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Stability and stabilization of 2D continuous state-delayed systems

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Abstract— **In this paper, we consider the problem of stability and stabilization of 2D continuous systems with state delays. The asymptotic stability of this class of systems described by the Roesser model is addressed *via* Lyapunov techniques. It is shown that linear matrix inequalities (LMIs) can be used to check the asymptotic stability of 2D linear delayed systems and this is applied to the case of state feedback stabilization. A numerical example is introduced to show the efficiency of the proposed criterion for a 2D linear delayed system.**

Keywords: 2D Roesser model, delayed systems, stabilization, Lyapunov-Krasovskii functional, LMI.

I. Introduction

As usual in the development of science, the theory of multidimensional systems (n-D systems) emerged from the growing complexity of modern technology particularly in image and signal processing, coding/decoding, filtering, etc. (for an overview, see [3]). Another interesting class of systems which appeared recently is the repetitive processes (or multipass processes) such as longwall coal cutting or metal rolling operations that can be modeled as n-D systems [14]. It was more than enough to catch the interest of the scientific community both in engineering and mathematical areas as the growing number of recent studies shows and it is not a surprise that the theory is now advancing faster than the applications. The field is not recent though. The first studies on multidimensional systems started in the early 70s with the introduction of two well known – in the multidimensional community – state space models to describe nD systems; the so called Roesser and Fornasini-Marchesini models [13], [5] still used today both for discrete systems. It is easy to show that both models are equivalent; one can be transformed into another with a simple state substitution (see for instance [11]). More recently, [2] introduced the first hybrid system based on an extension of the Roesser model adding a continuous part to the original discrete system.

In this paper, our attention is focused on the continuous part of this model, generalized to the delay case (for a survey on time delay systems, the reader can refer to [7], [6], [15]. Delayed multidimensional systems have been recently introduced but in the majority of the existing studies only the discrete case have been analyzed (see e.g. [12], [16], [10]) except for a few recent papers that inspired our work [1], [9] where a Lyapunov approach is applied to continuous Roesser models. The problem of stability/stabilization is addressed by the above authors but the results given in terms of LMIs (linear matrix inequalities, see [4]) are all delay independent. The aim of this work is to give less conservative results to design a state-feedback controller for 2D delayed systems which stabilizes the system.

The paper is organized as follows. In section II, we introduce the mathematical background we need to address the problem. In section III, we introduce our main results: a sufficient condition to check the asymptotic stability and the stabilization of a 2D delayed system. Our approach is based on Lyapunov techniques; stability and stabilization criteria are given in form of LMIs. Finally section IV presents an illustrative example.

II. Problem formulation

The class of 2-D systems with delays under consideration is represented by an extension of the Roesser model with (constant) state delays (see [13] and [2]) of the form:

$$\begin{bmatrix} \frac{\partial x^h(t_1,t_2)}{\partial t_1} \\ \frac{\partial x^v(t_1,t_2)}{\partial t_2} \end{bmatrix} = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} x^h(t_1,t_2) \\ x^v(t_1,t_2) \end{bmatrix} + \begin{bmatrix} A_{11d} & A_{12d} \\ A_{21d} & A_{22d} \end{bmatrix} \begin{bmatrix} x^h(t_1-\tau_1,t_2) \\ x^v(t_1,t_2-\tau_2) \end{bmatrix} + \begin{bmatrix} B_1 \\ B_2 \end{bmatrix} u(t_1,t_2) \quad (1)$$

where $x^h(t_1,t_2)$ is the horizontal state in $\mathbb{R}^{n_h}$, $x^v(t_1,t_2)$ is the vertical state in $\mathbb{R}^{n_v}$, $u(t_1,t_2)$ is the control vector in $\mathbb{R}^m$, τ_1 and τ_2 are the delays in horizontal and vertical directions respectively and A_{ii}, A_{iid} and B_i are real constant matrices of appropriate dimensions. The initial conditions are

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given by

$$x^h(\theta, t_2) = f(\theta, t_2), \forall t_2 \quad \text{and} \quad -\tau_{1max} < \theta < 0$$
$$x^v(t_1, \theta) = g(t_1, \theta), \forall t_1 \quad \text{and} \quad -\tau_{2max} < \theta < 0$$

where f and g are continuous functions. For such a system we denote

$$x(t_1, t_2) \equiv \begin{bmatrix} x^h(t_1, t_2) \\ x^v(t_1, t_2) \end{bmatrix},$$
$$x(t_1 - \tau_1, t_2 - \tau_2) \equiv \begin{bmatrix} x^h(t_1 - \tau_1, t_2) \\ x^v(t_1, t_2 - \tau_2) \end{bmatrix}$$
$$\dot{x}(t_1, t_2) \equiv \begin{bmatrix} \frac{\partial x^h(t_1,t_2)}{\partial t_1} \\ \frac{\partial x^v(t_1,t_2)}{\partial t_2} \end{bmatrix}$$

and

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, A_d = \begin{bmatrix} A_{11d} & A_{12d} \\ A_{21d} & A_{22d} \end{bmatrix}, B = \begin{bmatrix} B_1 \\ B_2 \end{bmatrix}$$

which allows us to write (1) in the usual form

$$\dot{x}(t_1, t_2) = Ax(t_1, t_2) + A_d x(t_1 - \tau_1, t_2 - \tau_2) + Bu(t_1, t_2) \quad (2)$$

Consider the state feedback control:

$$u(t_1, t_2) = Kx(t_1, t_2) \quad (3)$$

where the matrix

$$K = \begin{bmatrix} K_1 & K_2 \end{bmatrix}$$

is the state feedback gain to be determined. The problem is then to compute a static feedback control given by (3) such that the closed-loop 2D system (1) is asymptotically stable.

III. Main results

A. Stability criteria

In this section, we investigate stability conditions for the 2D delayed system (1).

Theorem 1: If there exist symmetric bloc diagonal matrices $P > 0$, $Q > 0$ and $R > 0$ such that the following LMI

$$\begin{bmatrix} A^T P + PA + Q - R & PA_d + R & \tau A^T R \\ * & -Q - R & \tau A_d^T R \\ * & * & -R \end{bmatrix} \prec 0 \quad (4)$$

where

$$\tau = \begin{bmatrix} \tau_1 I_{n_h} & 0 \\ 0 & \tau_2 I_{n_v} \end{bmatrix},$$

$$P = \begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}, Q = \begin{bmatrix} Q_1 & 0 \\ 0 & Q_2 \end{bmatrix}, R = \begin{bmatrix} R_1 & 0 \\ 0 & R_2 \end{bmatrix}$$

holds true, then the 2D linear delayed system (1) is asymptotically stable.

Proof: Proof of theorem 1 is given in the appendix VI-A. ■

B. Stabilization

The objective of this section is the design of a stabilizing state-feedback controller for system (1). Using the state-feedback control (3), (1) can be rewritten as:

$$\frac{\partial x^h(t_1,t_2)}{\partial t_1} = A_{11}x^h(t_1,t_2) + A_{12}x^v(t_1,t_2) + B_1K_1x^h(t_1,t_2) + B_1K_2x^v(t_1,t_2) + A_{11d}x^h(t_1-\tau_1,t_2) + A_{12d}x^v(t_1,t_2-\tau_2)$$
$$\frac{\partial x^v(t_1,t_2)}{\partial t_2} = A_{21}x^v(t_1,t_2) + A_{22}x^v(t_1,t_2) + B_2K_1x^v(t_1,t_2) + B_2K_2x^v(t_1,t_2) + A_{21d}x^v(t_1-\tau_1,t_2) + A_{22d}x^v(t_1,t_2-\tau_2)$$

or equivalently,

$$\dot{x}(t_1, t_2) = A_c x(t_1, t_2) + A_d x(t_1 - \tau_1, t_2 - \tau_2) \quad (5)$$

where:

$$A_c = \begin{bmatrix} (A_{11} + B_1K_1) & (A_{12} + B_1K_2) \\ (A_{21} + B_2K_1) & (A_{22} + B_2K_2) \end{bmatrix}$$

$$A_d = \begin{bmatrix} A_{11d} & A_{12d} \\ A_{21d} & A_{22d} \end{bmatrix}$$

Theorem 2: If there exist symmetric bloc diagonal matrices $X > 0$, $\bar{Q} > 0$ and $Y > 0$ such that the following LMI

$$\begin{bmatrix} XA^T + AX + Y^TB^T + BY + \bar{Q} - X & A_dX + X & \tau XA^T + \tau Y^TB^T \\ * & -\bar{Q} - X & \tau XA_d^T \\ * & * & -X \end{bmatrix} \prec 0 \quad (6)$$

with $X = P^{-1}$, $Y = KX$ and $Q = P\bar{Q}P$ is verified, then (5) is asymptotically stable.
The gains K_1 and K_2 of the controller law (3) are given by

$$K_1 = Y_1X_1^{-1}, K_2 = Y_2X_2^{-1} \quad (7)$$

Proof: Proof of theorem 2 is given in the appendix VI-B. ■

IV. Example

In order to show the applicability of our results, consider a 2D continuous system represented by (1) with:

$$A_{11} = \begin{bmatrix} 1 & -0.5 \\ 0 & -2 \end{bmatrix}, A_{12} = \begin{bmatrix} 0.1 & -1 \\ 0 & 0.1 \end{bmatrix},$$
$$A_{21} = \begin{bmatrix} -1 & 0 \\ 0 & 0.1 \end{bmatrix}, A_{22} = \begin{bmatrix} 0 & -3 \\ 1 & -0.6 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 1 & -1 \\ 1 & 0 \end{bmatrix}, B_2 = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix}.$$

The "delayed" matrices are given by:

$$A_{11d} = \begin{bmatrix} 0.30 & -0.15 \\ 0 & -0.60 \end{bmatrix}, A_{12d} = \begin{bmatrix} 0.03 & -0.30 \\ 0 & 0.03 \end{bmatrix},$$
$$A_{21d} = \begin{bmatrix} -0.30 & 0 \\ 0 & 0.03 \end{bmatrix}, A_{22d} = \begin{bmatrix} 0 & -0.90 \\ 0.30 & -0.18 \end{bmatrix}.$$

For $\tau_2 = 0.1$ and $\tau_1 = 1.2$, the solutions are given by:

$$P_1 = \begin{bmatrix} 1.5880 & 1.2137 \\ 1.2137 & 2.4878 \end{bmatrix}, P_2 = \begin{bmatrix} 0.0540 & -0.0116 \\ -0.0116 & 0.1396 \end{bmatrix},$$
$$Q_1 = \begin{bmatrix} 0.2717 & -0.0574 \\ -0.0574 & 1.0118 \end{bmatrix}, Q_2 = \begin{bmatrix} 0.0112 & -0.0328 \\ -0.0328 & 0.4322 \end{bmatrix}.$$

The stabilizing control law (5) is given by (7):

$$K_1 = \begin{bmatrix} -0.2691 & 1.3792 \\ 1.2283 & 0.4440 \end{bmatrix}, K_2 = \begin{bmatrix} -0.0410 & -0.1919 \\ 0.0243 & -1.2210 \end{bmatrix}$$

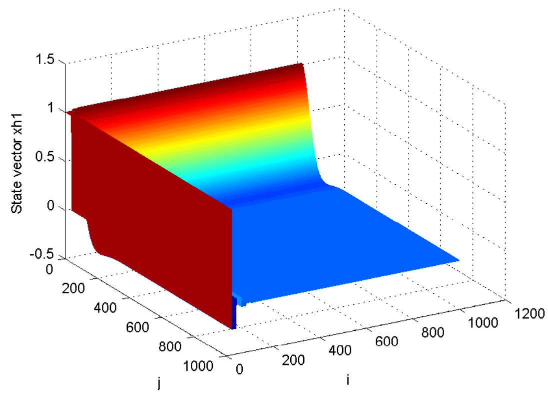

Fig. 1. The state evolution of the first component of $x^h(t_1, t_2)$

For $\tau_2 = 0.1$ and $\tau_1 = 0.2$, we have:

$$P_1 = \begin{bmatrix} 0.1817 & -0.0063 \\ -0.0063 & 0.0878 \end{bmatrix}, P_2 = \begin{bmatrix} 0.0904 & -0.0558 \\ -0.0558 & 0.2780 \end{bmatrix},$$
$$Q_1 = \begin{bmatrix} 0.5848 & -0.0172 \\ -0.0172 & 0.1000 \end{bmatrix}, Q_2 = \begin{bmatrix} 0.1053 & -0.2751 \\ -0.2751 & 1.3104 \end{bmatrix}$$

and

$$K_1 = \begin{bmatrix} -2.4597 & 0.9485 \\ 1.8430 & -0.7777 \end{bmatrix}, K_2 = \begin{bmatrix} 0.5039 & -1.2618 \\ 0.3839 & -3.4540 \end{bmatrix}$$

Now, in both cases, injecting the values of K_1 and K_2 back into the system, it is possible to check that the closed loop system is asymptotically stable using theorem 2. But it is also nice to have a look at the evolution of the components of states $x^h(t_1, t_2)$ and $x^v(t_1, t_2)$ in 3D graphics (Fig. 1, Fig. 2, Fig. 3 and Fig. 4). The curves are obtained using a simple discretization method (Euler type). We have used the sampling periods $T_h = 0.01s$ and $T_v = 0.01s$ and the following initial conditions:

$$x^h(\theta, t_2) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \forall t_2 \quad \text{and} \quad -0.2 < \theta < 0$$
$$x^v(t_1, \theta) = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \forall t_1 \quad \text{and} \quad -0.1 < \theta < 0$$

Note that i and j presented in these figures are the indexes of the discretized times t_1 and t_2 respectively ($t_1 = iT_h$, $t_2 = jT_v$). Both vertical and horizontal states quickly converge to the origin. This can be better seen in the next set of graphs. Indeed, Fig. 5 shows the evolution of the horizontal state $x^h(t_1, t_2)$ for a fixed value of t_2 equal to $6s$. Same goes for Fig. 6 which represents the evolution of the horizontal state $x^h(t_1, t_2)$ for $t_1 = 8s$.

V. Conclusion

To conclude, let us highlight the general contribution of this paper. We first developed a sufficient condition of asymptotic stability for 2D continuous systems with state delays. This system is based on the generalization of the (discrete) commonly used Roesser state space model. Using Lyapunov approach, we proposed the synthesis of a state feedback controller. In order to design the controller, it is necessary to solve an LMI. It is also important to note that this is the first time a delay dependant criteria is proposed (see introduction) thus leading to less conservative results. A numerical example is provided to illustrate the results.

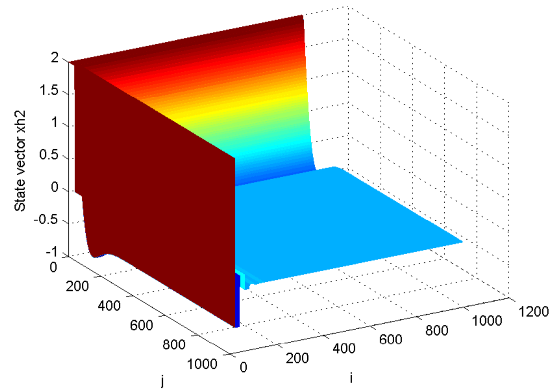

Fig. 2. The state evolution of the second component of $x^h(t_1, t_2)$

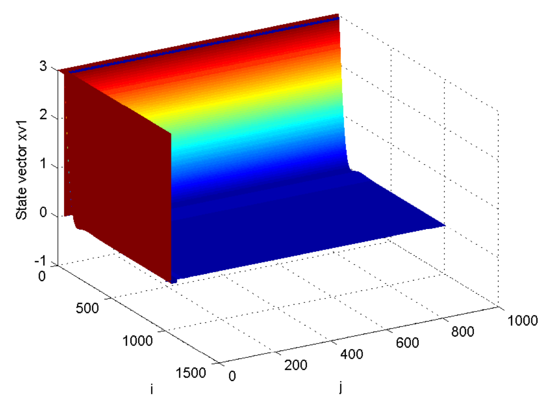

Fig. 3. The state evolution of the first component of $x^v(t_1, t_2)$

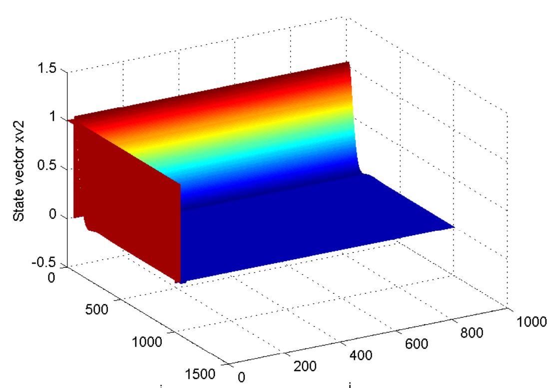

Fig. 4. The state evolution of the second component of $x^v(t_1, t_2)$

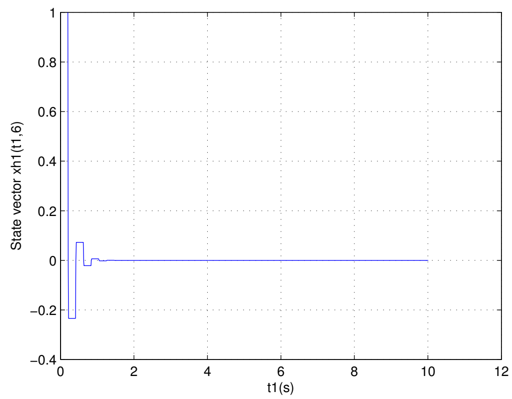

Fig. 5. The state evolution of the first component of $x^h(t_1,t_2)$ at time instant $t_2 = 6s$

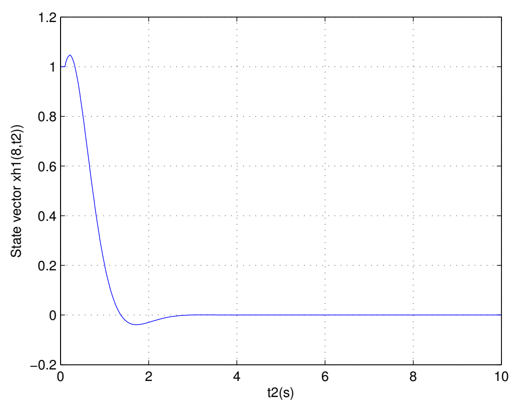

Fig. 6. The state evolution of the first component of $x^h(t_1,t_2)$ at time instant $t_1 = 8s$

VI. Appendix

A. Proof of Theorem 1

Proof: Let us define

$$V(x(t_1,t_2)) = V_1(t_1,t_2) + V_2(t_1,t_2) \tag{8}$$

as a possible LK functional candidate for the system (1) with:

$$\begin{aligned}
V_1(t_1,t_2) &= x^{hT}(t_1,t_2)P_1x^h(t_1,t_2) + \textstyle\int_{t_1-\tau_1}^{t_1} x^{hT}(\theta,t_2)Q_1x^h(\theta,t_2)d\theta \\
&+ \textstyle\int_{t_1-\tau_1}^{t_1}(\tau_1 - t_1 + \theta)\left(\frac{\partial x^h(\theta,t_2)}{\partial \boldsymbol{t_1}}\right)^T(\tau_1R_1)\left(\frac{\partial x^h(\theta,t_2)}{\partial \boldsymbol{t_1}}\right)d\theta \\
V_2(t_1,t_2) &= x^{vT}(t_1,t_2)P_2x^v(t_1,t_2) + \textstyle\int_{t_2-\tau_2}^{t_2} x^{vT}(t_1,\theta)Q_2x^v(t_1,\theta)d\theta \\
&+ \textstyle\int_{t_2-\tau_2}^{t_1}(\tau_2 - t_2 + \theta)\left(\frac{\partial x^v(t_1,\theta)}{\partial \boldsymbol{t_2}}\right)^T(\tau_2R_2)\left(\frac{\partial x^h(t_1,\theta)}{\partial \boldsymbol{t_2}}\right)d\theta
\end{aligned}$$

The derivative of function $V(x(t_1,t_2))$ along the vector

$$\zeta(t_1,t_2) = \begin{bmatrix} \dfrac{\partial x^h(t_1,t_2)}{\partial t_1} \\ \dfrac{\partial x^v(t_1,t_2)}{\partial t_2} \end{bmatrix} \tag{9}$$

(what we refer to by the trajectory of (1) in the sequel) is given by:

$$\begin{aligned}
\nabla_\zeta V(x(t_1,t_2)) &= (\nabla V)^T\zeta(t_1,t_2) = \begin{bmatrix} \dfrac{\partial V}{\partial x^h} & \dfrac{\partial V}{\partial x^v} \end{bmatrix}\zeta(t_1,t_2) \\
&= \frac{\partial V(t_1,t_2)}{\partial x^h(t_1,t_2)}\frac{\partial x^h(t_1,t_2)}{\partial t_1} + \frac{\partial V(t_1,t_2)}{\partial x^v(t_1,t_2)}\frac{\partial x^v(t_1,t_2)}{\partial t_2} \\
&= \frac{\partial V_1(t_1,t_2)}{\partial x^h(t_1,t_2)}\frac{\partial x^h(t_1,t_2)}{\partial t_1} + \frac{\partial V_2(t_1,t_2)}{\partial x^v(t_1,t_2)}\frac{\partial x^v(t_1,t_2)}{\partial t_2}
\end{aligned} \tag{10}$$

where ∇V is the gradient of the function V.

Remark 1: In [9], the authors introduce the notion of unidirectional derivative. In this paper we refer to this derivation as the derivative of the function V along the vector $\zeta(t_1,t_2)$ defined by (9).

In [9], the authors also proved that using this derivation, the "usual" Lyapunov theorem is true (roughly speaking $\dot{V} < 0$ implies asymptotic stability).

Computing[1] the derivative of $V_1(t_1,t_2)$ along the trajectories of (1) gives:

$$\begin{aligned}
&\tfrac{\partial V_1(t_1,t_2)}{\partial x^h(t_1,t_2)}\tfrac{\partial x^h(t_1,t_2)}{\partial \boldsymbol{t_1}} = \\
&\left(\tfrac{\partial x^h(t_1,t_2)}{\partial \boldsymbol{t_1}}\right)^T P_1x^h(t_1,t_2) + x^{hT}(t_1,t_2)P_1\left(\tfrac{\partial x^h(t_1,t_2)}{\partial \boldsymbol{t_1}}\right) \\
&+x^{hT}(t_1,t_2)Q_1x^h(t_1,t_2) - x^{hT}(t_1-\tau_1,t_2)Q_1x^h(t_1-\tau_1,t_2) \\
&+\left(\tfrac{\partial x^h(t_1,t_2)}{\partial \boldsymbol{t_1}}\right)^T(\tau_1^2R_1)\left(\tfrac{\partial x^h(t_1,t_2)}{\partial \boldsymbol{t_1}}\right) \\
&-\textstyle\int_{t_1-\tau_1}^{t_1}\left(\tfrac{\partial x^h(\theta,t_2)}{\partial \boldsymbol{t_1}}\right)^T(\tau_1R_1)\left(\tfrac{\partial x^h(\theta,t_2)}{\partial \boldsymbol{t_1}}\right)d\theta
\end{aligned}$$

Lemma 1 ([8]): For any constant matrix $W \in \mathbb{R}^{n\times n}, W = W^T \succ 0$, scalar $\gamma > 0$, and vector function $\dot{x} : [-\gamma, 0] \rightarrow \mathbb{R}^n$ such that the following integration is well defined, then

$$\begin{aligned}
&-\gamma\textstyle\int_{-\gamma}^{0}\dot{x}^T(t+\xi)W\dot{x}^T(t+\xi)d\xi \\
&\leq \begin{bmatrix} x^T(t) & x^T(t-\gamma) \end{bmatrix}\begin{bmatrix} -W & W \\ W & -W \end{bmatrix}\begin{bmatrix} x(t) \\ x(t-\gamma) \end{bmatrix}
\end{aligned}$$

Using lemma 1, we obtain

$$\begin{aligned}
&-\textstyle\int_{t_1-\tau_1}^{t_1}\left(\frac{\partial x^h}{\partial \boldsymbol{t_1}}\right)^T(\theta,t_2)(\tau_1R_1)\left(\frac{\partial x^h}{\partial \boldsymbol{t_1}}\right)(\theta,t_2)d\theta \\
&\leq \begin{bmatrix} x^h(t_1,t_2) \\ x^h(t_1-\tau_1,t_2) \end{bmatrix}^T\begin{bmatrix} -R_1 & R_1 \\ R_1 & -R_1 \end{bmatrix}\begin{bmatrix} x^h(t_1,t_2) \\ x^h(t_1-\tau_1,t_2) \end{bmatrix}
\end{aligned}$$

Let $J = \begin{bmatrix} -I & I \end{bmatrix}$, then

$$\begin{bmatrix} -R_1 & R_1 \\ R_1 & -R_1 \end{bmatrix} = J^T(-R_1)J$$

Hence

$$\begin{aligned}
&-\textstyle\int_{t_1-\tau_1}^{t_1}\left(\frac{\partial x^h}{\partial \boldsymbol{t_1}}\right)^T(\theta,t_2)(\tau_1R_1)\left(\frac{\partial x^h}{\partial \boldsymbol{t_1}}\right)(\theta,t_2)d\theta \\
&\leq \begin{bmatrix} x^h(t_1,t_2) \\ x^h(t_1-\tau_1,t_2) \end{bmatrix}^T J^T(-R_1)J\begin{bmatrix} x^h(t,t_2) \\ x^h(t_1-\tau_1,t_2) \end{bmatrix}
\end{aligned}$$

Now, let us introduce the augmented state ξ as:

$$\xi(t_1,t_2) = \begin{bmatrix} x(t_1,t_2) \\ x(t_1-\tau_1,t_2-\tau_2) \end{bmatrix} = \begin{bmatrix} x^h(t_1,t_2) \\ x^v(t_1,t_2) \\ x^h(t_1-\tau_1,t_2) \\ x^v(t_1,t_2-\tau_2) \end{bmatrix}$$

With this in mind, we can write

$$\begin{aligned}
&\left(\tfrac{\partial x^h}{\partial \boldsymbol{t_1}}\right)^T(t_1,t_2)(\tau_1^2R_1)\left(\tfrac{\partial x^h}{\partial \boldsymbol{t_1}}\right)(t_1,t_2) = \\
&\xi(t_1,t_2)^T\begin{bmatrix} A_{11}^T \\ A_{12}^T \\ A_{11d}^T \\ A_{12d}^T \end{bmatrix}(\tau_1^2R_1)\begin{bmatrix} A_{11}^T \\ A_{12}^T \\ A_{11d}^T \\ A_{12d}^T \end{bmatrix}^T\xi(t_1,t_2)
\end{aligned}$$

[1]To make the proof easier to follow for the reader, we will only explicit the calculations in one dimension, the horizontal one, as the expressions are similar in the vertical one.

Then, following the same logic with the other dimension, we can show that

$$\dot{V}(t_1,t_2) \le \xi^T(t_1,t_2)\Theta\xi(t_1,t_2)$$

where

$$\Theta = \begin{bmatrix} A^TP+PA+Q-R+A^T(\tau^2R)A & PA_d+A^T(\tau^2R)A_d+R \\ A_d^TP+R+A_d^T(\tau^2R)A & -Q-R+A_d^T(\tau^2R)A_d \end{bmatrix} \prec 0$$

which can be written as

$$\Theta = \begin{bmatrix} \Theta_{11} & \Theta_{12} & \Theta_{13} & \Theta_{14} \\ * & \Theta_{22} & \Theta_{23} & \Theta_{24} \\ * & * & \Theta_{33} & \Theta_{34} \\ * & * & * & \Theta_{44} \end{bmatrix}$$

with

$$\Theta_{11} = A_{11}^TP_1+P_1A_{11}+Q_1-R_1+A_{11}^T(\tau_1^2R_1)A_{11}+A_{21}^T(\tau_2^2R_2)A_{21}$$
$$\Theta_{12} = P_1A_{12}A_{21}^TP_2+A_{11}^T(\tau_1^2R_1)A_{12}+ A_{21}^T(\tau_2^2R_2)A_{22}$$
$$\Theta_{13} = P_1A_{11d}+A_{11}(\tau_1^2R_1)A_{11d}+R_1+ A_{21}^T(\tau_2^2R_2)A_{21d}$$
$$\Theta_{14} = P_1A_{12d}+A_{11}(\tau_1^2R_1)A_{12d}+ A_{21}^T(\tau_2^2R_2)A_{22d}$$
$$\Theta_{22} = A_{22}^TP_2+P_2A_{22}+Q_2-R_2+A_{12}^T(\tau_1^2R_1)A_{12}+A_{22}^T(\tau_2^2R_2)A_{22}$$
$$\Theta_{23} = P_2A_{21d}+A_{12}^T(\tau_1^2R_1)A_{11d}+A_{22}^T(\tau_2^2R_2)A_{21d}$$
$$\Theta_{24} = P_2A_{22d}+R_2+A_{12}^T(\tau_1^2R_1)A_{12d}+A_{22}^T(\tau_2^2R_2)A_{22d}$$
$$\Theta_{33} = A_{11d}^T(\tau_1^2R_1)A_{11d}-R_1-Q_1+A_{21d}^T(\tau_2^2R_2)A_{21d}$$
$$\Theta_{34} = A_{11d}^T(\tau_1^2R_1)A_{12d}+A_{21d}^T(\tau_2^2R_2)A_{22d}$$
$$\Theta_{44} = -R_2-Q_2+A_{12d}^T(\tau_1^2R_1)A_{12d}+ A_{22d}^T(\tau_2^2R_2)A_{22d}$$

As we still have nonlinear terms in these last inequalities, we need to apply the Schur complement twice. After the first one, we have

$$\Theta = \begin{bmatrix} \phi_{11} & \phi_{12} & \phi_{13} & \phi_{14} & \tau_1A_{11}^TR_1 \\ * & \phi_{22} & \phi_{23} & \phi_{24} & \tau_1A_{12}^TR_1 \\ * & * & \phi_{33} & \phi_{34} & \tau_1A_{11d}^TR_1 \\ * & * & * & \phi_{44} & \tau_1A_{12d}^TR_1 \\ * & * & * & * & -R_1 \end{bmatrix} \prec 0 \qquad (11)$$

where

$$\phi_{11} = \Gamma_{11}+A_{21}^T(\tau_2^2R_2)A_{21}$$
$$\phi_{12} = \Gamma_{12}+A_{21}^T(\tau_2^2R_2)A_{22}$$
$$\phi_{13} = \Gamma_{13}+A_{21}^T(\tau_2^2R_2)A_{21d}$$
$$\phi_{14} = \Gamma_{14}+A_{21}^T(\tau_2^2R_2)A_{22d}$$
$$\phi_{22} = \Gamma_{22}+A_{22}^T(\tau_2^2R_2)A_{22}$$
$$\phi_{23} = \Gamma_{23}+A_{22}^T(\tau_2^2R_2)A_{21d}$$
$$\phi_{24} = \Gamma_{24}+A_{22}^T(\tau_2^2R_2)A_{22d}$$
$$\phi_{33} = \Gamma_{33}+A_{21d}^T(\tau_2^2R_2)A_{21d}$$
$$\phi_{34} = \Gamma_{34}+A_{21d}^T(\tau_2^2R_2)A_{22d}$$
$$\phi_{44} = \Gamma_{44}+A_{22d}^T(\tau_2^2R_2)A_{22d}$$

with

$$\Gamma_{11} = (A_{11}^T+\alpha I_{n_h})P_1+P_1(A_{11}+\alpha I_{n_h})+Q_1-R_1$$
$$\Gamma_{12} = P_1A_{12}+A_{21}^TP_2$$
$$\Gamma_{13} = P_1A_{11d}e^{\alpha\tau_1}+R_1$$
$$\Gamma_{14} = P_1A_{12d}e^{\alpha\tau_2}$$
$$\Gamma_{22} = (A_{22}^T+\alpha I_{n_v})P_2+P_2(A_{22}+\alpha I_{n_v})+Q_2-R_2$$
$$\Gamma_{23} = P_2A_{21d}e^{\alpha\tau_1}$$
$$\Gamma_{24} = P_2A_{22d}e^{\alpha\tau_2}+R_2$$
$$\Gamma_{33} = -R_1-Q_1$$
$$\Gamma_{34} = 0$$
$$\Gamma_{44} = -R_2-Q_2$$

Applying the Schur complement a second time to eliminate the last non linear terms gives

$$\Theta = \begin{bmatrix} \Gamma_{11} & \Gamma_{12} & \Gamma_{13} & \Gamma_{14} & \tau_1A_{11}^TR_1 & \tau_2A_{21}^TR_2 \\ * & \Gamma_{22} & \Gamma_{23} & \Gamma_{24} & \tau_1A_{12}^TR_1 & \tau_2A_{22}^TR_2 \\ * & * & \Gamma_{33} & \Gamma_{34} & \tau_1A_{11d}^TR_1 & \tau_2A_{21d}^TR_2 \\ * & * & * & \Gamma_{44} & \tau_1A_{12d}^TR_1 & \tau_2A_{22d}^TR_2 \\ * & * & * & * & -R_1 & 0 \\ * & * & * & * & * & -R_2 \end{bmatrix} \prec 0 \qquad (12)$$

which corresponds to the LMI (4) and thus concludes the proof. ■

B. *Proof of Theorem2*

Proof: Using the same Lyapunov functional as mentioned in the section III, we have

$$\dot{V}(t_1,t_2) \le \xi^T(t_1,t_2)\Gamma\xi(t_1,t_2)$$

where

$$\Gamma = \begin{bmatrix} A_c^TP+PA_c+Q-R+A_c^T(\tau^2R)A_c & PA_d+A_c^T(\tau^2R)A_d+R \\ A_d^TP+R+A_d^T(\tau^2R)A_c & -Q-R+A_d^T(\tau^2R)A_d \end{bmatrix}$$

Consider the case where $R=P$, then

$$\Gamma = \begin{bmatrix} A_c^TP+PA_c+Q-P+A_c^T(\tau^2P)A_c & PA_d+A_c^T(\tau^2P)A_d+P \\ A_d^TP+P+A_d^T(\tau^2P)A_c & -Q-P+A_d^T(\tau^2P)A_d \end{bmatrix}$$

In view of Schur complement, $\Gamma \prec 0$ if there exist real matrices $P \succ 0, Q \succ 0$ such that

$$\begin{bmatrix} A_c^TP+PA_c+Q-P & PA_d+P & \tau A_c^TP \\ * & -Q-P & \tau A_d^TP \\ * & * & -P \end{bmatrix} \prec 0 \qquad (13)$$

Note that this last condition is bilinear matrix variables P and K and therefore it may be considered as a BMI problem. To obtain the LMI (6), it is necessary to pre- and post-multiply inequality (13) by $diag\{P^{-1},P^{-1},P^{-1}\}$. ■

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