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Bifurcation Analysis of a Heterogeneous Mean-Field Oscillator Game Model

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Abstract— **This paper studies the phase transition in a heterogeneous mean-field oscillator game model using methods from bifurcation theory. In our earlier paper [1], we had obtained a coupled PDE model using mean-field approximation and described linear analysis of the PDEs that suggested possibility of a Hamiltonian Hopf bifurcation. In this paper, we simplify the analysis somewhat by relating the solutions of the PDE model to the solutions of a certain nonlinear eigenvalue problem. Both analysis and computations are much easier for the nonlinear eigenvalue problem. Apart from the bifurcation analysis that shows existence of a phase transition, we also describe a Lyapunov-Schmidt perturbation method to obtain asymptotic formulae for the small amplitude bifurcated solutions. For comparison, we also depict numerical solutions that are obtained using the continuation software AUTO.**

I. Introduction

The dynamics of a large population of coupled heterogeneous nonlinear systems is of interest in a number of applications, including neuroscience, communication networks, power systems, and economic markets. Game theory provides a powerful set of tools for analysis and design of strategic behavior in controlled multi-agent systems. In economics, for example, game-theoretic techniques provide a foundation for analyzing the behavior of rational agents in markets.

In practice, a fundamental problem is that controlled multi-agent systems can exhibit phase transitions with often undesirable outcomes. In economics, an example of this is the so-called *rational irrationality*: "behavior that, on the individual level, is perfectly reasonable but that, when aggregated in the marketplace, produces calamity [2]".

A prototypical example of a multi-agent heterogeneous nonlinear system that exhibits phase transition is the coupled oscillator model of Kuramoto [3]. Motivated by the Kuramoto model, we introduced a *mean-field oscillator game* model in our earlier paper [1]. The model comprises of a large number (N) of oscillators that are coupled via performance objectives in a non-cooperative game.

In general, establishing the existence and uniqueness of Nash equilibrium for large N is an intractable problem. The key idea for complexity mitigation is to use the *Nash Certainty Equivalence* (NCE) principle introduced in the seminal work of Huang et. al. [4], wherein the i^{th} oscillator optimizes by using local information consisting of (i) its own state (θ_i) and (ii) the *mass-influence* of the population. The mass influence arises as a result of mean-field approximation: In the limit of large population size (as $N \to \infty$), the population affects the i^{th} oscillator in a nearly deterministic fashion. The NCE based analysis results in a coupled PDE model which comprises of a backward PDE, the Hamilton-Jacobi-Bellman (HJB) equation, coupled with a forward PDE, the Fokker-Planck-Kolmogorov (FPK) equation.

In this paper, we use the coupled PDE model for analysis of phase transition in a *heterogeneous* mean-field oscillator game. The phase transition is important in a number of applications [5]. For example, in thalamocortical circuits in the brain, transition to the synchronized state is associated with diseased brain states such as epilepsy [6].

The phase transition is studied by using methods from bifurcation theory. In our earlier paper [1], we had described linear analysis of the coupled PDE model that suggested possibility of a Hamiltonian Hopf bifurcation. In this paper, we simplify the analysis somewhat by relating the solutions of the PDE model to the solutions of a certain nonlinear eigenvalue problem. Both analysis and computations are much easier for the nonlinear eigenvalue problem. In particular, the bifurcation result reduces to a straightforward application of the standard steady state bifurcation theorem.

Apart from the bifurcation analysis that shows existence of a phase transition, we also describe a Lyapunov-Schmidt perturbation method to obtain asymptotic formulae for the small amplitude bifurcated solutions. For comparison, we obtain numerical solutions of the nonlinear eigenvalue problem. The numerical solutions are obtained using the continuation software AUTO.

The remainder of the paper is organized as follows: In Sec. II, we briefly introduce the mean-field oscillator game problem and the coupled PDE model. The solutions of the PDE model are related to a nonlinear eigenvalue problem. In Sec. III, we describe the bifurcation and perturbation analyses of the nonlinear eigenvalue problem via Lyapunov-schmidt reduction. Finally, we discuss the numerical results obtained by using AUTO in Sec. IV.

II. The mean-field oscillator game

We consider a set of N oscillators, denoted by $\mathscr{N} := \{1,\ldots,N\}$. The dynamics for the i^{th} oscillator is described by the stochastic differential equation (SDE):

$$\mathrm{d}\theta_i(t) = (\omega_i + u_i(t))\,\mathrm{d}t + \sigma\,\mathrm{d}\xi_i(t), \qquad (1)$$

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where $\theta_i(t) \in [0, 2\pi]$ is the phase of the i^{th} oscillator at time t, $u_i(t)$ is the control input, and $\{\xi_i(t),\, i \in \mathcal{N}\}$ are mutually independent standard Wiener processes. The frequencies $\{\omega_i\}$ are chosen independently according to a fixed distribution with density g, which is supported on an interval of the form $\Omega = [1-\gamma, 1+\gamma]$ for some $\gamma < 1$. In this paper we consider the uniform distribution, i.e., $g(\omega) = \frac{1}{2\gamma}$.

We consider an $N-$player noncooperative game, denoted by $\mathcal{G}_N$, where we assume that the i^{th} oscillator minimizes its own performance objective, given the decisions of (competing) oscillators:

$$\eta_i^{\text{(POP)}}(u_i; u_{-i}) = \lim_{T\to\infty} \frac{1}{T} \int_0^T [c(\theta_i; \theta_{-i}) + \tfrac{1}{2} R u_i^2]\, \mathrm{d}s, \qquad (2)$$

$u_{-i} = (u_j)_{j\neq i}$, $R > 0$ denotes the control penalty, $\theta_{-i} = (\theta_j)_{j\neq i}$, $c(\cdot)$ is the cost function of the following separable form:

$$c(\theta_i; \theta_{-i}) := \frac{1}{N} \sum_{j\neq i} c^\bullet(\theta_i, \theta_j(t)), \qquad (3)$$

and the following assumption is made for $c^\bullet$:

Assumption (A1) The function $c^\bullet$ introduced in (3) is assumed to be a bounded function that is

1) spatially invariant, i.e., $c^\bullet(\vartheta, \theta) = c^\bullet(\vartheta - \theta)$,
2) 2π-periodic, i.e., $c^\bullet(\theta) = c^\bullet(\theta + 2\pi)$,
3) non-negative, i.e., $c^\bullet(\theta) \geq 0$,
4) even, i.e., $c^\bullet(\theta) = c^\bullet(-\theta)$.

We write the Fourier series of the cost function as

$$c^\bullet(\theta) = C_0^\bullet + \sum_{k=1}^{\infty} C_k^\bullet \cos(k\theta).$$

As an example, the function

$$c^\bullet(\theta, \vartheta) = \frac{1}{2} \sin^2\left(\frac{\theta - \vartheta}{2}\right) \qquad (4)$$

satisfies the assumption (A1) with $C_0^\bullet = \frac{1}{4}$, $C_1^\bullet = -\frac{1}{4}$ and $C_k^\bullet = 0$ for $k = 2, 3, \ldots$.

The form of the function c and the value of R are assumed to be common to the entire population. A *Nash equilibrium* in control policies is given by $\{u_i^*\}_{i\in\mathcal{N}}$ such that u_i^* minimizes $\eta_i^{\text{(POP)}}(u_i; u_{-i}^*)$ for $i = 1, \ldots, N$.

Our interest in this paper is in the large-population limit, where $N \to \infty$. We denote the limiting dynamic game as $\mathcal{G}_\infty$. As shown in [1], a mean-field approximation leads to the following PDE-based characterization of the solutions

$$\partial_t h + \omega \partial_\theta h = \frac{1}{2R}(\partial_\theta h)^2 - \bar{c}(\theta, t) + \eta^* - \frac{\sigma^2}{2} \partial^2_{\theta\theta} h, \qquad (5)$$

$$\partial_t p + \omega \partial_\theta p = \frac{1}{R} \partial_\theta \left[p(\partial_\theta h)\right] + \frac{\sigma^2}{2} \partial^2_{\theta\theta} p, \qquad (6)$$

$$\bar{c}(\theta, t) = \int_\Omega \int_0^{2\pi} c^\bullet(\theta, \vartheta) p(\vartheta, t, \omega) g(\omega)\, \mathrm{d}\vartheta\, \mathrm{d}\omega, \quad (7)$$

where $h(\theta, t, \omega)$ is the relative value function, $p(\theta, t, \omega)$ is intended to approximate probability density of the random variable $\theta_i(t)$, evolving according to the SDE (1) with the optimal control function

$$u(\theta, t; \omega) = -\frac{1}{R} \partial_\theta h(\theta, t; \omega). \qquad (8)$$

In this paper, we restrict our attention to solutions of the following type:

$$\begin{aligned} p(\theta, t; \omega) &= \tilde{p}(\tilde{\theta}; \omega), \\ h(\theta, t; \omega) &= \tilde{h}(\tilde{\theta}; \omega), \\ u(\theta, t; \omega) &= \tilde{u}(\tilde{\theta}; \omega), \end{aligned} \qquad (9)$$

where $\tilde{\theta} = \theta - at$, a denotes the wave speed, and $\tilde{p}$, $\tilde{h}$ are 2π-periodic functions with respect to $\tilde{\theta}$. The form (9) leads to the following two types of solution:

1) If $a = 0$, the solution is referred to as the *equilibrium* solution in which the cost function, the relative value function, and the density are independent of time.
2) If $a > 0$, the solution is referred to as the *periodic* solution in which $p(\theta, t; \omega)$, $h(\theta, t; \omega)$, and $\bar{c}(\theta, t)$ are periodic in time, with period $\tau = 2\pi/a > 0$.

The equilibrium and periodic solutions are considered for the following reasons:

1) These solutions define approximate Nash equilibrium of the game with a finite large number of oscillators [1].
2) For certain values of the parameter R, these solutions represent the steady-state solutions of the PDE model.
3) These solutions potentially represent the incoherence and synchrony solutions described in the coupled oscillators literature [5], [7].

For solutions of type (9),

$$\begin{aligned} \partial_t h + \omega \partial_\theta h &= (\omega - a) \partial_{\tilde{\theta}} \tilde{h}, \\ \partial_t p + \omega \partial_\theta p &= (\omega - a) \partial_{\tilde{\theta}} \tilde{p}. \end{aligned}$$

Using FPK equation (6), we obtain a formula for optimal control in terms of density function $\tilde{p}$.

Lemma 1: Suppose (p, h) is a solution of the coupled PDE model (5) - (7) of the type (9). Then the optimal control function (see (8)) is given by

$$\tilde{u}(\tilde{\theta}; \omega) = \frac{\sigma^2}{2} \partial_{\tilde{\theta}} \ln \tilde{p} + (a - \omega) \left(1 - \frac{2\pi}{\tilde{p} \int_0^{2\pi} (\tilde{p})^{-1}\, \mathrm{d}\tilde{\theta}}\right), \quad (10)$$

where $\tilde{\theta} = \theta - at$.

Proof: Using the ansatz (9), Eqn. (6) can be written as

$$(\omega - a) \partial_{\tilde{\theta}} \tilde{p} = -\partial_{\tilde{\theta}} [\tilde{p}\tilde{u}] + \frac{\sigma^2}{2} \partial^2_{\tilde{\theta}\tilde{\theta}} \tilde{p}.$$

Integrating both sides of the equation with respect to $\tilde{\theta}$, we have

$$\tilde{u} = \frac{\sigma^2}{2} \frac{\partial_{\tilde{\theta}} \tilde{p}}{\tilde{p}} + (a - \omega) + \frac{K(\omega)}{\tilde{p}}, \qquad (11)$$

where K denotes the constant of integration. Integrating both sides of the resulting equation (11) from 0 to 2π once more, we obtain

$$\int_0^{2\pi} \tilde{u}\, \mathrm{d}\tilde{\theta} = \int_0^{2\pi} \frac{\sigma^2}{2} \partial_{\tilde{\theta}} \ln \tilde{p}\, \mathrm{d}\tilde{\theta} + K \int_0^{2\pi} \frac{1}{\tilde{p}}\, \mathrm{d}\tilde{\theta} + (a - \omega) 2\pi.$$

Using the fact that $\tilde{h}$ and $\tilde{p}$ are 2π-periodic in $\tilde{\theta}$, we have

$$0 = 0 + K \int_0^{2\pi} \frac{1}{\tilde{p}}\, \mathrm{d}\tilde{\theta} + (a - \omega) 2\pi,$$

which implies

$$K(\omega)=\frac{(\omega-a)2\pi}{\int_0^{2\pi}(\tilde{p})^{-1}\,\mathrm{d}\tilde{\theta}}. \tag{12}$$

We obtain the result (10) by substituting (12) back in (11). ■

Since $\tilde{p}$ is non-negative valued, we write $\tilde{p}(\theta;\omega)=v^2(\theta,\omega)$; for notational ease, we drop "$\sim$" in the remainder of this paper. Using Eqn. (10) and Eqn. (8), we can simplify the coupled PDE model (5)-(7) to a nonlinear eigenvalue problem:

$$G(v,\eta,R,a):=\partial^2_{\theta\theta}v+\frac{2}{R\sigma^4}(\eta-\bar{c})v \\ -\frac{(\omega-a)^2}{\sigma^4}\left(1-\left(\frac{2\pi}{v^2\int v^{-2}}\right)^2\right)v=0, \tag{13}$$

$$B(v):=\int_0^{2\pi}v^2(\theta,\omega)\,\mathrm{d}\theta-1=0, \tag{14}$$

where $\bar{c}$ is defined as

$$\bar{c}(\theta)=\int_\Omega\int_0^{2\pi}c^\bullet(\theta,\vartheta)v^2(\vartheta,\omega)g(\omega)\,\mathrm{d}\vartheta\,\mathrm{d}\omega. \tag{15}$$

By solving the eigenvalue problem (13)-(15), one can obtain solution of coupled PDE model (5)-(7). The result is summarized in the following without proof.

Lemma 2: Suppose (h,p,η^*) is a traveling wave solution of the form (9) of the coupled PDE model (5)-(7) with wave-speed 'a'. Let $v=\sqrt{p}$. Then (v,η^*) is the solution of the nonlinear eigenvalue problem (13)-(15). Conversely, suppose (v,η^*) is the solution of the nonlinear eigenvalue problem (13)-(15). Let

$$p(\theta,t,\omega)=v^2(\theta-at,\omega),$$

and h satisfy

$$\partial_\theta h=-\frac{R\sigma^2}{2}\partial_\theta\ln p-R(a-\omega)\left(1-\frac{2\pi}{p\int p^{-1}}\right).$$

Then (h,p,η^*) is a solution to the PDEs (5)-(7). ■

In the remainder of the paper, we consider solutions of the eigenvalue problem (13)-(15). The solutions of coupled PDEs (5)-(7) are obtained by using Lemma 2.

III. Bifurcation analysis

In this section, we describe bifurcation analysis for the nonlinear eigenvalue problem (13)-(15).

We denote $\mathbb{T}:=[0,2\pi]$, $\mathbf{X}:=C^2_{2\pi}(\mathbb{T}\times\Omega,\mathbb{R})$, the space of twice continuously differentiable real-valued periodic functions on $\mathbb{T}\times\Omega$, $\mathbf{Y}:=C^0_{2\pi}(\mathbb{T}\times\Omega,\mathbb{R})$ and $\mathbf{E}$ as the space of functions $\eta:\Omega\to\mathbb{R}^+$. The nonlinear maps (13)-(14), $G:\mathbf{X}\times\mathbf{E}\times\mathbb{R}^2_+\to\mathbf{Y}$ and $B:\mathbf{X}\to\mathbb{R}$. For any fixed $R\in\mathbb{R}^+$, we are interested in obtaining *solutions* $(v,\eta,a)\in\mathbf{X}\times\mathbf{E}\times\mathbb{R}^+$ such that $G(v,\eta,R,a)=0$ and $B(v)=0$.

We begin by noting that there is a trivial solution given by

$$v(\theta,\omega)=v_0:=\frac{1}{\sqrt{2\pi}}, \\ \eta(\omega)=\eta_0:=C^\bullet_0=\frac{1}{2\pi}\int_0^{2\pi}c^\bullet(\theta)\,\mathrm{d}\theta. \tag{16}$$

About the trivial solution, the linearization of G in (13) is given by

$$\mathscr{L}(R,a)\tilde{v}:=\partial^2_{\theta\theta}\tilde{v}-\frac{2}{\sigma^4R}\frac{1}{\pi}\int_\mathbb{T}\int_\Omega c^\bullet(\theta,\vartheta)\tilde{v}(\theta,\omega)g(\omega)\,\mathrm{d}\omega\,\mathrm{d}\theta \\ -\frac{4(\omega-a)^2}{\sigma^4}\left(\tilde{v}-v_0^2\int_\mathbb{T}\tilde{v}\right), \tag{17}$$

with $\tilde{v}\in\mathbf{X}$. The spectrum of the linear operator $\mathscr{L}(R,a):\mathbf{X}\to\mathbf{Y}$ is summarized in the following:

Theorem 3: For the linear operator $\mathscr{L}(R,a):\mathbf{X}\to\mathbf{Y}$,

(i) The continuous spectrum equals the union of sets $\{S_c^{(k)}\}_{k=1,2,\ldots}$

$$S_c^{(k)}:=\{\lambda\in\mathbb{R}|\lambda=-k^2-\frac{4}{\sigma^4}(\omega-a)^2 \text{ for all } \omega\in\Omega\}.$$

(ii) The discrete spectrum equals the union of sets $\{S_d^{(k)}\}_{k=1,2,\ldots}$. We have the following two cases:

1) If $C^\bullet_k=0$, the set $S_d^{(k)}$ is empty and
2) if $C^\bullet_k\neq0$,

$$S_d^{(k)}:=\left\{\lambda\in\mathbb{C}\Big|\frac{C^\bullet_k}{2R}\int_\Omega\frac{g(\omega)}{(\omega-a)^2+\frac{\sigma^4}{4}(k^2+\lambda)}\,\mathrm{d}\omega=-1\right\}.$$

The eigenspace for the k^{th} eigenvalue, $\lambda_k\in S_d^{(k)}$, is given by $\operatorname{span}\{\cos(k\theta),\sin(k\theta)\}$. ■

As the parameter R varies, the potential bifurcation points are where a discrete eigenvalue crosses zero. The k^{th} such bifurcation point is given by

$$R=-\frac{C^\bullet_k}{2}\int_\Omega\frac{g(\omega)}{(\omega-a)^2+\frac{\sigma^4}{4}k^2}\,\mathrm{d}\omega,$$

provided $C^\bullet_k<0$. In the following, we consider the first bifurcation point where $k=1$ (we assume $C^\bullet_1<0$). The analysis for other bifurcation points is similar.

The first bifurcation point is given by,

$$r_0(a):=-\frac{C^\bullet_1}{2}\int_\Omega\frac{g(\omega)}{\rho^2(\omega,a)}\,\mathrm{d}\omega, \tag{18}$$

where $\rho(\omega,a):=\sqrt{(\omega-a)^2+\sigma^4/4}$. Note that $r_0(a)$ achieves its maximum value at $a=a_0:=1$. Fig. 1 depicts the plot of $r_0(a)$ in the neighborhood of a_0 for $c^\bullet=\frac{1}{2}\sin^2(\frac{\theta-\vartheta}{2})$. We denote

$$R_c:=r_0(a_0)$$

and have the following eigen-speed property at (R_c,a_0). The proof appears in Appendix V-A:

Lemma 4: Suppose $\lambda(R,a)\in S_d^{(1)}$ is the 1st discrete eigenvalue of linear operator $\mathscr{L}(R,a)$. Then

$$\frac{\partial\lambda}{\partial R}(R_c,a_0)=-\frac{32}{\sigma^4K(a_0)}<0, \tag{19}$$

where

$$K(a):=\int_\Omega\frac{1}{\rho^4(\omega,a)}g(\omega)\,\mathrm{d}\omega. \tag{20}$$

■

As R decreases from a large value, a (double) real-valued eigenvalue, λ_1, crosses imaginary axis with non-zero speed.

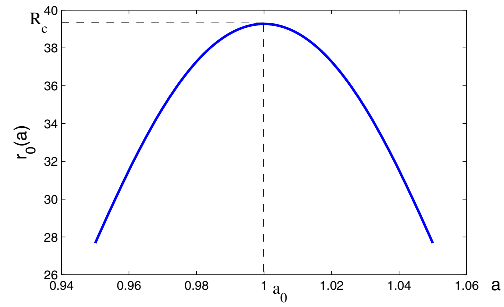


Fig. 1. The plot of r_0 as a function of a when $C_1^\bullet = -\frac{1}{4}$.

One would thus expect existence of a bifurcated solution branch [8]. One problem, however, is that the eigenvalue is double. This is dealt with by using the symmetry properties. In particular, because of Assumption (A1), G is equivariant with respect to the spatial symmetry group $O(2)$:

$$SO(2): \quad G(v(\theta+\vartheta),\eta,R,a) = G(v,\eta,R,a)(\theta+\vartheta), \quad (21)$$
$$Z^2: \quad G(v(-\theta),\eta,R,a) = G(v,\eta,R,a)(-\theta). \quad (22)$$

These two properties (21)-(22) allow us to look for solutions with respect to even (or odd) functions $v(\theta,\omega)$. In particular, denote $\mathbf{X}^e := \{v \in \mathbf{X} : v(\theta,\omega) = v(-\theta,\omega)\}$ and similarly for $\mathbf{Y}^e$. Then we have

$$G : \mathbf{X}^e \to \mathbf{Y}^e, \text{ and } B : \mathbf{X}^e \to \mathbb{R}. \quad (23)$$

Henceforth, we consider these restricted map and seek solutions of

$$G(v,\eta,R,a) = 0 \quad \text{and} \quad B(v) = 0$$

for $(v,\eta,R,a) \in \mathbf{X}^e \times \mathbf{E} \times \mathbb{R}_+^2$ near (v_0,η_0,R_c,a_0). Using property (21), a more general family of solutions is obtained by applying an arbitrary phase shift $\vartheta \in \mathbb{R}(\text{mod } 2\pi)$ to these even solutions.

A. The Lyapunov-Schmidt reduction

We define the inner product for any $v,w \in \mathbf{Y}^e$ as

$$\langle v,w\rangle := \frac{1}{\pi}\int_{\mathbb{T}}\int_{\Omega} v(\theta,\omega)w(\theta,\omega)g(\omega)\,\mathrm{d}\omega\,\mathrm{d}\theta. \quad (24)$$

One can verify that the operator $\mathscr{L}(R,a)$ is self-adjoint with respect to this inner product because the convolution kernel $c^\bullet$ is spatially invariant and even. Define the function

$$\phi(\theta,\omega,a) := \frac{1}{\sqrt{K(a)}}\frac{1}{\rho^2(\omega,a)}\cos(\theta), \quad (25)$$

where $\rho(\omega,a)$ is defined in (18) and $K(a)$ is defined in (20). Then, $\mathscr{L}(r_0,a)\phi = 0$ and $\langle\phi,\phi\rangle = 1$.

We denote

$$\mathscr{L}_o := \mathscr{L}(R_c,a_0) \text{ and } \zeta = \phi(\theta,\omega,a_0).$$

Then $\mathscr{L}_o\zeta = 0$. So the kernel of the operator $\mathscr{L}_o$ is given by

$$\ker(\mathscr{L}_o) = \{v : v = x\zeta, x \in \mathbb{R}\}. \quad (26)$$

Because $\mathscr{L}_o$ is self-adjoint, the range of $\mathscr{L}_o$ is given by

$$\mathbf{R}(\mathscr{L}_o) = \{y \in \mathbf{Y}^e : \langle y,\zeta\rangle = 0\}. \quad (27)$$

We consider the direct-sum decomposition:

$$\mathbf{X}^e = \ker(\mathscr{L}_o) \oplus \mathbf{X}_0,$$
$$\mathbf{Y}^e = R(\mathscr{L}_o) \oplus \ker(\mathscr{L}_o),$$

where $\mathbf{X}_0 := \{v \in \mathbf{X}^e : \langle v,\zeta\rangle = 0\}$. These decompositions define projection $P : \mathbf{Y}^e \to \ker(\mathscr{L}_o)$ through

$$Py := \langle\zeta,y\rangle\zeta, \quad \forall y \in \mathbf{Y}^e.$$

We rewrite Eqn. (13) as

$$PG(v_0 + x\zeta + w,\eta,R,a) = 0, \quad (28)$$
$$(I-P)G(v_0 + x\zeta + w,\eta,R,a) = 0, \quad (29)$$

where $w \in \mathbf{X}_0$. The range of $I-P$ equals $\mathbf{R}(\mathscr{L}_o)$. Now, $\mathscr{L}_o : \mathbf{X}_0 \to \mathbf{R}(\mathscr{L}_o)$ is invertible and by the implicit function theorem, Eqn. (29) can be solved uniquely for $w = \hat{w}(x,\eta,R,a)$ in some neighborhood of $(v,\eta,R,a) = (v_0,\eta_0,R_c,a_0) \subset \mathbf{X}^e \times \mathbf{E} \times \mathbb{R}_+^2$. Because $(v,\eta) = (v_0,\eta_0)$ solves Eqn. (29), it is clear that $\hat{w}(0,\eta_0,R,a) \equiv 0$, and it can also be shown that $\hat{w}_v(0,\eta_0,R_c,a_0) = 0$; cf. [9].

Substituting $\hat{w}$ into (28), we obtain a scalar equation,

$$PG(v_0 + x\zeta + \hat{w}(x,\eta,R,a),\eta,R,a) = 0.$$

Or explicitly,

$$s_1(x,\eta,R,a) := \langle\zeta, G(v_0 + x\zeta + \hat{w}(x,\eta,R,a),\eta,R,a)\rangle = 0.$$

Note by construction there exists a trivial solution $(x,\eta) = (0,\eta_0)$ such that $s_1(0,\eta_0,R,a) = 0$. Using the property (21), it also follows that

$$s_1(-x,\eta,R,a) = -s_1(x,\eta,R,a).$$

So, we write

$$s_1(x,\eta,R,a) = \tilde{s}_1(x^2,\eta,R,a)x.$$

The non-trivial solution of R is obtained by solving $\tilde{s}_1(x^2,\eta,R,a) = 0$. A direct calculation shows that

$$\begin{aligned}
\frac{\partial\tilde{s}_1}{\partial R}(0,\eta_0,R_c,a_0) &= \frac{\partial^2 s_1}{\partial x\partial R}(0,\eta_0,R_c,a_0) \\
&= \langle\zeta, \frac{\partial}{\partial R}\mathscr{L}(R,a)[\zeta + \hat{w}_v(0,\eta_0,R_c,a_0)\zeta]\rangle\Big|_{R=R_c,a=a_0} \\
&\quad + \langle\zeta, \mathscr{L}_o[\hat{w}_{v,R}(0,\eta_0,R_c,a_0)\zeta]\rangle \\
&= \langle\zeta, \frac{\partial}{\partial R}\mathscr{L}(R,a)[\zeta]\rangle\Big|_{R=R_c,a=a_0} \\
&= \frac{\partial\lambda}{\partial R}(R_c,a_0)\langle\zeta,\zeta\rangle < 0,
\end{aligned}$$

where the last inequality follows from (19). Using the implicit function theorem, we find a local branch of nontrivial solutions of $R = \hat{R}(x^2,\eta,a)$ in the neighborhood of $(0,\eta_0,a_0)$ such that $\hat{R}(0,\eta_0,a_0) = R_c$ and $\tilde{s}_1(x^2,\eta,\hat{R}(x^2,\eta,a),a) \equiv 0$.

Now, we consider the constraint $B(v) = 0$ to solve for η. Instead of solving it directly, we integrate both sides of

$G(v,\eta,R,a)=0$ from 0 to 2π and substitute $B(v)=0$. This results in an equivalent constraint

$$
\begin{aligned}
&B'(v,\eta,R,a)\\
&:=\eta - C_0^\bullet - \frac{1}{\int_0^{2\pi} v(\theta,\omega)\,\mathrm{d}\theta}\times\Bigg[\sum_{k=1}^{\infty} C_k^\bullet\int_0^{2\pi} v(\theta,\omega)\cos(k\theta)\,\mathrm{d}\theta\\
&\int_{\mathbb{T}}\int_{\Omega} v^2(\theta,\omega)\cos(k\theta)g(\omega)\,\mathrm{d}\omega\,\mathrm{d}\theta\\
&\quad -\frac{R}{2}(\omega-a)^2\int_{\mathbb{T}}\left(v(\theta,\omega)-\frac{4\pi^2}{v^3(\int v^{-2})^2}\right)\mathrm{d}\theta\Bigg]=0.
\end{aligned}
$$

By substituting $v=v_0+x\zeta+\hat{w}(x,\eta,R,a)$ and $R=\hat{R}(x^2,\eta,a)$ into $B'(v,\eta,R,a)$, we obtain a scalar equation

$$
\begin{aligned}
&s_2(x,\eta,a)\\
&:=B'(v_0+x\zeta+\hat{w}(x,\eta,\hat{R}(x^2,\eta,a),a),\eta,\hat{R}(x^2,\eta,a),a)=0.
\end{aligned}
$$

Expanding B', we find locally that $s_2(x,\eta,a)=\tilde{s}_2(x^2,\eta,a)$ and that $\frac{\partial\tilde{s}_2}{\partial\eta}(0,\eta_0,a_0)=1$. So by implicit function theorem, one obtains a unique solution $\eta=\hat{\eta}(x^2,a)$ in the neighborhood of (v_0,a_0) such that $\tilde{s}_2(x^2,\hat{\eta}(x^2,a),a)\equiv 0$ and $\hat{\eta}(0,a_0)=\eta_0$. We conclude:

Theorem 5: Consider the nonlinear eigenvalue problem (13)-(15) with Assumption (A1) and (A2). Let (v_0,η_0) denote the incoherence solution. Then from $R=R_c=r_0(a_0)$ bifurcates a branch of non-constant solutions (v,η) of (13)-(15). More precisely, there exists a neighborhood $J(x,a)\subset\mathbb{R}\times\mathbb{R}^+$ of $(0,a_0)$, functions $\hat{\eta}(x,a;\omega)$, $\hat{R}(x,a)\in C^1(J(x,a))$, and a family $v(x,\omega)$ of non-constant solutions of (13)-(15) in $\mathbf{X}$ such that, for all $\omega\in\Omega$,

(i) $\eta=\hat{\eta}(x,a;\omega)$ and $\hat{\eta}(x,a;\omega)\to\eta_0$, $R=\hat{R}(x,a)$ and $\hat{R}(x,a)\to R_c$ as $x\to 0$ and $a\to a_0\equiv 1$,
(ii) $v(x,\omega)-v_0$ tends to zero as $x\to 0$.

B. Perturbation analysis

We next describe Lyapunov-Schmidt reduction based perturbation analysis to obtain asymptotic formulae for the solution (v,η,R); We use $c^\bullet(\theta,\vartheta)=\frac{1}{2}\sin^2\left(\frac{\theta-\vartheta}{2}\right)$ to carry out the calculations. Specifically, we consider a series expansion in the small parameter ε:

$$
\begin{aligned}
R&=r_0+\varepsilon r_1+\varepsilon^2 r_2+\dots\\
v&=v_0+\varepsilon v_1+\varepsilon^2 v_2+\dots\\
\eta&=\eta_0+\varepsilon\eta_1+\varepsilon^2\eta_2+\dots
\end{aligned}
$$

We substitute the series into the nonlinear eigenvalue problem (13) - (14) and collect the terms according to different orders of ε.

At $O(1)$, we recover the incoherence solution (16). At $O(\varepsilon)$,

$$\mathscr{L}(r_0,a)v_1=-\frac{2v_0}{\sigma^4 r_0}\eta_1. \tag{30}$$

Its solution is given by $v_1=\phi(\theta,\omega,a)$, where ϕ is given by (25), $\eta_1=0$ and r_0 is given by (18).

At $O(\varepsilon^2)$,

$$
\begin{aligned}
&\mathscr{L}(r_0,a)v_2\\
&=-\frac{r_1}{r_0}\partial^2_{\theta\theta}v_1-\frac{2}{\sigma^4 r_0}\left(v_0\eta_2-\frac{v_0^{-1}}{8}+\frac{2r_0v_0^{-1}}{\sqrt{K(a)}}v_1\cos(\theta)\right)\\
&\quad+\frac{(\omega-a)^2}{\sigma^4}\left(\frac{4r_2v_1}{r_0}-6v_0^{-1}v_1^2+6v_0\int_{\mathbb{T}}v_1^2\,\mathrm{d}\theta\right),
\end{aligned}
\tag{31}
$$

whose solution is given by

$$r_1=0, \tag{32}$$

$$\eta_2=-\frac{r_0v_0^{-2}}{K(a)}\frac{1}{\rho^2(\omega,a)}, \tag{33}$$

$$v_2=v_{20}+v_{22}\cos(2\theta), \tag{34}$$

where

$$
\begin{aligned}
v_{20}&=-\frac{v_0^{-1}}{4K(a)\rho^4(\omega,a)},\\
v_{22}&=\frac{v_0^{-1}}{4K(a)\rho^4(\omega,a)}\frac{5(\omega-a)^2+\frac{\sigma^4}{2}}{(\omega-a)^2+\sigma^4}.
\end{aligned}
$$

At $O(\varepsilon^3)$,

$$
\begin{aligned}
&\mathscr{L}(r_0,a)v_3\\
&=-\frac{r_2}{r_0}\partial^2_{\theta\theta}v_1-\frac{2}{\sigma^4 r_0}\Bigg(v_0\eta_3+v_1\eta_2-2v_0\int_{\mathbb{T}}\int_{\Omega}c^\bullet v_1v_2\\
&\quad -v_1\int_{\mathbb{T}}\int_{\Omega}c^\bullet v_1^2-2v_0v_1\int_{\mathbb{T}}\int_{\Omega}c^\bullet v_2-2v_0v_2\int_{\mathbb{T}}\int_{\Omega}c^\bullet v_1\Bigg)\\
&\quad+\frac{(\omega-a)^2}{\sigma^4}\Bigg(4\frac{r_2}{r_0}v_1-12v_0^{-1}v_1v_2+10v_0^{-2}v_1^3\\
&\quad -18v_1\int_{\mathbb{T}}v_1^2+12v_0\int_{\mathbb{T}}v_1v_2+12v_0v_1\int_{\mathbb{T}}v_2\Bigg).
\end{aligned}
\tag{35}
$$

We use (35) to find the function r_2 by noting that $\langle v_1,\mathscr{L}(r_0,a)v_3\rangle=\langle\mathscr{L}(r_0,a)v_1,v_3\rangle=0$. This yields

$$r_2(a)=\frac{v_0^{-2}}{16K(a)}\int_{\Omega}\frac{g(\omega)}{\rho^6(\omega,a)}\frac{5(\omega-a)^2-\frac{7}{4}\sigma^4}{(\omega-a)^2+\sigma^4}\,\mathrm{d}\omega. \tag{36}$$

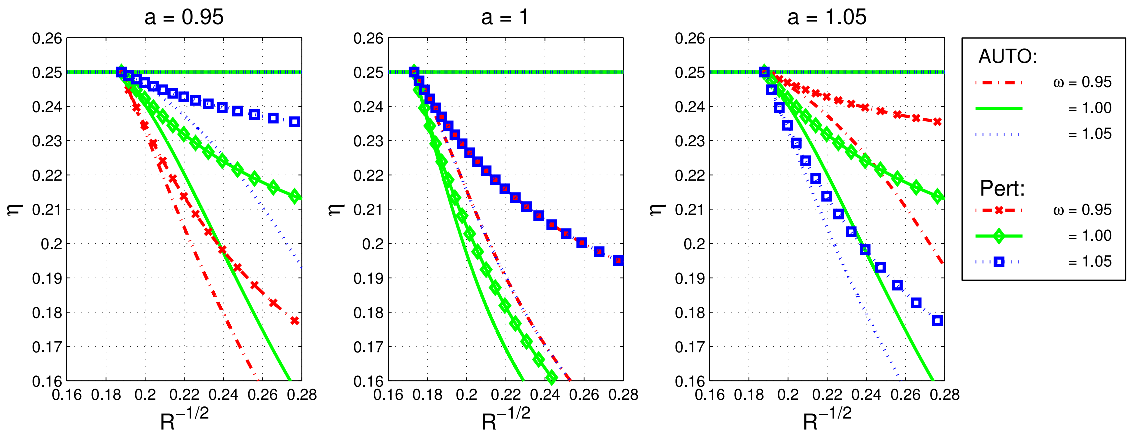
In summary, the solution is given by the asymptotic formulae,

$$
\begin{aligned}
R&=r_0(a)+\varepsilon^2 r_2(a)+o(\varepsilon^2),\\
v&=v_0+\varepsilon v_1(\theta,\omega,a)+\varepsilon^2 v_2(\theta,\omega,a)+o(\varepsilon^2),\\
\eta&=\eta_0+\varepsilon^2\eta_2(\omega,a)+o(\varepsilon^2),
\end{aligned}
\tag{37}
$$

where $r_0(a)$ is obtained in (18), $r_2(a)$ is obtained in (36), η_0, v_0 are defined in (16), $v_1=\phi(\theta,\omega,a)$ which is defined in (25), v_2 is obtained in (34) and $\eta_2(\omega,a)$ is obtained in (33).

IV. Numerical results

In this section, we provide computation of the bifurcated solutions obtained using the perturbation method (37) and using the continuation-based software AUTO [10]. We assume $g(\omega)=\frac{1}{3}\delta(\omega-\omega_1)+\frac{1}{3}\delta(\omega-\omega_2)+\frac{1}{3}\delta(\omega-\omega_3)$, where $\omega_1=0.95$, $\omega_2=1.00$ and $\omega_3=1.0$. The cost function $c^\bullet(\theta,\vartheta)=\frac{1}{2}\sin^2(\frac{\theta-\vartheta}{2})$.



(a) (b) (c)

Fig. 2. Average cost as a function of $R^{-1/2}$ for different values of frequency ω and wave speed a.

TABLE I

CRITICAL VALUE OF R FOR DIFFERENT WAVE SPEED.

	a		
	ω_1	ω_2	ω_3
$r_0(a)$	28.33	33.33	28.33
$r_0(a)^{-1/2}$	0.1879	0.1732	0.1879

Fig. 2 depicts a companion of average cost $\eta(\omega)$ for three wave speeds $a = \omega_1$, ω_2 and ω_3. Table I lists the critical value $R = r_0(a)$ for the three solutions. The bifurcation plots depicted in Fig. 2 help validate the local behavior of the solution obtained using the perturbation formulae (37).

V. APPENDIX

A. Proof of Lemma 4

Because $\lambda(R,a) \in S_d^{(1)}$, it satisfies the equation

$$\frac{C_1^\bullet}{2R}\int_\Omega \frac{g(\omega)}{(\omega-a)^2+\frac{\sigma^4}{4}(1+\lambda)}\,\mathrm{d}\omega + 1 = 0. \tag{38}$$

Denote the left-hand side of (38) as $F(\lambda,R,a)$. Then, $F(\lambda,R,a) = 0$. So

$$\begin{aligned} 0 &= \frac{\mathrm{d}}{\mathrm{d}R}F(\lambda(R_c,a_0),R_c,a_0) & (39)\\ &= \frac{\partial}{\partial R}F(\lambda,R,a) + \frac{\partial F}{\partial \lambda}\frac{\partial}{\partial R}\lambda(R,a)\Big|_{R=R_c,a=a_0}. & (40)\end{aligned}$$

From (38), we obtain

$$\begin{aligned} \frac{\partial F}{\partial R}\Big|_{R=R_c,a=a_0} &= -\frac{C_1^\bullet}{2R_c^2}\int_\Omega \frac{g(\omega)}{(\omega-a_0)^2+\frac{\sigma^4}{4}(1+\lambda(R_c,a_0))}\,\mathrm{d}\omega \\ &= -\frac{C_1^\bullet}{2R_c^2}\int_\Omega \frac{g(\omega)}{(\omega-a_0)^2+\frac{\sigma^4}{4}}\,\mathrm{d}\omega \\ &= -\frac{4C_1^\bullet}{R_c}, \end{aligned} \tag{41}$$

where the second equality uses the fact that $\lambda(R_c,a_0) = 0$. Next

$$\begin{aligned} \frac{\partial F}{\partial \lambda}\Big|_{R=R_c,a=a_0} &= \frac{C_1^\bullet}{2R_c}\int_\Omega \frac{g(\omega)(-\frac{\sigma^4}{4})}{\left((\omega-a_0)^2+\frac{\sigma^4}{4}\right)^2}\,\mathrm{d}\omega \\ &= -\frac{\sigma^4 C_1^\bullet}{8R_C}\int_\Omega \frac{g(\omega)}{\rho^4(\omega,a_0)}\,\mathrm{d}\omega. \end{aligned} \tag{42}$$

Denote $K(a) = \int_\Omega \frac{g(\omega)}{\rho^4(\omega,a)}\,\mathrm{d}\omega$ and substitute (41)-(42) in (40) to obtain the result (19).

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