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Necessary Conditions of Optimality for State Constrained Infinite Horizon Differential Inclusions

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Abstract— **This article presents and discusses necessary conditions of optimality for infinite horizon dynamic optimization problems with inequality state constraints and set inclusion constraints at both endpoints of the trajectory. The cost functional depends on the state variable at the final time, and the dynamics are given by a differential inclusion. Moreover, the optimization is carried out over asymptotically convergent state trajectories. The novelty of the proposed optimality conditions for this class of problems is that the boundary condition of the adjoint variable is given as a weak directional inclusion at infinity. This improves on the currently available necessary conditions of optimality for infinite horizon problems.**

I. INTRODUCTION

This article concerns necessary conditions of optimality for infinite horizon dynamic optimization problems with state constraints. The infinite horizon optimal control problem has been considered since the early seventies and a lot of effort has been spent on how to define the transversality conditions to be satisfied by the adjoint variable so that the optimality conditions remain informative.

The challenges posed by transversality conditions in infinite horizon control problems were already identified in [4] where a problem with an integral cost functional was considered. After defining an appropriate solution concept, a maximum principle without transversality conditions is derived. Later, it is shown in [7], that, under a certain controllability assumption, the Hamiltonian tends to zero as time goes to infinity. Inspired by stability theory, a regularity assumption formulated in terms of Lyapunov exponents to be satisfied by the adjoint variable is required in [12] in order to derive necessary and sufficient conditions of optimality for infinite horizon control problems with a transversality condition. A nonsmooth maximum principle encompassing final time transversality conditions was derived in [11] for nonsmooth optimal control problems with final state dependent cost functional as well as final time state constraints. However, a linear structure is required for both of these ingredients. In [14], strong hypotheses implying that the adjoint variable remains bounded were assumed on the data of an infinite horizon discounted optimal control problem in order to derive a maximum principle with a transversality condition.

In [6], a new type of transversality condition - directional weak inclusion at infinity - is proposed by the authors for optimal control problems with endpoint state constraints and state dependent cost functional at infinity which are weaker than the usual ones. This new concept enables the derivation of necessary conditions of optimality benefiting from the wealth of information provided by the boundary conditions of the adjoint equation at infinity, and, at the same time, does not require strong assumptions on the data of the optimal control problem that strongly restrict their applicability. This work has been developed along the lines of the one in [5].

State constraints pose formidable challenges in the derivation of necessary conditions of optimality in dynamic optimization, even for finite horizon optimal control problems. To appreciate the most sophisticated results addressing the deeper issues arising in this context, albeit in a context different of the one of this article, you should consider the book of A. Arutyunov, [1].

The literature on necessary conditions for infinite horizon optimal control problems with state constraints is relatively limited. In [9], some results are obtained that reveal the formidable challenges intrinsic to the derivation of necessary conditions of optimality for this problem: in order to propagate in a informative way the final time boundary condition of the adjoint variable to any given finite time, one needs to impose very strict assumptions and this implies restricting the range of applicability of the derived optimality conditions.

We consider a dynamic optimization problem over the set of asymptotically convergent state trajectories. Although, this might sound a somewhat unusual setting in dynamic optimization, for many applications, the asymptotic convergence to equilibria at "infinity"corresponds to a functional requirement of the system, and, thus, it is a reasonable property to be enforced via the overall system design. Our dynamic optimization problem can be stated as follows:

$$(P)\ \text{Minimize}\quad g(\xi) \tag{1}$$
$$\text{subject to}\quad \dot{x}(t) \in F(t, x(t)),\ \mathcal{L} - a.e. \tag{2}$$
$$(x(0), \xi) \in C_0 \times C_\infty, \tag{3}$$
$$h(t, x(t)) \le 0,\ \forall t \ge 0, \tag{4}$$
$$\xi = \lim_{t\to\infty} x(t), \tag{5}$$

where $C_0 \subset \boldsymbol{R}^n$ and $C_\infty \subset \boldsymbol{R}^n$, $g : \boldsymbol{R}^n \to \boldsymbol{R}$, $F : [0, \infty) \times \boldsymbol{R}^n \to \mathcal{P}(\boldsymbol{R}^n)$, $h : [0, \infty) \times \boldsymbol{R}^n \to \boldsymbol{R}^q$ satisfy the following set of basic assumptions:

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B1 g is continuously differentiable.

B2 The compact and convex valued set-valued map F is measurably Lipschitz, i.e., F is Lebesgue measurable with respect to time and, $\forall\, t \in [0,\infty)$, Lipschitz continuous in x in the Hausdorff sense with time independent Lipschitz constant.

B3 h is Lipschitz continuous in x and continuous in t.

B4 The endpoint state constraint sets C_0, and C_∞ are closed.

Although assumption $(B1)$ can be weakened to mere Lipschitz continuity, we keep it in order to facilitate some developments discussed later in this article. With the $(B1)$ weakened to Lipschitz continuity, and $(B2)$ somewhat further weakened, $(B1)-(B4)$ are the standing assumptions usually considered for finite horizon problems.

The article is organized as follows. In the next section, we present some preliminary concepts and definitions. Some of these concern additional assumptions on the data of the problem, as well as the class of solutions on which the optimization is to be considered, and some others concern a refinement of the framework for the new concept of boundary condition first introduced in [6]. Then, in section 3, we present and discuss the necessary conditions of optimality, as well as some specific technical assumptions required on the data of the problem in order to prove the stated result. The main points of the proof are provided in its outline given in section 4 and some brief conclusions are given in the last section.

II. PRELIMINARY DEFINITIONS

In this section, we list a number of definitions required to formulate and prove our main result.

Since the optimization is carried out over all feasible control processes that converge asymptotically to some point in the infinite time horizon, we need to define equilibrium at infinity.

Definition 1. We say that the point $\xi \in \boldsymbol{R}^n$ is an equilibrium as $t \to \infty$ if $\exists$ a feasible trajectory $x(\cdot)$ such that

$$\lim_{t\to\infty} x(t) = \xi, \text{ and } 0 \in \liminf_{t\to\infty} F(t, x(t)),$$

where the limit is in the sense of Hausdorff and the limit set is assumed to be nonempty.

Definition 2. A trajectory is said to be feasible if it satisfies $x(0) \in C_0$, $\dot{x}(t) \in F(t, x(t))$, $\mathcal{L}$-a.e., $\lim_{t\to\infty} x(t) = \xi$ for some equilibrium $\xi \in C_\infty$ and $h(t, x(t)) \leq 0$ for all $t \geq 0$.

Definition 3. The state constraints are said to be compatible with the end-point constraints at the pair $(\bar{x}(0), \bar{\xi})$ if any $(x(0), \xi) \in (C_0 \times C_\infty) \cap \left[(\bar{x}(0), \bar{\xi}) + \varepsilon B_{2n}\right]$ with $\xi = \lim_{t\to\infty} x(t)$, satisfies $h(0, x(0)) \leq 0$, and $\lim_{t\to\infty} h(t, x(t)) \leq 0$ for some $\varepsilon > 0$.

Definition 4. The state constraints are regular if $\forall (t, x) \in \boldsymbol{R}^{n+1}$, such that $h(t, x) \leq 0$, $\exists \pi \in \boldsymbol{R}^n$ satisfying $\langle \pi, \zeta^i \rangle > 0$, $\forall \zeta^i \in \partial_x h^i(t, x)$, $\forall i$ such that $h_i(t, x) = 0$. Here, the generalized derivative $\partial_x h^i(t, x)$ is considered in the sense of Clarke, (see [2] for details).

These two last definitions correspond to technical assumptions to be imposed on the data of the problem enabling the derivation of nondegenerate nontrivial multipliers.

In order to capture the behavior of the adjoint variable as time goes to infinity, we specify the final endpoint transversality condition in terms of directional inclusion at infinity. This will call for a number of concepts introduced in [10] enabling to deal with the extended $\boldsymbol{R}^n$.

We consider a direction to be a ray, i.e., a closed half-line emanating from the origin. We think of rays as abstract direction points which lie beyond $\boldsymbol{R}^n$ and form the horizon of $\boldsymbol{R}^n$, denoted by hzn$\boldsymbol{R}^n$. We represent a direction point by dir$x$, where x is any nonzero vector in the ray representing the direction point in question. The cosmic space csm $\boldsymbol{R}^n$ is the union of the $\boldsymbol{R}^n$ with its horizon hzn $\boldsymbol{R}^n$. With this definition, it becomes clear that the cosmic $\boldsymbol{R}^n$ is a compact space.

A sequence of points $x_k \in \boldsymbol{R}^n$ converges to a direction point dir x, written $x_k \to$ dir x, $x \neq 0$, if $\lambda_k x_k \to x$ for some choice of $\lambda_k \searrow 0$, i.e., $\lambda_k > 0$ and $\lambda_k \to 0$.

Given a set $C \subset \boldsymbol{R}^n$, the cosmic closure closure of C is given by

$$\text{csm}C := \text{cl}C \cup \text{hzn}C,$$

where cl$C$ is the usual closure of C in $\boldsymbol{R}^n$ while the hzn$C$ is the collection of all direction points obtained with limits of sequences of points in C.

Given a cone $K \subset \boldsymbol{R}^n$, denote the set of direction points defined by the rays of K by dir$K$.

For a given nonempty set C in $\boldsymbol{R}^n$, the horizon cone representing the direction set hzn$C$, is defined by

$$C^\infty = \{x : \exists x_k {\in} C, \lambda_k {\searrow} 0, \text{ with } \lambda_k x_k \to x\}.$$

Observe that C is bounded if and only if $C^\infty = \{0\}$. With this notation, we have that hzn$C$ = dirC^∞ and csm$C$ = cl$C \cup$ dirC^∞.

A subset of csm $\boldsymbol{R}^n$, written as $C \cup$dir$K$, for a set $C \subset \boldsymbol{R}^n$ and a cone $K \subset \boldsymbol{R}^n$, is closed in csm $\boldsymbol{R}^n$ if C and K are closed in $\boldsymbol{R}^n$ and $C^\infty \subset K$. The cosmic closure of $C \cup$dir$K$ is

$$\text{csm}(C \cup \text{dir}K) = \text{cl}C \cup \text{dir}(C^\infty \cup \text{cl}K).$$

Now, we are in position to define the concept of directional inclusion at infinity. This enables us to state boundary conditions involving variables which may either become unbounded or persist in a certain set as time goes to infinity.

Definition 5. Let $y : [0,\infty) \to \boldsymbol{R}^n$ be a continuous function. Let $\boldsymbol{P}(y) := \boldsymbol{P}_L(y) \cup \text{dir}\boldsymbol{P}_\infty(y)$, also alluded to as the set of persistency points of y, where

- $\boldsymbol{P}_L(y) := \{\xi {\in} \boldsymbol{R}^n : \exists\ t_i \to \infty, \lim_{i\to\infty} y(t_i) = \xi\}$
- dir$\boldsymbol{P}_\infty(y) {:=} \{\xi {\in} \boldsymbol{R}^n {:} \exists t_i {\to} \infty, \lambda_i {\searrow} 0, \lim_{i\to\infty} \lambda_i y(t_i) {=} \xi\}$.

Given a function $y : [0,\infty) \to \boldsymbol{R}^n$ and a set $C \subset \boldsymbol{R}^n$ we say that y satisfies the weak directional inclusion in C at ∞

if $\boldsymbol{P}(y) \cap \text{csm}\, C \neq \emptyset$. In order to shorten the notation, this relation will be referred to by $y \in_\infty^* C$.

III. NECESSARY CONDITIONS OF OPTIMALITY

This problem is cast in the context of nonsmooth analysis (see [3]) due to both the assumptions on its data and the approach used to derive the optimality conditions.

We consider the following additional assumptions on the data of the problem:

A1 The state and endpoint constraints are compatible (see Definition 3).

A2 The state constraints are regular (see Definition 4).

A3 The $\liminf_{t\to\infty} F(t, x(t))$ exists in the sense of Hausdorff and is denoted by $F_\infty(\xi)$, where $\xi := \lim_{t\to\infty} x(t)$.

A4 Let $\xi^* := \lim_{t\to\infty} x^*(t)$. There exists $\delta > 0$ such that, $\forall\, x \in \xi^* + \delta B$,

$$0 \in \text{Int} \liminf_{t\to 0} F(t, x^*(t)).$$

A5 $F_0(x^*(0)) := \liminf_{t\to 0} F(t, x^*(t))$ has nonempty interior, and $\exists\, v_0 \in \text{Int}F_0(x^*(0))$ satisfying:

$$\begin{cases} \text{either} & x^*(0) \in \text{Int}C_0, \\ \text{or} & \langle \zeta_0, v_0\rangle < 0,\ \forall \zeta_0 \in N_{C_0}(x^*(0)). \end{cases}$$

Conditions $(A1) - (A5)$ are additional technical assumptions required to prove the stated necessary conditions of optimality. While $[A1]$ and $[A2]$ ensures the derivation of a nonempty set of multipliers that do not degenerate, that is, they remain always informative, $(A3)$ reflects a kind of persistence of the velocity set at the limiting value of the state variable that enables the extraction of limits of the necessary conditions of optimality as time goes to infinity. Another technical property required in the proof is $(A4)$ which implies the controllability in a neighborhood of the optimal reference trajectory as time goes to ∞, and $(A5)$ reflecting an initial point controllability condition with respect to the initial state constraint. Conditions $(A3) - (A5)$ could be somewhat simplified if the continuity of F in t was assumed.

Our necessary conditions of optimality for (P) are stated in the form of a maximum principle and they involve the <u>Hamiltonian</u> defined as

$$H(t, x, p) := \sup\{\langle p, v\rangle : v \in F(t, x)\}.$$

The adjoint variable $p : [0, \infty) \to \boldsymbol{R}^n$ satisfies a boundary condition at $t = \infty$. This is stated as the existence of a non empty subset of its persistence points, $\boldsymbol{P}(p)$, on the cosmic closure of the right hand side set of the usual transversality conditions. Moreover p can be regarded as a subgradient of the value function V at the optimum state trajectory value, which, for the subset of trajectories being considered for optimization, is defined by

$$V(t, z) \;:=\; \text{Min}\{g(\xi) : \text{ all admissible } x \text{ on } [t, \infty) \text{ s.t. } x(t) = z,\ \lim_{\tau\to\infty} x(\tau) = \xi\}.$$

In particular, if p converges asymptotically to some point $\bar{p}$, then $\boldsymbol{P}(p) = \{\bar{p}\}$. If p approaches a limit cycle C_L at infinite time, then $\boldsymbol{P}(p) = C_L$. The pattern of realization of the limiting approach towards a given infinitely often visited set of points might not be periodic. Below, $\partial_x f$, $\partial_x^P f$ and $\partial_x^M f$ denote the generalized gradients of f with respect to x, in the Clarke (see [2]), proximal (see [2]), and Mordukhovich (also known as limiting gradient) (see [8]) senses, respectively.

Next, we state the main result of this article.

<u>Theorem</u>.

Let x^* be an optimal trajectory for problem (P).

Then, there exists a multiplier (p, ν, λ_0), with $\lambda_0 \geq 0$, $p \in AC([0, \infty), \boldsymbol{R}^n)$, and $\nu \in C^*([0, \infty), \boldsymbol{R}^{q+})$ supported on the set $\{t \geq 0 : h(t, x^*(t)) = 0\}$ and a ν-measurable function $\gamma : [0, \infty) \to \boldsymbol{R}^{n\times q}$, with $\gamma(t) \in \partial_x^> h(t, x^*(t))$, ν-a.e., satisfying:

a) $\lambda_0 + \|p\| + \|\nu\|_{TV} \neq 0$ (nontriviality).

b) $\exists p(0) \in N_{C_0}(x^*(0))$ for which there is a solution to

$$-\dot{p}(t) \in \partial_x H(t, x^*(t), \bar{p}(t)),\ \mathcal{L}\text{-a.e.},$$

satisfying:

$$(i) - \bar{p}(t) - \gamma(t)\nu(\{t\}) \in \lambda_0 \partial_x^P V(t, x^*(t)),$$
$$\mathcal{L} \text{ and } \nu\text{-a.e. on } [0, \infty);$$

and $(ii) \boldsymbol{P}(-\bar{p} - \tilde{\gamma}) \cap \text{csm}\left[\lambda_0 \partial^M g(\xi^*) + N_{C_\infty}(\xi^*)\right] \neq \emptyset$,

where $\bar{p}(t) = p(t) + \int_{[0,t)} \gamma(s)\nu(ds)$, the set valued map $\partial_x^> h(t, x)$ is defined as in [2] by

$$co\left\{\lim_{i\to\infty} \gamma_i : \gamma_i \in \partial_x h(t_i, x_i),\ t_i \to t, x_i \to x, h(t_i, x_i) > 0\right\},$$

and $\tilde{\gamma}$ is such that $\alpha\tilde{\gamma} \in \partial_x \tilde{h}(\xi^*)$ for some $\alpha > 0$, with $\tilde{h}(\xi^*) = \lim_{t\to\infty} h(t, x^*(t))$, being $\xi^* = \lim_{t\to\infty} x^*(t)$.

Remark that $\boldsymbol{P}(-\bar{p} - \tilde{\gamma}) \cap \text{csm}\left(\lambda_0 \partial^M g(\xi^*) + N_{C_\infty}(\xi^*)\right)$ can be interpreted as $\exists \zeta \in \lambda_0 \partial g(\xi^*) + N_{C_\infty}(\xi^*)$ for which

- either $\zeta \in \boldsymbol{P}_L(-\bar{p})$, if $\bar{p}$ is bounded,
- or $\zeta \in \text{dir}\boldsymbol{P}^\infty(\bar{p})$, otherwise.

The information provided by this concept is certainly weaker than the one given by the boundary condition of the adjoint variable for finite time horizon dynamic optimization problems. In general, there are many functions p that persist in an absolute or a directional sense towards a point of $\lambda_0 \partial^M g(\xi^*) + N_{C_\infty}(\xi^*)$ at infinite time. Nevertheless, this information is still useful in delimiting the number of multipliers which satisfy the maximum condition.

IV. OUTLINE OF THE PROOF

The proof is based on considering a sequence of auxiliary finite time horizon optimal dynamic optimization problems without state constraints approximating (P) for which known results can be applied to yield an associated sequence of multipliers satisfying necessary conditions of optimality. Then, under the assumptions considered here, we are able to extract a subsequence of multipliers converging in a certain sense to another one satisfying the conditions stated in our main result.

Take $\{T_k\}$, $T_k \uparrow \infty$ and consider the following auxiliary problem.

$$
\begin{aligned}
(P_{T_k})\ \text{Minimize}\quad & V(T_k, x(T_k)) \\
\text{subject to}\quad & \dot{x}(t) \in F(t, x(t)),\ \mathcal{L}\text{-a.e. in } [0, T_k], \\
& h(t, x(t)) \leq 0, \forall t \in [0, T_k], \\
& x(0) \in C_0,
\end{aligned}
$$

where $V(t,z) : [0,\infty) \times \boldsymbol{R}^n \to \boldsymbol{R}$ is defined by

$$
\begin{aligned}
V(t,z)\ := \min\{g(\xi) : \dot{x} \in F(\tau, x),\ \mathcal{L}\text{-a.e. in } [t,\infty), \\
h(\tau, x(\tau)) \leq 0,\ \forall \tau \geq 0,\ x(t) = z, \\
\lim_{\tau\to\infty} x(\tau) = \xi \in C_\infty\}.
\end{aligned}
$$

Notice that, by the principle of optimality, the optimal trajectory to (P_{T_k}), denoted by x_k^*, coincides with x^*, the optimal trajectory to (P) on $[0, T_k]$. Unfortunately, this auxiliary sequence of problems does not serve our purpose of supplying a sequence of multipliers from which a conveniently converging subsequence can be extracted. This is due to the facts that on the one hand, V is, in general, merely lower semi-continuous due to the presence of state constraints, and, on the other hand, measures are components in the multipliers associated with the sequence auxiliary problems from which a convergent subsequence has to be extracted.

Thus, an adequate sequence of auxiliary dynamic optimization optimal control problems overcoming these obstacles has to be constructed.

Consider some $\delta > 0$ and choose $T_k = \dfrac{1}{\delta}$. For $t \leq T_k$, let $\bar{h}(t,z) = \max\{0, h_1(t,z), \ldots, h_q(t,z)\}$, $\tilde{h}_t(x) = \sup_{\tau\in[t,T_k)} \{\bar{h}(\tau, x(\tau))\}$ and, for any feasible state trajectory x s.t. $\lim_{\tau\to\infty} x(\tau) = \xi$, for some $\xi \in C_\infty$, let

$$
\bar{g}_t^\delta(x) = g(\xi) + \frac{1}{\delta}\tilde{h}_t(x).
$$

Now, for $t \leq \dfrac{1}{\delta}$, let

$$
\begin{aligned}
V^\delta(t,z)\ = \min\{\bar{g}_t^\delta(x) : \dot{x} \in F(\tau, x) \text{ a.e. on } [t, \tfrac{1}{\delta}], \\
\lim_{\tau\to\infty} x(\tau) = \xi \in C_\infty,\ x(t) = z\}.
\end{aligned}
$$

In the limit above, x is, for $t > \dfrac{1}{\delta}$ a feasible trajectory for the original dynamical control system that converges asymptotically to ξ.

Notice that, now in the absence of state constraints and under our assumptions, $V^\delta(t,z)$ is Lipschitz continuous in z with a constant that depends on δ.

Let

$$
\begin{aligned}
S(t,z)\ := \{(x,\xi) : x{:}[t,\infty) \to \boldsymbol{R}^n,\ \dot{x}(\tau) \in F(\tau, x(\tau)), \\
x(t) = z,\ x(\tau) \to \xi,\ \text{for some } \xi \in C_\infty\}.
\end{aligned}
$$

By introducing an appropriate topology in $S(t,z)$, it is possible to prove that bounded subsets of $S(t,z)$ are sequentially compact. Indeed, for each compact set $K \subset [t,\infty)$, consider the norm $\|(x,\xi)\|_K := \|x\|_\infty + \|\xi\|$, where $\|x\|_\infty := \sum \sup_K \|x(s)\|$, and $\|\xi\|$ is the Euclidean norm. Endow $S(t,z)$ with the topology τ for which the convergence of a sequence $(x^N, \xi^N) \in S(t,z)$ to a pair $(x,\xi) \in S(t,z)$ means that $(x^N, \xi^N) \to (x,\xi)$ with respect to the norm $\|(x,\xi)\|_K$ for all compact set $K \subset [t,\infty)$. If $C_\infty \subset \boldsymbol{R}^n$ is closed, then the set of admissible arcs $S(t,z)$ with the τ topology just defined is a complete space. With the additional assumption that C_∞ is compact, we can now prove the following result.

Proposition 2. Under the above assumptions, we have that:

- $\bar{g}_t^\delta(\cdot)$ is continuous on $S(t,z)$ w.r.t. the τ topology and consequently $V^\delta(t,z)$ is everywhere finite;
- the value function $V^\delta(t,z)$ is lower semi-continuous in t; and
- if $\bar{g}_t^\delta(\cdot)$ is Lipschitz continuous, then so is $V^\delta(t,\cdot)$.

Moreover, it can be shown that $\exists \bar{\delta} > 0$ s.t. $\forall\, z \in x^*(t) + \bar{\delta}B \cap \{z \in \boldsymbol{R}^n : h(t,z) \leq 0\}$,

$$
\lim_{\delta\to 0} V^\delta(t,z) = V(t,z),
$$

where V is the value function defined for the original problem with state constraints.

Moreover, if x_t^* is the trajectory on $[t,\infty)$, with $\lim_{\tau\to\infty} x_t^*(\tau) = \xi^* \in C_\infty$, such that $V(t, x_t^*(t)) = g(\xi^*)$, and if x_t^δ is the trajectory on $[t,\infty)$, with $\lim_{\tau\to\infty} x_t^\delta(\tau) = \xi^\delta \in C_\infty$, such that $V^\delta(t, x_t^\delta(t)) = \bar{g}_t^\delta(x)$, then $\lim_{\delta\to 0} x_t^\delta = x_t^*$.

Now, for a sequence $\{\delta_k\}$, s.t. $\delta_k > 0$, and $\delta_k \downarrow 0$, let us consider the following auxiliary, standard finite horizon dynamic optimization problem.

$$
\begin{aligned}
(P^{\delta_k})\ \text{Minimize}\quad & V^{\delta_k}(1/\delta_k, x(1/\delta_k)) \\
\text{subject to}\quad & \dot{x}(\tau) \in F(\tau, x(\tau)),\ \mathcal{L}\text{-a.e. in } [0, 1/\delta_k], \\
& h(\tau, x(\tau) \leq 0, \forall t \in [0, 1/\delta_k], \\
& x(0) \in C_0,
\end{aligned}
$$

For any given $\delta_k > 0$ sufficiently small, there is $\varepsilon[:= \varepsilon(\delta_k)]$, such that x^* (the solution to the original infinite horizon problem restricted to the interval $[0, \frac{1}{\delta_k}]$ is a "ε-solution"to (P_k^δ). Since the underlying trajectory space can be endowed with a complete metric, Δ_k, we may apply Ekeland's variational principle and apply this well known proof methodology, [2]. Ekeland's theorem allows us to obtain a a trajectory $x^{\delta_k,\varepsilon}$ solving an auxiliary problem $(P^{\delta_k,\varepsilon})$ with cost functional $V^{\delta_k,\varepsilon}\left(\frac{1}{\delta_k}, x(\frac{1}{\delta_k})\right)$, which results from (P_k^δ) by an well known appropriate penalization, i.e.,

$$
V^{\delta_k,\varepsilon}\left(\frac{1}{\delta_k}, x(\frac{1}{\delta_k})\right) = V^\delta\left(\frac{1}{\delta_k}, x(\frac{1}{\delta_k})\right) + \varepsilon\Delta_k(x^{\delta_k}, x^{\delta_k,\varepsilon}).
$$

Now, we can apply, the standard necessary conditions of optimality for finite time dynamic optimization problems with dynamics given by differential inclusions and whose state trajectory satisfies both state constraints and endpoint constraints.

Let $x^{\delta_k,\varepsilon}$ be a solution to $(P^{\delta_k,\varepsilon})$.

Then, there exists a multiplier $(p^{\delta_k,\varepsilon},\nu^{\delta_k,\varepsilon},\lambda_0^{\delta_k,\varepsilon})$, with $p^{\delta_k,\varepsilon}\in AC(\boldsymbol{R}^+,\boldsymbol{R}^n)$, $\lambda_0^{\delta_k,\varepsilon}\geq 0$, and $\nu^{\delta_k,\varepsilon}\in C^*(\boldsymbol{R}^+,\boldsymbol{R}^{q+})$ supported on the set $\{t\in[0,\frac{1}{\delta_k}] : h(t,x^{\delta_k,\varepsilon}(t))\leq 0\}$, and a $\nu^{\delta_k,\varepsilon}$-measurable function $\gamma^{\delta_k,\varepsilon}:[0,\frac{1}{\delta_k}]\to\boldsymbol{R}^{n\times q}$, with $\gamma^{\delta_k,\varepsilon}(t)\in\partial_x^> h(t,x^{\delta_k,\varepsilon}(t))$, $\nu^{\delta_k,\varepsilon}$-a.e., satisfying:

1. $\lambda_0^{\delta_k,\varepsilon}+\|p^{\delta_k,\varepsilon}\|+\|\nu^{\delta_k,\varepsilon}\|_{TV}\neq 0$ (nontriviality).
2. $\exists p^{\delta_k,\varepsilon}(0)\in N_{C_0}(x^{\delta_k,\varepsilon}(0))$ for which there is a solution to

$$-\dot{p}^{\delta_k,\varepsilon}(t)\in\partial_x H(t,x^{\delta_k,\varepsilon}(t),\bar{p}^{\delta_k,\varepsilon}(t)),\ \mathcal{L}\text{-a.e.},$$

satisfying:

$$-\bar{p}^{\delta_k,\varepsilon}(t)-\gamma^{\delta_k,\varepsilon}(t)\nu^{\delta_k,\varepsilon}(\{\frac{1}{\delta_k}\}) \in\lambda_0^{\delta_k,\varepsilon}\partial_x V^{\delta_k,\varepsilon}\left(\frac{1}{\delta_k},x^{\delta_k,\varepsilon}(\frac{1}{\delta_k})\right).$$

where $\bar{p}^{\delta_k,\varepsilon}(t)=p^{\delta_k,\varepsilon}(t)+\int_{[0,t)}\gamma^{\delta_k,\varepsilon}(s)\nu^{\delta_k,\varepsilon}(ds)$.

It can be shown that $\lim_{\varepsilon,\delta_k\to 0}x^{\delta_k,\varepsilon}=x^*$ uniformly on any finite subinterval $[0,t]$. We can also show that the ingredients of the multiplier $p^{\delta_k,\varepsilon}$, $\nu^{\delta_k,\varepsilon}$, and $\lambda_0^{\delta_k,\varepsilon}$ converge in appropriate senses, as δ_k, and ε go to 0, respectively, to some p, ν and λ_0 that satisfy $-\bar{p}(t)+\gamma_k(t)\nu(\{t\})\in\lambda_0\partial_x^P V(t,x^*(t))$.

Now, we need to determine an the estimate of $\partial_x^P V^{\delta_k,\varepsilon}\left(\frac{1}{\delta_k},x^{\delta_k,\varepsilon}(\frac{1}{\delta_k})\right)$ in order to show our transversality conditions in the limit.

To simplify notation, let us put $T_k=\dfrac{1}{\delta_k}$ and drop the indexes δ_k and ε whenever there is no ambiguity.

Proposition 3. Under the assumptions $(H1)-(H6)$, we have that $\partial_x^P V(T_k,x^*(T_k))$ contains the set

$\{\bar{p}_k\in\boldsymbol{R}^n : \exists(\bar{p},\bar{\nu},\bar{\lambda})$ satisfying :

$(i)\ \|\bar{p}(\cdot)\|+\|\bar{\nu}\|+\bar{\lambda}\neq 0,\ \bar{\lambda}\geq 0$

$(ii)-\dot{\bar{p}}(t)\in\partial_x H(x^*(t),\tilde{p}(t)),\ [T_k,\infty)$-a.e.

$(iii)\ \bar{p}(T_k)=\bar{p}_k$

$(iv)\ \boldsymbol{P}(-\tilde{p}-\tilde{\gamma})\cap\text{csm}\left[\lambda_0\partial^M g(\xi^*)+N_{C_\infty}(\xi^*)\right]\neq\emptyset$

$(v)\ \dot{x}^*(t)$ maximizes in $F(t,x^*(t))$, $[T_k,\infty)$-a.e., the map $v\to\langle\tilde{p}(t),v\rangle\}$,

where

- $\alpha\tilde{\gamma}\in\partial_x\tilde{h}(\xi^*)$ for some $\alpha>0$, being $\xi^*=\lim_{t\to\infty}x^*(t)$ and $\tilde{h}(\xi^*)=\lim_{t\to\infty}h(t,x^*(t))$,
- $\tilde{p}(t)=\bar{p}(t)+\int_{[T_k,t)}\gamma(\tau)\nu(d\tau)$,
- $\gamma(t)\in\partial_x^> h(t,x^*(t))$, ν-a.e., and
- $\nu\in C^*([T_k,\infty),\boldsymbol{R}^q)$ is supported on the set

$$\{t\in[T_k,\infty) : h(t,x^*(t))=0\}.$$

Let $x\in AC([0,\infty);\boldsymbol{R}^n)$ be such that $x(T_k)=z$, $\dot{x}(t)\in F(t,x(t))$ a.e., $h(t,x(t))\leq 0$, $\forall t>T_k$, and $\lim_{t\to\infty}x(t)=\xi$ asymptotically. Then, by using the fact that g is assumed to be C_1, we have

$$g(\xi)=g(z)+\int_{T_k}^{\infty}\nabla g(x(t))\dot{x}(t)dt.$$

We also need an additional auxiliary variable y satisfying $\dot{y}=0$ with $y(T_k)\in C_\infty$ and also $\lim_{t\to\infty}(y(t)-x(t))=0$. Note that, since $\tilde{C}:=\{(x,y) : x=y\}$, we have that, for any $(x,y)\in\tilde{C}$, $N_{\tilde{C}}(x,y)=\{(\bar{p}_x,\bar{p}_y) : \bar{p}_x=-\bar{p}_y\}$.

Now, notice that $V(T_k,z)$ is the minimum cost of the following auxiliary optimal control problem

$$\begin{array}{ll}\text{Minimize} & \int_{T_k}^{\infty}\nabla g(x(t))\dot{x}(t)dt\\ \text{subject to} & \dot{x}(t)\in F(t,x(t)),\ \dot{y}(t)=0,\ [T_k,\infty)\text{-a.e.},\\ & h(t,x(t))\leq 0,\ \forall t\in[T_k,\infty),\\ & \lim_{t\to\infty}(x(t),y(t))\in\tilde{C},\\ & (x(T_k),y(T_k))\in\{z\}\times C_\infty.\end{array}$$

Observe that the generalized gradient of V with respect to x at time T_k at $x^*(T_k)$ is given by the set of values of the (symmetric of the) adjoint variable at time T_k. Remark also that the cost functional of this problem does not depend on state at the final time (∞). The final endpoint constraint does not cause any difficulty since it is affine in the state variable and always active.

By applying the maximum principle to this auxiliary problem, and, then, by expressing the obtained conditions in terms of the data of the original problem, it is straightforward to derive the intended characterization of the estimate of $\partial_x^P V\left(\frac{1}{\delta_k},x^*(\frac{1}{\delta_k})\right)$.

Indeed, we have

$$H(t,x,y,p_x,p_y,\lambda_0)=\sup_{v\in F(t,x)}\{\langle p_x,v\rangle-\lambda_0\nabla g(x)v\}$$

and, thus:

- $-\dot{p}_x(t)\in\partial_x H(t,x^*(t),y^*(t),\bar{p}_x(t),p_y(t),\lambda_0)$.
- $-\dot{p}_y(t)\equiv 0$, and $p_y(t)\equiv p_y(T_k)\in N_{C_\infty}(x^*(T_k))$.
- $\exists\{t_i\}$, $t_i\uparrow\infty$, $\exists\{\alpha_i\}$, $\alpha_i>0$, $\alpha_i\to\alpha_\infty\geq 0$, such that

$$\lim_{i\to\infty}\alpha_i p_x(t_i)=-p_y(T_k),$$

where $\bar{p}_x(t)=p_x(t)+\int_{[T_k,t)}\gamma(\tau)\nu(d\tau)$, being $\gamma(t)$ a ν-a.e. measurable selection of $\partial_x^> h(t,x^*(t))$, with $\nu\in C^*([T_k,\infty);\boldsymbol{R}^q)$ supported on the set $\{t\in[T_k,\infty) : h(t,x^*(t))=0\}$.

Notice that the third item arises naturally from the fact that the adjoint equation relative to p_x, involving also λ_0, can be scaled down by some positive number.

Now, by putting $p(t)=p_x(t)-\lambda_0\nabla g(x^*(t))$, we have that

$$-\dot{p}(t)\in\partial_x H(t,x^*(t),p(t)),$$

and, by considering sequences $\{t_i\}$ and $\{\alpha_i\}$ with either $\alpha_\infty>0$ or $\alpha_\infty=0$ we have the stated transversality conditions.

To complete the proof of Theorem 1, it is enough to show that the desired conditions are obtained as the limit of the necessary conditions derived for (P_{T_k}). By once more recalling the principle of optimality, the properties of V, and by using the characterization of the estimate of its generalized gradient in a proximal sense derived in the above proposition, we readily obtain the desired conclusions, i.e., the necessary conditions of optimality for (P).

V. CONCLUSIONS

In this article, necessary conditions of optimality in the form of the Hamiltonian inclusions and featuring a novel transversality condition were given for an infinite horizon dynamic optimization problem with dynamics given by a differential inclusion and whose state trajectories have to satisfy state constraints, endpoint constraints, and are assumed to converge asymptotically to an equilibrium point is constrained to a given closed set. This result extends previous work of the author for optimal control problems without state constraints. Various comments relating the obtained result are included.

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