen state $x[r]$ is observed and assume that the conditions for consensus (Assumptions A.1 and A.2) are satisfied. The decentralised problem is to compute the asymptotic value of this state $\phi = \lim_{k \rightarrow \infty} x_k[r]$ using only its own previously observed values $y_k = x_k[r]$.

Consider the vector of successive discrete-time values at node r , $X_{0,1,\dots,2k}[r] = \{x_0[r], x_1[r], \dots, x_{2k}[r]\}$, and its

associated Hankel matrix:

$$\Gamma\{X_{0,1,\dots,2k}[r]\} \triangleq \begin{bmatrix} x_0[r] & x_1[r] & \dots & x_k[r] \\ x_1[r] & x_2[r] & \dots & x_{k+1}[r] \\ \vdots & \vdots & \ddots & \vdots \\ x_k[r] & x_{k+1}[r] & \dots & x_{2k}[r] \end{bmatrix} \quad k \in \mathbb{Z}. \quad (22)$$

We also define the vector of differences between successive values of $x[r]$:

$$\bar{X}_{0,1,\dots,2k}[r] = \{x_1[r] - x_0[r], \dots, x_{2k+1}[r] - x_{2k}[r]\}.$$

The following algorithm then allows us to compute the final consensus value in a minimal number of steps.

Algorithm 1 Decentralised minimal-time consensus value computation

Data: Successive observations of $x_i[r]$, $i = 0, 1, \dots$

Result: Final consensus value: ϕ .

Step 1: Increase the dimension k of the square Hankel matrix $\Gamma\{\bar{X}_{0,1,\dots,2k}[r]\}$ until it loses rank and store the first defective Hankel matrix.

Step 2: The kernel $\beta = [\beta_0 \ \dots \ \beta_{D_r-1} \ 1]^T$ of the first defective Hankel matrix gives the coefficients of eq. (6).

Step 3: Compute the final consensus value ϕ using eq. (6).

To understand Algorithm 1, consider a Vandermonde factorisation [19] of the Hankel matrix (22):

$$\Gamma\{X_{0,1,\dots,2k}[r]\} = V(0, k) T_r V^T(0, k), \quad (23)$$

in which we have defined the confluent Vandermonde matrix

$$V(0, k)_{(k+1) \times (D_r+1)} = \begin{bmatrix} E_r^T \\ E_r^T \mathbf{J}_r \\ \vdots \\ E_r^T \mathbf{J}_r^k \end{bmatrix}, \quad (24)$$

in terms of the elements defined in eq. (21). As shown in [19], the $(D_r + 1) \times (D_r + 1)$ block diagonal matrix

$$T_r = \text{diag}\{T_{r,1}, \dots, T_{r,l_r}\}, \quad T_{r,i} \in \mathbb{R}^{n_i^r \times n_i^r},$$

has the following symmetric upper anti-diagonal form:

$$T_{r,i} = \begin{bmatrix} * & * & * & * & t_i \\ * & * & * & t_i & \\ * & * & \ddots & & \\ * & t_i & & 0 & \\ t_i & & & & \end{bmatrix},$$

where t_i and $*$ are determined from the values of y_k .

Without loss of generality, consider $\lambda_1 = 1$ so that $T_{r,1} \in \mathbb{R}$. We then have

$$\begin{aligned} &\Gamma\{\bar{X}_{0,1,\dots,2k}[r]\} \\ &= \Gamma\{X_{1,2,\dots,2k+1}[r]\} - \Gamma\{X_{0,1,\dots,2k}[r]\} \\ &= V T_r \text{diag}\{\lambda_1, \dots, \lambda_{l_r}\} V^T - V T_r V^T \\ &= V T_r \text{diag}\{0, \lambda_2 - 1, \dots, \lambda_{l_r} - 1\} V^T \\ &= V \text{diag}\{0, (\lambda_2 - 1) T_{r,2}, \dots, (\lambda_{l_r} - 1) T_{r,l_r}\} V^T \\ &= V' \text{diag}\{(\lambda_2 - 1) T_{r,2}, \dots, (\lambda_{l_r} - 1) T_{r,l_r}\} V'^T, \end{aligned}$$

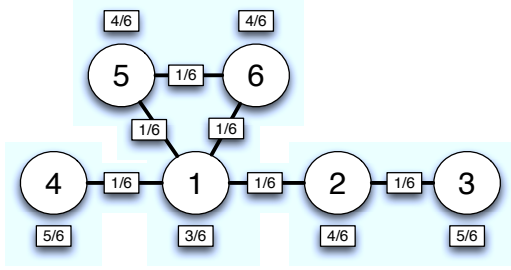


Fig. 1. Underlying topology for Example 1 with sampling time $\epsilon = 1/6$.

where $V' = V[2 : k + 1, 2 : D_r + 1]$. From the last equation, it is easy to see that for $\Gamma\{\bar{X}_{0,1,\dots,2k}[r]\}$ to be defective, one must have $k \geq D_r + 1$.

Theorem 1: Consider the system in (1) and assume that the conditions for consensus (Assumptions A.1 and A.2) are satisfied. Then the minimal number of successive discrete-time values, starting from step i , for the arbitrarily chosen state $x[r]$, is $2(D_r + 1) - \delta_r - \min\{i, \delta_r\}$, where δ_r is the number of zero roots in $q_r(t) = 0$.

Proof: Combining the above derivations and performing a proof similar to the one presented in [Corollary 1, [14]] (by taking $z_k \triangleq x_{k+1}[r] - x_k[r]$ as y_k in that Corollary) yields the result. ■

More elaborate versions of the results presented here can be obtained by modifying the model in eq. (1) so as to encompass more complex situations, e.g., time-delays in the model, noise in the observations or packet drops in the observations. Due to space constraints, we will not address them here and present them in more detail in a future paper. In the present paper, we only focus on the ideal model in eq. (1). For simplicity of exposition, we further make the following assumption in the rest of this paper:²

A.3 The matrix A in eq. (1) does not possess any eigenvalue at 0.

Under Assumption A.3, Theorem 1 establishes that the minimal number of steps for node r to compute the final consensus value is $2D_r + 2$.

Example 1: Consider the network topology in Fig. 1 under dynamics (1) with $A \triangleq I_n - \epsilon L$ and a sampling time $\epsilon = 1/6$. The topology is undirected and connected and A satisfies assumptions A.1, A.2, and A.3. Therefore the final value of each node is the average of the initial state values. For the randomly chosen initial state $x_0 = [1.3389 \ 2.0227 \ 1.9872 \ 6.0379 \ 2.7219 \ 1.9881]^T$, the final consensus value is thus 2.6828. We now apply Algorithm 1 to node $r = 1$.

Step 1: We increase the dimension k of the square Hankel matrix $\Gamma\{\bar{X}_{0,1,\dots,2k}[1]\}$ until it loses rank. This happens for

²When A has some eigenvalues at 0, the expression of the minimal number of steps for node r to compute the final consensus value takes a more complicated form, see [14].

$k = 4$. We then store the first defective Hankel matrix:

$$\Gamma\{\bar{X}_{0,1,\dots,8}[1]\} = \begin{bmatrix} 1.2358 & 0.2050 & 0.0367 & 0.0047 \\ 0.2050 & 0.0367 & 0.0047 & -0.0037 \\ 0.0367 & 0.0047 & -0.0037 & -0.0067 \\ 0.0047 & -0.0037 & -0.0067 & -0.0079 \end{bmatrix}.$$

Step 2: The normalised kernel of the first defective Hankel matrix is

$$\beta = [-0.0833 \ 0.7778 \ -1.6667 \ 1]^T.$$

This gives the coefficients of eq. (6).

Step 3: We compute the final consensus value $\phi = 2.6828$ using eq. (6).

As shown here for node $r = 1$, the value of ϕ obtained in a decentralised manner is equal to the average of the initial states.

Repeating this procedure for each of the six nodes gives the same value ϕ . However, the number of steps required by each node to compute the final consensus value ϕ differs. This is summarised in Table I.

	Ref. [13]	Our result
Node 1	$6 \times 7 = 42$	$2 \times 4 = 8$
Node 2	$6 \times 7 = 42$	$2 \times 4 = 8$
Node 3	$6 \times 7 = 42$	$2 \times 4 = 8$
Node 4	$6 \times 7 = 42$	$2 \times 5 = 10$
Node 5	$6 \times 7 = 42$	$2 \times 6 = 12$
Node 6	$6 \times 7 = 42$	$2 \times 6 = 12$

TABLE I

COMPARISON OF THE MINIMAL NUMBER OF SUCCESSIVE VALUES NEEDED BY EACH NODE TO COMPUTE THE FINAL CONSENSUS VALUE OF THE NETWORK IN FIG. 1 WITH $n = 6$ NODES.

While the method proposed in [13] requires a total of $n(n + 1)$ successive values of $x[r]$, our algorithm shows that the minimal number of successive values of $x[r]$ is just $2(D_r + 1)$ for *almost all* initial conditions. Furthermore, our algorithm is completely decentralised, i.e., our result does not require that the arbitrarily chosen state $x[r]$ has any knowledge of the total number of nodes in the network, n , or any other kind of global (centralised) information about the network (contrary to what is assumed in [13, Section V]).

As can be noticed in Table I, some nodes need fewer successive observations of their own state to compute the final consensus value of the whole network. In what follows, we call such nodes *dominant nodes*. An important question arises at this point: given a consensus-guaranteed network, can we identify the dominant nodes? Below, we answer this question based on an algebraic characterisation of the minimal number of steps which we then link to a specific graph partition of the consensus network around the chosen node.

V. CHARACTERISATION ON THE MINIMAL NUMBER OF STEPS

We now provide an answer to the question raised at the end of the last section from two perspectives. First, in Section V-A, we provide an algebraic characterisation of the minimal recursion length $D_r + 1$ for node r by performing an analysis of the Laplacian of the graph. Second, in Section V-B, we

relate $D_r + 1$ to the number of cells in a special partition of the graph called the *minimal external equitable partition with respect to node r* .

For simplicity of exposition, we only consider undirected graphs in the following sections, i.e., we assume:

A.4 The matrices W , L , A in eq. (1) are symmetric.

A. Algebraic characterisation

An algebraic characterisation of the degree of the minimal polynomial of $[A, e_r^T]$ can be obtained based on the Jordan block decomposition described in Section III. The symmetry of the Laplacian matrix in undirected graphs simplifies the analysis since the Jordan matrix in Eq. (12) becomes diagonal. The following Corollary provides a relationship between the minimal number of successive values required by a node to compute the final consensus value of the network and algebraic properties of the underlying graph. Before presenting the main result, we introduce the following notation, which will be used extensively in the remainder of the paper.

Definition 3 (D-cardinality of a set): Let Λ be a finite set, potentially containing repeated elements, with cardinality $\text{card}\{\Lambda\}$. The d-cardinality of the set, denoted $\text{dcard}\{\Lambda\}$, is defined as the number of *distinct* elements in the set.

Example 2: Let $\Lambda = \{1, 2, 3, 1, 3, 5\}$. Then $\text{card}\{\Lambda\} = 6$ and $\text{dcard}\{\Lambda\} = 4$.

Our first algebraic characterisation of the minimal recursion length at node r relates $D_r + 1$ to the number of distinct eigenvalues of the Laplacian matrix whose eigenvectors have non-zero components for node r , as given by the following Corollary.

Corollary 1: Consider the dynamics (1) where A is associated with an unweighted and undirected graph. Denote the eigenvalues of the symmetric matrix A by λ_i and their corresponding right eigenvector by u_i . Let $\Lambda = \{\lambda_i(A) \mid i = 1, \dots, n\}$ and $\Psi_r = \{\lambda_i(A) \mid u_i[r] = 0\}$. Then

$$D_r + 1 = \text{dcard}\{\Lambda/\Psi_r\},$$

where Λ/Ψ_r is the relative complement of Ψ_r in Λ .

Proof: Since A is symmetric, all the eigenvalues of A are real. The proof then follows from Lemma 1 and the PBH-test [22]. ■

Consider now the following well-known lemma:

Lemma 2: [23, Theorem 9.5.1] Let A be a symmetric matrix in $\mathbb{R}^{n \times n}$ and let $R \in \mathbb{R}^{n \times m}$ be such that $R^T R = I_m$. Define $\Theta = R^T A R$ and let $\{v_1, v_2, \dots, v_m\}$ be an orthogonal set of eigenvectors for Θ such that $\Theta v_i = \lambda_i(\Theta) v_i$, where $\lambda_i(\Theta)$ is the i^{th} eigenvalue of Θ associated with the eigenvector v_i . Then, we have the following result: if $\lambda_i(\Theta) = \lambda_i(A)$ for $i = 1, \dots, l$ then, $R v_i$ is an eigenvector of A with associated eigenvalue $\lambda_i(\Theta)$ for $i = 1, \dots, l$.

Our second algebraic characterisation relates $D_r + 1$ with the number of eigenvalues shared by the Laplacian matrix and the r -grounded Laplacian matrix.

Theorem 2: [21] Consider the system in Eq. (1) satisfying Assumptions A.1–A.4. The rank of the observability matrix for the pair $[A, e_r^T]$ is equal to $n - \mu_r$, i.e.,

$$D_r + 1 = n - \mu_r,$$

where μ_r is the number of eigenvalues shared between A and A_r , where A_r is the r -grounded Laplacian matrix, i.e., the submatrix of A obtained by deleting the r^{th} row and the r^{th} column.

Proof: Due to the page limitation, we refer the reader to the proof in [21] ■

B. Graph-theoretical characterisation

In this section, we consider the following question: given an undirected network, can we directly identify the dominant node(s) from the graph without any algebraic computation?

We adopt definitions and notations from [24]. A partition of a graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ is defined as a mapping from vertices to subsets of vertices called *cells*: $\pi : \mathcal{V} \rightarrow \{C_1, \dots, C_K\}$ where $C_i \subseteq \mathcal{V}$, $\forall i$. Let $\text{Im}(\pi)$ denote the image of π , i.e., $\text{Im}(\pi) = \{C_1, \dots, C_K\}$ and $\text{deg}_\pi(i, C_j)$ denote the node-to-cell degree. $\text{deg}_\pi(i, C_j)$ characterises the number of nodes in cell C_j that share an edge with node v_i under partition π :

$$\text{deg}_\pi(i, C_j) = \text{card}\{k \in \mathcal{V} \mid \pi(k) = C_j \text{ and } (i, k) \in \mathcal{E}\}.$$

We define $\pi^{-1}(C_i) = \{j \in \mathcal{V} \mid \pi(j) = C_i\}$, i.e., the set of nodes that are mapped to cell C_i .³

In what follows, we use the concept of external equitable partition (EEP) [24]. As we will show below, EEPs partition the graph into cells while neglecting the internal interconnection structure inside a cell. We will show that the EEP with respect to a node is directly related to the minimal number of steps necessary for this node to calculate the final consensus value.

Definition 4 (External equitable partition (EEP) [24]):

A partition π^* of the set of nodes \mathcal{V} consisting of $s > 1$ cells $\{C_1, \dots, C_s\}$ is external equitable if the number of neighbours $\in C_j$ of a vertex $v \in C_i$ depends only on the choice of C_i and C_j ($i \neq j$), i.e.,

$$\text{deg}_{\pi^*}(l, C_j) = \text{deg}_{\pi^*}(k, C_j), \forall k, l \in \pi^{*-1}(C_i).$$

Definition 5 (Minimal EEP with respect to a node): A partition π_r of \mathcal{V} consisting of cells $\{C_1, \dots, C_s\}$ is external equitable with respect to node r if the partition is external equitable and the node r is in a cell alone, i.e., $\pi_r(v_r) = v_r$. The minimal EEP of a graph with respect to node r , π_r^* , is such that $\text{card}\{\text{Im}(\pi_r^*)\}$ is minimal.

Theorem 3: Consider the system in (1). Solely based on observations of node r , the minimal length of recursion necessary to obtain the final consensus value is equal to the number of cells s_r in π_r^* , the minimal external equitable partition with respect to node r , i.e.

$$D_r + 1 = \text{card}\{\text{Im}(\pi_r^*)\} \triangleq s_r. \quad (25)$$

Proof: Without loss of generality, let $r = 1$. We use a Breadth-First-Search (BFS) algorithm to label the cells, as follows. We start from node 1 (i.e., cell 1) and explore all the neighbouring cells. For each of those nearest cells, we consider their own neighbouring cells and so on, until we have labelled all the cells in the minimal EEP with respect to cell 1 [21].

³Note that π is not a one-to-one mapping but a one-to-many mapping. However, we can still define a new function to map back from C_j to \mathcal{V} . We adopt this notation from [24].

Consider now the block matrix obtained by permuting and partitioning A according to π_1^* , the minimal EEP with respect to node 1:

$$A_{\pi_1^*} = \begin{bmatrix} A_{11} & A_{12} & \dots & A_{1s_1} \\ A_{21} & A_{22} & \dots & A_{2s_1} \\ \vdots & \vdots & \ddots & \vdots \\ A_{s_11} & A_{s_12} & \dots & A_{s_1s_1} \end{bmatrix}.$$

Here, $A_{ii} \in \mathbb{R}^{l_i \times l_i}$ contains the interconnections between any two nodes in cell C_i^* and l_i denotes the number of nodes in cell C_i^* . Hence, $l_1 = 1$ and $\sum_{i=1}^s l_i = n$. The off-diagonal submatrices $A_{ij} \in \mathbb{R}^{l_i \times l_j}$ contain the interconnections between nodes in C_i^* and C_j^* . In particular, we will consider the following submatrices:

$$A_1 \triangleq A_{\pi_1^*}[2:n, 2:n]$$

$$f_1^T \triangleq A_{\pi_1^*}[1, 1:n] = [A_{12} \quad \dots \quad A_{1j} \quad 0 \quad \dots \quad 0].$$

Note that there are only j neighbouring cells to cell 1, i.e., $A_{1(j+1)}, \dots, A_{1s_1} = 0$ for some $j > 1$.

The observability matrix associated with the pair $[A_{\pi_1^*}, e_1^T]$ is:

$$\Omega = \begin{bmatrix} 1 & 0 & \dots & 0 \\ A_{11} & A_{12} & \dots & A_{1s_1} \\ \vdots & \vdots & \ddots & \vdots \\ * & * & \dots & * \end{bmatrix}, \quad (26)$$

where $*$ is a placeholder representing a real value.

Let Ξ be the observability matrix associated with the pair $[A_1, f_1^T]$. According to [20], [24], the rank of the observability matrix is equal to the dimension of the following span

$$\text{rank}(\Xi) = \dim\text{-span} \left\{ \begin{bmatrix} \mathbf{1}_{r_2} \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ \mathbf{1}_{r_3} \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ \mathbf{1}_{r_{s_1}} \end{bmatrix} \right\}, \quad (27)$$

with $r_i = \text{card}\{C_i^*\}$. Hence,

$$\text{rank}(\Xi) = s_1 - 1,$$

from whence it follows that

$$D_1 + 1 = \text{rank}(\Omega) = \text{rank} \begin{bmatrix} 1 & 0 & \dots & 0 \\ * & & & \\ \vdots & & \Xi & \\ * & & & \end{bmatrix}$$

$$= \text{rank}(\Xi) + 1 = \text{card}\{\text{Im}(\pi_1^*)\}$$

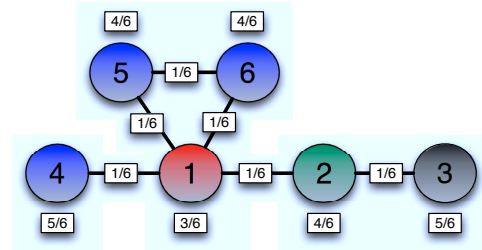
Remark 4: Definition 5 implies that the number of cells in π_r^* , s_r , is greater or equal than the longest distance from node r to all other nodes in the graph \mathcal{G} , $d(\mathcal{G}, r)$. Therefore,

$$D_r + 1 \geq d(\mathcal{G}, r).$$

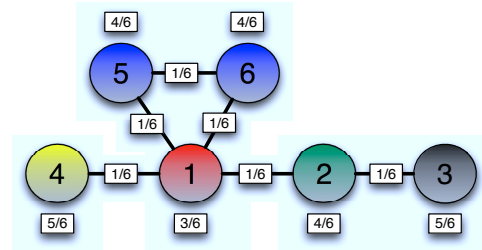
Remark 5: Theorem 3 provides a link between local observations, i.e., the minimal number of successive values that a node r needs to accumulate to compute the final

consensus value of the network) and a global property, i.e., the underlying minimal EEP of the network with respect to node r . Based on this theorem, one can directly identify the dominant nodes in the network without resorting to algebraic numerical manipulations.

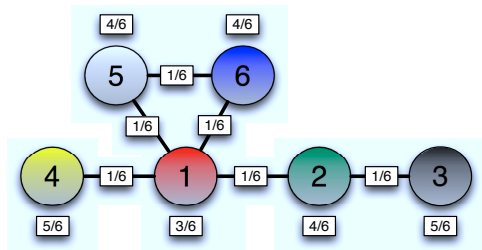
Example 3: As shown numerically in Example 1, nodes 1, 2 and 3 are the dominant nodes since they only require 8 steps, i.e., $D_r + 1 = 4$ for $r = 1, 2, 3$. It is easy to check in Fig. 2(a) that the minimal external equitable partition with respect to these nodes has 4 cells. Similarly, Figs. 2(b) and 2(c) show the minimal EEPs for node 4 and for nodes 5 and 6, respectively. The number of cells in the corresponding minimal EEPs is consistent with the numerical results in Example 1 which indicate that these nodes require respectively 10 and 12 successive values of their own state to compute the final consensus value of the network according to Algorithm 1.



(a) 4-cell based minimal external equitable partition with respect to nodes 1, 2, 3. As illustrated in Example 1, nodes 1, 2, 3 require $2 \times 4 = 8$ steps to compute the final consensus value.



(b) 5-cell based minimal external equitable partition with respect to node 4. As illustrated in Example 1, node 4 requires $2 \times 5 = 10$ steps to compute the final consensus value.



(c) 6-cell based minimal external equitable partition with respect to nodes 5, 6. As shown in Example 1, nodes 5, 6 require $2 \times 6 = 12$ steps to compute the final consensus value.

Fig. 2. Minimal EEP with respect to the different nodes in Example 1. Different colours correspond to different cells (colour online).

VI. CONCLUSION

This paper formulates and analyses the decentralised minimal time consensus problem. In contrast to other tools in the literature, our algorithm computes consensus from the history of any node in a completely decentralised, local manner. The necessary information for any node is its own history and is therefore exclusively local. The algorithm does not require global knowledge, such as the total number of nodes in the system, information about the neighbourhood of the node, or specific edge weights. After characterising the minimal number of steps required for any given node to compute the final consensus value, we provided algebraic, graph-theoretical and local informative interpretations of the minimal number of steps.

There are a number of interesting directions for future research in terms of network design. For instance, we are currently working on the problem of computing a minimal EEP with respect to a node in polynomial time. Also it is important to mention that the EEP-based results provided here for undirected graphs can be extended to directed graphs at the price of a more elaborate exposition.

Design of network topologies that minimise algebraic connectivity was presented in [15], [25], [29]. Instead of minimising the second smallest eigenvalue of a network, we aim here at minimising the d -cardinality of the Laplacian spectrum. An interesting question in this context is: given a constraint on the number of edges in the network, what are the network structures that minimise the d -cardinality of the Laplacian spectrum? Constructing Laplacian matrices with small spectra has been intensively studied in the graph theoretic community [26], [27]. In the Appendix of [28], the author computed all the Laplacian spectra for trees up to $n = 10$ vertices and connected graphs up to $n = 6$ vertices. Interestingly, in a recent paper [25], the authors minimised the second smallest eigenvalue of a weighted Laplacian given a constraint on the number of edges in the graph. It turned out in both examples that the obtained optimal Laplacian matrix had only 2 (resp. 4) distinct eigenvalues for 5- (resp. 10-) node networks. Future work lies in formulating the optimal minimal-time consensus network problem as a standard optimisation problem.

On the analysis part, future work will consist in extending the model in eq. (1) so as to encompass more complex situations, e.g., time-delay in the model, noise/quantisation error in the communication links, packet drop in the observations. Yet another extension lies in the reconstruction of agent-network from minimal amount of observed data as illustrated in [30], [31].

VII. ACKNOWLEDGEMENT

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