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# Controllability and Observability of Networked Systems of Linear Hyperbolic Partial Differential Equations

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*Abstract*— **In this paper, we consider the controllability and the observability of a networked system of linear hyperbolic partial differential equations with coupled boundary conditions. Using the method of characteristics, we characterize them by the controllability/observability of a discrete-time system which is defined on the boundaries of the PDE system, has low-dimension, and hence the analysis is easy for. We also show the effectiveness of our approach using an simple example.**

## I. INTRODUCTION

A lot of distributed parameter systems are described by hyperbolic partial differential equations (PDE), for example, networks of open-channels [1], chemical processes (plug-flow reactors [2]), electric circuits [3], networked systems of conservation laws [4], [5] and so on. The feature of hyperbolic PDEs is possessing infinite-dimensional modes of nearly the same amount of energy, hence, it cannot be accurately represented by a finite number of modes. For such hyperbolic systems, various control problems have been studying: Control theory for linear hyperbolic PDE has been well-studied [6], [7] since Russell's pioneer work [6]; Recently, researchers are attracted to more complicated behavior caused by nonlinearity or complex system structure, which appear in, for example, quasilinear PDE [1], [8]–[11], boundary conditions [3], [12]–[14], networked PDE systems [15]. The authors have addressed stabilizing/synchronizing problems for a class of PDE systems producing spatio-temporal chaos [12]–[14], where a stabilizing/synchronizing control law has been proposed for each specific class of problem. However, more fundamental issues such as controllability have not been addressed there.

Thus in this paper, for networked systems of first-order linear PDEs, which includes the systems treated in [12]–[14] as special cases, we derive a characterization of the controllability and the observability in terms of a kind of discrete-time dynamics on the boundaries. We assume that the PDE system to be studied here has non-intersecting characteristics and explicit solution formulas, the multiple linear PDEs are coupled at their boundaries in nonlinear ways, and some boundaries have control inputs and sensors. To analyze such properties, at first, using the method of characteristics [16], we reduce the PDE system to a discrete-time system under an assumption for the propagation periods of the signal in the media. Next, we show that the PDE system and the discrete-time system are equivalent in the sense of the controllability and the observability. Thus we can can determine if the PDE system is controllable or observable by using the finite-dimensional discrete-time system on the boundaries, which has smaller size than the finite-dimensional discrete-time system derived by the conventional discretization scheme. Moreover, as shown in processes com firming the equivalency in this paper, we can apply control laws designed for the discrete-time system to the original PDE system, which is practical from the viewpoint of designing controller. At the end of this paper, we deal with a coupled time-delayed Chua's circuit as an application example, which consists of two transmission lines with nonlinear (piecewise affine) resistances called Chua's diode.

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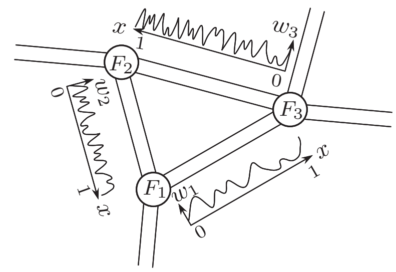


Fig. 1. Multi PDE systems with coupled boundary conditions.

## II. PROBLEM FORMULATION

### A. Networked system of linear hyperbolic PDEs

Consider a system of $n$ linear hyperbolic PDEs nonlinearly coupled at the boundaries, as in Fig. 1. We denote such a system as follows.

$$\partial_t w + \Lambda(x)\partial_x w = 0, \quad x \in (0,1),\ t \geq 0 \tag{1}$$

$$F(w(0,t), w(1,t), u(t)) = 0, \quad t \geq 0 \tag{2}$$

$$w(x,0) = \eta(x), \quad x \in [0,1], \tag{3}$$

where $w(x,t) = [w_1\ w_2\ \cdots\ w_n]^{\mathrm{T}}(x,t) \in \mathbb{R}^n$, $\Lambda(x) = \mathrm{diag}(\lambda_1(x), \lambda_2(x), \cdots, \lambda_n(x))$, $\lambda_i$ is a Lipschitz function such that $\lambda_i(x) > 0$ for $x \in [0,1]$, $F : \mathbb{R}^{2n} \times \mathbb{R}^m \to \mathbb{R}^n$ is a continuous (nonlinear) map describing boundary conditions, $u : \mathbb{R}_+ \to \mathbb{R}^m$ is a continuous input map and $\eta : [0,1] \to \mathbb{R}^n$ is an initial function.

**Example 1** The PDE system of the form (1) includes the following PDE:

$$\partial_t z + A(x)\partial_x z = 0,\ x \in (0,1),\ t \geq 0,$$

where, for $x \in (0,1)$, $A(x)$ is a real continuously differentiable symmetric $n \times n$ matrix function whose eigenvalues satisfy that $\nu_1(x) \leq \cdots \leq \nu_p(x) < 0 < \nu_{p+1}(x) \leq \cdots \leq \nu_n(x)$ for all $x \in [0,1]$ and that if there exist $x \in [0,1]$ and $i$ such that $\nu_i(x) = \nu_{i+1}(x)$ then $\nu_i(x) = \nu_{i+1}(x)$ for all $x \in [0,1]$. According to a lemma proved by Phillips [6], [17], there is a continuously differentiable orthogonal $n \times n$ matrix function $O(x)$ such that

$$O(x)^{-1}A(x)O(x) = \mathrm{diag}(\nu_1(x), \nu_2(x), \cdots, \nu_n(x)),$$

for all $x \in [0,1]$. Let us change the variable as follows.

$$w_i(1-x,t) := [O(x)^{-1}]_i z(x,t),\ i = 1, \cdots, p$$
$$w_i(x,t) := [O(x)^{-1}]_i z(x,t),\ i = p+1, \cdots, n.$$

Here, for elements of a new variable, $w_i,\ i = 1, \cdots, p$, we use inverse spatial coordinate $1-x$ rather than the original one $x$. Using the new variable $w$, we derive an equation in the form (1), where $\lambda_i(x) := -\nu_i(1-x),\ i = 1, \cdots, p$ and $\lambda_i(x) := \nu_i(x),\ i = p+1, \cdots, n$. □

Using *the method of characteristics* for (1), we find that, for each $i = 1, 2, \cdots, n$, the solution $w_i(x,t)$ takes constant values along curves called the characteristics, which is given as solutions of

$$dx/dt = \lambda_i(x),\ x(t_0) = x_0. \tag{4}$$

We denote the characteristic by $c_i(t, x_0, t_0)$. Since $\lambda_i(x) > 0,\ \forall x \in [0,1]$, $c_i$ is monotone increasing with respect to $t$. Therefore, it turns out that initial data and boundary data are transported from left to right, and reach to the right boundary in finite time[1]. Motivated by this fact, we define a flow incoming into the right boundary (i.e., an incoming flow at the right boundary) and a flow outgoing from the left boundary (i.e., an outgoing flow at the left boundary) by

$$w_{ic}(t) = [w_1(1,t)\ w_2(1,t)\ \cdots\ w_n(1,t)]^{\mathrm{T}},$$
$$w_{og}(t) = [w_1(0,t)\ w_2(0,t)\ \cdots\ w_n(0,t)]^{\mathrm{T}},$$

respectively. From the requirement of existence and uniqueness of solutions of the mixed initial-boundary value problem (1)–(3), it is natural to suppose that the outgoing flow should be determined by the incoming flow (and the input). Then we consider the following assumptions on the boundary condition and the initial condition.

**Assumption 1** We suppose that the boundary condition (2) can be rewritten as follows:

$$w_{og}(t) = \tilde{F}(w_{ic}(t), u(t)), \tag{5}$$

where $\tilde{F}$ is a Lipschitz continuous map. Furthermore, we assume that an initial condition $\eta$ is given in a function space consisting of functions in $H^1([0,1], \mathbb{R}^n)$ which satisfy boundary condition (5) (i.e., $\eta$ is compatible with (5))[2].

[1] propagation periods are specified later.

[2] $H^1([0,1], \mathbb{R}^n)$ is the Sobolev space of $n$-dimensional vector functions whose derivatives defined in the sense of distributions are square integrable.

Under these assumptions and the Lipschitz continuous property of $\Lambda$, for every initial conditions, the system (1), (3), (5) has a unique weak solution $w$ for all $t \geq 0$, where, for every $t \geq 0$, $w(\cdot,t)$ lies in $H^1([0,1], \mathbb{R}^n)$ and is strongly continuous in time in the sense that

$$\lim_{\varepsilon \to 0} \|w(\cdot, t+\varepsilon) - w(\cdot,t)\|_{L^2([0,1],\mathbb{R}^n)} = 0.$$

In a similar way to literature [6], [8], [18], [19], the proof can be given using properties that characteristics are always defined and do not intersect each other. (For a general theory, see also [20]).

*Output of the system:* We suppose that the system (1), (3), (5) has sensors only at some of boundaries, and the output signal $y(t)$ is given as follows.

$$y(t) = h(w_{ic}(t)),\ t \geq 0. \tag{6}$$

We denote the system (1), (3), (5) and (6) by $\Sigma$.

### B. Problem statement

Now, we give the definition of controllability and observability for the PDE system.

**Definition 1** The system $\Sigma$ is said to be (exactly) controllable at $t_f$ if, for any given $\eta, \zeta \in H^1([0,1], \mathbb{R}^n)$, there exist a time $t_f$ and an input $u \in L^2([0,t_f], \mathbb{R}^m)$ such that the solution $w$ satisfies

$$w(\cdot,0) = \eta,\ w(\cdot,t_f) = \zeta.$$

**Definition 2** Let solutions of the system $\Sigma$ for initial values $\eta_1,\ \eta_2$ be $w_{\eta_1}(\cdot,t),\ w_{\eta_2}(\cdot,t)$, respectively. The system $\Sigma$ is said to be observable at $t_f$ if, for any given initial functions $\eta_1,\ \eta_2 (\neq \eta_1)$, there exist a time $t_f$ and an input $u \in L^2([0,t_f], \mathbb{R}^m)$ such that $h(w_{\eta_1}(1,t_f)) \neq h(w_{\eta_2}(1,t_f))$.

The problem to be tackled in this paper is as follows.

**Problems** Determine if the system $\Sigma$ is controllable and/or observable.

## III. SYSTEM ANALYSIS

In this section, we investigate the controllability and the observability of the system $\Sigma$. There are two steps. At first, we introduce an extended PDE system that conserves these properties from the original system. Next, we derive a finite-dimensional discrete-time system from the extended PDE system and confirm that the discrete-time system also conserve the properties.

### A. Introduction of an extended PDE system

As mentioned above, from the method of characteristics, each state $w_i(x,t)$ behaves as like a wave propagating without changing its value. We can calculate the propagation periods $\{\tau_i\}$ from $x=0$ to $x=1$ as follows.

$$\tau_i = \int_0^1 \lambda_i^{-1}(x)dx,\ i = 1, \cdots, n.$$

Then, we find that the value of the left boundary (i.e. the outgoing flow) at time $t$ means that of the right boundary at time $t + \tau_i$, that is,

$$w_{og}(t) = [w_1(1,t+\tau_1)\ w_2(1,t+\tau_2)\ \cdots\ w_n(1,t+\tau_n)]^{\mathrm{T}}.$$

Therefore, the boundary condition (5) is represented as

$$\begin{aligned}&[w_1(1,t+\tau_1)\ w_2(1,t+\tau_2)\ \cdots\ w_n(1,t+\tau_n)]^{\mathrm{T}}\\&\quad=\tilde{F}([w_1(1,t)\ w_2(1,t)\ \cdots\ w_n(1,t)]^{\mathrm{T}},u(t)),\end{aligned}\tag{7}$$

which can be regarded as an evolution equation with respect to the state $w(1,t),\ t\geq 0$. Although this equation with some appropriate initial condition is well-posed, capturing the dynamics is difficult because the time differences $\tau_i,\ i=1,\cdots,n$ are different.

To overcome this difficulty, we divide the media of the PDE system and adjust the propagation periods. For each $i$, we give $l_i$ dividing points $\{x_i^j\}_{j=0}^{l_i}$ in $[0,1]$ such that

$$0=x_i^0<x_i^1<\cdots<x_i^{l_i-1}<x_i^{l_i}=1.$$

Then, dividing the space for each $i$ according to these points $\{x_i^j\}$, we extend the PDE with respect to $w_i$ in (1)

$$\partial_t w_i+\lambda_i(x)\partial_x w_i=0,\ x\in(0,1)$$

to $l_i$ PDEs

$$\partial_t\bar{w}_i^j+\lambda_i(x)\partial_x\bar{w}_i^j=0,\ x\in(x_i^{j-1},x_i^j),\ j=1,\cdots,l_i.\tag{8}$$

The number of the new PDEs is $N=\sum_{i=1}^n l_i$. Boundary conditions are also extended by setting virtual boundary conditions at the new boundaries, that is, conditions each of which transfers state values from the left interval to the right one without changing their values:

$$\begin{cases}\bar{w}_i^1(0,t)=\tilde{F}_i(\bar{w}_{ic}(t),u(t))\\\bar{w}_i^2(x_i^1,t)=\bar{w}_i^1(x_i^1,t)\\\qquad\vdots\\\bar{w}_i^{l_i}(x_i^{j-1},t)=\bar{w}_i^{l_i-1}(x_i^{j-1},t),\end{cases}\tag{9}$$

where $\bar{w}_{ic}(t)=[\bar{w}_1^{l_1}(1,t)\ \bar{w}_2^{l_2}(1,t)\cdots\bar{w}_n^{l_n}(1,t)]^{\mathrm{T}}$. In addition, the corresponding output equation is written by

$$y(t)=h(\bar{w}_{ic}(t)).\tag{10}$$

We denote the extended PDE system (8), (9), (10) with an appropriate initial condition by $\bar{\Sigma}$. It is easy to confirm the following proposition.

**Proposition 1** The followings are equivalent:

- The original PDE system $\Sigma$ is controllable (observable) at $t_f$.
- The extended PDE system $\bar{\Sigma}$ is controllable (observable) at $t_f$.

Thus, in this extension, the controllability and the observability are preserved. Moreover, we can adjust the propagation periods which are shorter than the original ones. Nevertheless, we have to consider the following assumption to make all periods coincide rigorously.

**Assumption 2** We suppose that, for any pair of $i$ and $j$, $\tau_i/\tau_j$ is rational. In other words, there exist $\Delta T\in\mathbb{R}_+$ and $\{l_i\}_{i=1}^n\subset\mathbb{N}_+$ such that

$$\tau_i=l_i\Delta T.\tag{11}$$

**Remark 1** Since the eigenvalues $\{\lambda_i(x)\}$, which mean propagation velocities, are independent from the initial condition, the propagation periods $\{\tau_i\}$ are so. Therefore, the value of $\Delta T$ and $\{l_i\}$ is also determined independently from the initial condition if they exist. It turns out easily that, if a pair $(\Delta T,\{l_i\})$ satisfies (11), then pairs $(\Delta T/n,\{l_i'\mid l_i'=nl_i\}),\ n=2,3,\cdots$ also satisfy (11). Since $\{l_i\}$ of small values means that the system introduced below has low dimension, we chose the values $\Delta T$ so that it is as large as possible .

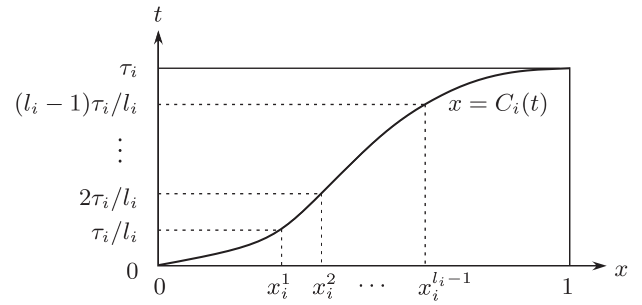


Fig. 2. Division of the interval $[0,1]$ using the coordinate function of the $i$th characteristics.

Under this assumption, we set dividing points $\{x_i^j\}$ as follows. Consider the characteristics $c_i$ again. For each $i=1,\cdots,n$, define a coordinate function $C_i:[0,\tau_i]\to[0,1]$ by

$$C_i(t)=c_i(t,0,0),$$

which means how long the wave propagates after $t\in[0,\tau_i]$ (see Fig. 2). We readily find that $C_i$ is a bijection, therefore the inverse $C_i^{-1}$ exists. Then, we define points $\{x_i^j\}$ in the interval $[0,1]$ for each $i$th medium.

$$x_i^j:=C_k^{-1}(j\tau_i/l_i),\ j=0,1,\cdots,l_i.$$

Since the propagation periods in the intervals $[x_i^{j-1},x_i^j]$ are $\Delta T$ in common, using the method of characteristics, we derive the following equation:

$$\begin{cases}\bar{w}_i^1(x_i^1,t+\Delta T)=\tilde{F}_i(\bar{w}_{ic}(t),u(t))\\\bar{w}_i^2(x_i^2,t+\Delta T)=\bar{w}_i^1(x_i^1,t)\\\qquad\vdots\\\bar{w}_i^{l_i}(1,t+\Delta T)=\bar{w}_i^{l_i-1}(x_i^{l_i-1},t)\end{cases},\ i=1,\cdots,n.$$

For brevity of notation, we denote $\bar{w}_i^j(x_i^j,\cdot)$ as $\alpha_i^j(\cdot)$. Then, we derive

$$\begin{cases}\alpha_i^1(t+\Delta T)=\tilde{F}_i(D\alpha(t),u(t))\\\alpha_i^2(t+\Delta T)=\alpha_i^1(t)\\\qquad\vdots\\\alpha_i^{l_i}(t+\Delta T)=\alpha_i^{l_i-1}(t)\end{cases},\ i=1,\cdots,n,\tag{12}$$

where[3] $D\alpha(t)=[\alpha_1^{l_1}(t)\cdots\alpha_n^{l_n}(t)]^{\mathrm{T}}$, $D\in\mathbb{R}^{n\times N}$. Furthermore, we denote (12) more simply as follows.

$$\alpha(t+\Delta T)=G(\alpha(t),u(t))$$

[3] $D:=[d_{ij}],\ d_{ij}=\begin{cases}1, & \text{when } j=\sum_{k=1}^i l_k\\0, & \text{otherwise.}\end{cases}\quad i=1,2,\cdots,n$

where $G$ is the $N$-dimensional vector function described by $[G_1^1\ G_1^2\ \cdots\ G_n^{l_n}]^{\mathrm{T}}$. Using the same description, we have the output equation represented by

$$y(t) = h(D\alpha(t)).$$

Since the independent valuable $t$ is real number, (12) is called the difference equation with continuous argument. Compared to $\Sigma$, it is easier to analyze the equation (12).

### *B. Induced discrete-time system on boundaries*

The difference equation with continuous argument (12) is similar to the so-called discrete-time system. Although they are different systems because of the difference of the independent variables (i.e. $t \in \mathbb{R}$ and $k \in \mathbb{N}$ ), some properties may be closely related each other. If we find such properties, we may analyze them using the finite-dimensional discrete-time system, which is much easier than using the original system defined on the function space.

Let us consider a discrete-time system [4] ($k \in \mathbb{N}_+$)

$$\begin{cases} \alpha_i^1[k+1] = \tilde{F}_i(\alpha_{ic}[k], u[k]) \\ \alpha_i^2[k+1] = \alpha_i^1[k] \\ \quad\vdots \\ \alpha_i^{l_i}[k+1] = \alpha_i^{l_i-1}[k]. \end{cases}, \ i = 1, \cdots, n, \quad (13)$$

which corresponds to (12). In a similar way, we also denote (13) by

$$\alpha[k+1] = G(\alpha[k], u[k]),$$

and, for the corresponding output equation, we give

$$v[k] = h(D\alpha[k]). \quad (14)$$

The controllability and the observability of the discrete-time system are defined as follows.

**Definition 3** The discrete-time system (13) is said to be controllable at $k_f \subset \mathbb{N}$ if, for any given $\alpha_0,\ \alpha_f \in \mathbb{R}^N$, there exist a time $t_f$ and $\{u[k]\}_{k=0}^{k_f}$ such that the solution $\alpha$ satisfies $\alpha[0] = \alpha_0$ and $\alpha[k_f] = \alpha_f$.

**Definition 4** Let solutions of the discrete-time system (13) for initial values $z_1$ and $z_2$ be $\alpha_{z_1}[k]$ and $\alpha_{z_2}[k]$, respectively. The system (13), (14) is said to be observable at $k_f$ if, for any given initial states $z_1$, $z_2$ ($\neq z_1$), there exist a time $k_f$ and an input $\{u[k]\}_{k=0}^{k_f}$ such that $h(D\alpha_{z_1}[k_f]) \neq h(D\alpha_{z_2}[k_f])$.

At first, we confirm that the controllability is preserved in reducing the original PDE system $\Sigma$ to the discrete-time system (13).

**Theorem 1** Under Assumption 2, the original PDE system $\Sigma$ is controllable at $k_f \varDelta T$ if and only if the discrete-time system (13) is controllable at $k_f$.

[4]This system can be also regarded as a system reduced from the continuous-time system by an upwind discretization with a non-uniform grid. However, since we give the virtual boundaries as few as possible (see Remark 1), the size of the system is small, which means the discretization is quite coarse.

*Proof:* It is sufficient to show that the extended PDE system $\bar{\Sigma}$ is controllable at $k_f \varDelta T$ if and only if the discrete-time system (13) is controllable at $k_f$. Since the space coordinates in (8) are different among each media, they are awkward. Introducing coordinate transformations $\Phi_{ij} : [x_i^{j-1}, x_i^j] \to [0,1]$

$$\Phi_{ij}(x) = \frac{1}{\varDelta T} C_{ij}^{-1}(x) =: \hat{x},$$

we identify each intervals $[x_i^{j-1}, x_i^j]$ with $[0,1]$. Here, $C_{ij} : [0, \varDelta T] \to [x_i^{j-1}, x_i^j]$ means a sub-arc of the characteristics defined below:

$$C_{ij}(t) = c_i(t, x_i^{j-1}, 0).$$

To begin with, we show the sufficiency. we first choose an initial condition $\bar{\eta} \in H^1([0,1], \mathbb{R}^N)$ and a terminal condition $\bar{\zeta} \in H^1([0,1], \mathbb{R}^N)$ arbitrarily for the extended PDE system $\bar{\Sigma}$. Then, fix $\hat{x} \in [0,1]$, and give two $N$-dimensional vectors as follows:

$$\alpha_0(\hat{x}) = \left[\bar{\eta}_1^1(\Phi_{11}^{-1}(\hat{x}))\ \bar{\eta}_1^2(\Phi_{12}^{-1}(\hat{x}))\ \cdots\ \bar{\eta}_n^{l_n}(\Phi_{n,l_n}^{-1}(\hat{x}))\right]^{\mathrm{T}},$$

$$\alpha_f(\hat{x}) = \left[\bar{\zeta}_1^1(\Phi_{11}^{-1}(\hat{x}))\ \bar{\zeta}_1^2(\Phi_{12}^{-1}(\hat{x}))\ \cdots\ \bar{\zeta}_n^{l_n}(\Phi_{n,l_n}^{-1}(\hat{x}))\right]^{\mathrm{T}}.$$

Since we assume that the discrete-time system (13) is controllable at $k_f$, there exists an input sequence $\{u_{\hat{x}}[k]\}_{k=0}^{k_f}$ such that the initial and the terminal states satisfy $\alpha[0] = \alpha_0(\hat{x}),\ \alpha[k_f] = \alpha_f(\hat{x})$, respectively. Similarly, for all $\hat{x} \in [0,1]$, the input sequences $\{u_{\hat{x}}[k]\}_{k=0}^{k_f}$ can be constructed. Now, for the system $\bar{\Sigma}$, we give the boundary input by

$$u(t) = u_{\bar{x}}[\bar{k}],\ \bar{k} = \left\lfloor \frac{t}{\varDelta T} \right\rfloor, \bar{x} = 1 - \frac{t}{\varDelta T} + \bar{k} \quad (15)$$

where $\lfloor\cdot\rfloor$ is the floor function. Then, for each $i, j$, $x \in [x_i^{j-1}, x_i^j]$, it turns out that

$$\begin{aligned}
&\bar{w}_i^j(x, k_f \varDelta T) \\
&\quad = \alpha_i^j(k_f \varDelta T + (\varDelta T - C_{ij}^{-1}(x))) \\
&\quad = \alpha_i^j((k_f - \hat{x} + 1)\varDelta T) \\
&\quad = G_i^j(\alpha((k_f - \hat{x})\varDelta T), u((k_f - \hat{x})\varDelta T)) \\
&\quad = G_i^j(F(\cdots(G(\alpha((1-\hat{x})\varDelta T), u((1-\hat{x})\varDelta T)), \\
&\qquad\quad u((2-\hat{x})\varDelta T)), \cdots), u_{\hat{x}}((k_f - \hat{x})\varDelta T)) \\
&\quad = G_i^j(F(\cdots(G(\alpha_0(\hat{x}), u_{\hat{x}}[0]), u_{\hat{x}}[1]), \cdots), u_{\hat{x}}[k_f]) \\
&\quad = \bar{\zeta}_i^j(\Phi_{ij}^{-1}(\hat{x})) \\
&\quad = \bar{\zeta}_i^j(x).
\end{aligned}$$

Therefore, by the input (15), the extended PDE system $\bar{\Sigma}$ satisfies

$$\bar{w}(\cdot, 0) = \bar{\eta},\ \bar{w}(\cdot, k_f \varDelta T) = \bar{\zeta}.$$

The necessity is proven as follows. Choosing $\alpha_0,\ \alpha_f \in \mathbb{R}^N$ arbitrarily, for the extended PDE system, we let the initial condition and the terminal condition be

$$\bar{\eta}_i^j(\Phi_{ij}^{-1}(\hat{x})) \equiv \alpha_0,\ \bar{\zeta}_i^j(\Phi_{ij}^{-1}(\hat{x})) \equiv \alpha_f, \quad \text{for } \forall \hat{x} \in [0,1].$$

Then, from the controllability of the system $\bar{\Sigma}$, there exists an input $\{u'(t) \mid t \in [0, k_f \Delta T]\}$ to steer $\bar{\eta}$ to $\bar{\zeta}$. We choose and fix $\beta \in (0, \Delta T)$, and give an input for the discrete-time system as follows.

$$u[k] = u'(k\Delta T + \beta),\ k = 0, 1, \cdots, k_f - 1$$

Then, in the discrete-time system, we can steer the states from $\alpha_0$ to $\alpha_f$. ■

**Remark 2** Since the classes of control input are different between the PDE system $\Sigma$ and the discrete-time system (13), it is not trivial to show the equivalence of the controllability as much as Proposition 1. Nevertheless, one procedure constructing control inputs from $\Sigma$ to (13) (and vice versa) is shown in this proof and therefore the equivalence is confirmed. (13) is regarded as a discrete-time dynamics on the boundaries which has small dimension and the controllability can be checked by. This means that one can analyze the controllability using the reduced model.

Next, we verify that the observability for the original system $\Sigma$ is equivalent to that for the discrete-time system (13), (14).

**Theorem 2** Under Assumption 2, the original PDE system $\Sigma$ is observable at $\kappa \in [k_f \Delta T, (k_f + 1)\Delta T]$ if and only if the discrete-time system (13), (14) is observable at $k_f$.

*Proof:* In a similar way to Theorem 1, it is sufficient to show that the extended PDE system $\bar{\Sigma}$ is observable at $\kappa \in [k_f \Delta T, (k_f + 1)\Delta T]$ if and only if the discrete-time system (13), (14) is observable at $k_f$.

We first show the sufficiency. Consider two different functions $\bar{\eta}_1$, $\bar{\eta}_2$, where there exists $\hat{x} \in [0, 1]$ such that $\bar{\eta}_1([\Phi_{ij}^{-1}(\hat{x})]) \neq \bar{\eta}_2([\Phi_{ij}^{-1}(\hat{x})]) \in \mathbb{R}^N$ [5]. From the observability of the discrete-time system, for two states given by

$$z_1 := \bar{\eta}_1([\Phi_{ij}^{-1}(\hat{x})]),\ z_2 := \bar{\eta}_2([\Phi_{ij}^{-1}(\hat{x})]),$$

there exist $k_f$ and an input $\{\check{u}[k]\}_{k=0}^{k_f}$ such that $h(D\alpha_{z_1}[k_f]) \neq h(D\alpha_{z_2}[k_f])$. Then, giving an input $\{u(t) \mid t \in [0, (k_f + 1)\Delta T]\}$ for the extended PDE system so that it satisfies

$$u(t) = \check{u}[k],\ \text{when}\ t = (k - \hat{x} + 1)\Delta T,\ k = 0, 1, \cdots, k_f,$$

we have

$$h(D\bar{w}_{\bar{\eta}_1}(1, (k_f - \hat{x} + 1)\Delta T) = h(D\alpha_{z_1}[\kappa])$$
$$h(D\bar{w}_{\bar{\eta}_2}(1, (k_f - \hat{x} + 1)\Delta T) = h(D\alpha_{z_2}[\kappa]).$$

Therefore, it turns out that $h(D\bar{w}_{\bar{\eta}_1}(1, \kappa)) \neq h(D\bar{w}_{\bar{\eta}_2}(1, \kappa))$, where $\kappa := (k_f - \hat{x} + 1)\Delta T \in [k_f \Delta T, (k_f + 1)\Delta T]$.

Next,we prove the necessity. Let $z_1,\ z_2 \in \mathbb{R}^N$ be two different states. Choose functions $\bar{\eta}_1$, $\bar{\eta}_2$ so that

$$\bar{\eta}_1(\Phi_{ij}^{-1}(\hat{x})) \equiv z_1,\ \bar{\eta}_1(\Phi_{ij}^{-1}(\hat{x})) \equiv z_2,\ \hat{x} \in [0, 1].$$

[5]We denote $[(\bar{\eta}_a)_1^1(\Phi_{11}^{-1}(\cdot))\ (\bar{\eta}_a)_1^2(\Phi_{12}^{-1}(\cdot))\ \cdots\ (\bar{\eta}_a)_n^{l_n}(\Phi_{n,l_n}^{-1}(\cdot))]^{\mathrm{T}}$ by $\bar{\eta}_a([\Phi_{ij}^{-1}(\cdot)])$.

There exist a time $\mathcal{T} \in [k'\Delta T, (k' + 1)\Delta T]$ and a control input $\{\tilde{u}(t) \mid t \in [0, k'\Delta T]\}$ so that $h(D\bar{w}_{\bar{\eta}_1}(1, \mathcal{T})) \neq h(D\bar{w}_{\bar{\eta}_2}(1, \mathcal{T}))$. We denote $\mathcal{T}$ by the following form:

$$\mathcal{T} = k'\Delta T + \gamma,\ \gamma \in [0, 1) \subset \mathbb{R}.$$

Then, giving an input for the discrete-time system by

$$u[k] = \tilde{u}(k\Delta T + 1 - \gamma),\ k = 0, 1, \cdots, k' - 1$$

we can prove that $h(D\alpha_{z_1}[k']) \neq h(D\alpha_{z_2}[k'])$. ■

## IV. APPLICATIONS

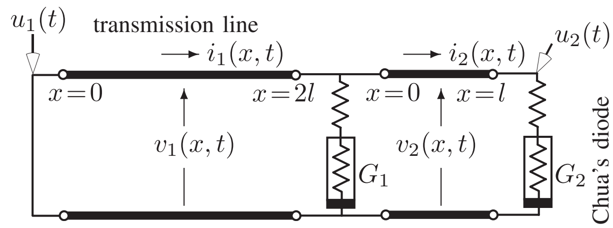


Fig. 3. Lossless transmissions line with Chua's diodes

Let us consider a circuit consisting of two transmission lines and Chua's diodes as shown in Fig. 3. This circuit is called the time-delayed Chua's circuit, in which spatio-temporal chaos occurs for some parameters[3]. We suppose that this circuit has voltage control inputs at the left boundary and the right boundary (See Fig. 3). These transmission lines are denoted by wave equations

$$\begin{cases} L\ \partial_t i_1 + \partial_x v_1 = 0 \\ C\ \partial_t v_1 + \partial_x i_1 = 0 \end{cases},\ x \in [0, 2l],\ t \in \mathbb{R}_+,$$
$$\begin{cases} L\ \partial_t i_2 + \partial_x v_2 = 0 \\ C\ \partial_t v_2 + \partial_x i_2 = 0 \end{cases},\ x \in [0, l],\ t \in \mathbb{R}_+,$$

and nonlinear boundary conditions

$$v_1(0,t) = u_1(t)$$
$$i_1(2l,t) - i_2(0,t) = G_1(v_1(2l,t) - R_1(i_1(2l,t) - i_2(0,t)) - E)$$
$$v_2(0,t) = v_1(2l,t)$$
$$i_2(l,t) = G_2(v_2(l,t) - Ri_2(l,t) - E + u_2(t)) + u_2(t)/(Z + R_2),$$

where $G_i$ represents the voltage-current characteristic of Chua's diode given as follows.

$$G_i(\xi) = m_{i,1}\xi + \frac{1}{2}(m_{i,0} - m_{i,1})[|\xi + B_i| - |\xi - B_i|].$$

By mean of a variable transformation

$$v_i = p_i - q_i,\ i_i = (p_i + q_i)/Z,\ i = 1, 2,$$

we derive PDEs whose new variables $p_i$, $q_i$ are separated.

$$\begin{cases} \partial_t p_1 + \lambda_1\ \partial_x p_1 = 0 \\ \partial_t q_1 - \lambda_1\ \partial_x q_1 = 0 \end{cases},\ x \in [0, 1],\ t \in \mathbb{R}_+,$$
$$\begin{cases} \partial_t p_2 + \lambda_2\ \partial_x p_2 = 0 \\ \partial_t q_2 - \lambda_2\ \partial_x q_2 = 0 \end{cases},\ x \in [0, 1],\ t \in \mathbb{R}_+.$$

Here, $\lambda_1 = \nu/(2l)$, $\lambda_2 = \nu/l$, $Z = \sqrt{L/C}$ and $\nu = 1/\sqrt{LC}$. Moreover, defining the state by

$$\begin{aligned} w(x,t) &= [w_1(x,t)\ w_2(x,t)\ w_3(x,t)\ w_4(x,t)]^{\mathrm{T}} \\ &:= [p_1(x,t)\ q_1(1-x,t)\ p_2(x,t)\ q_2(1-x,t)]^{\mathrm{T}}, \end{aligned}$$

we have

$$\partial_t w + \Lambda\ \partial_x w = 0,\ x \in [0,1],\ t \in \mathbb{R}_+, \tag{16}$$

where $\Lambda = \mathrm{diag}\{\lambda_1, \cdots, \lambda_4\}$. Then, the above boundary conditions become

$$\begin{aligned}
w_1(0,t) &= w_2(1,t) + u_1(t)\\
w_2(0,t) &= f_2(w_1(1,t), w_4(1,t))\\
w_3(0,t) &= f_3(w_1(1,t), w_4(1,t))\\
w_4(0,t) &= f_4(w_3(1,t)) + (Z/(Z+R_2))u_2(t),
\end{aligned} \tag{17}$$

where

$$f_2(x,y) = \begin{cases} a_1 x + a_2 y + b_1, & x - y - E < -\delta_1 \\ a_3 x + a_4 y + b_2, & |x - y - E| \le \delta_1 \\ a_1 x + a_2 y + b_3, & x - y - E > \delta_1 \end{cases}$$

$$f_3(x,y) = x + y - f_2(x,y)$$

$$f_4(x) = \begin{cases} a_5 x + b_4, & x - E/2 < -\delta_2 \\ a_6 x + b_5, & |x - E/2| \le \delta_2 \\ a_5 x + b_6, & x - E/2 > \delta_2 \end{cases}$$

$$\begin{aligned}
a_1 &= \tfrac{m_{1,1}Z}{m_{1,1}(Z+2R_1)+2},\ a_2 = \tfrac{2m_{1,1}R_1+2}{m_{1,1}(Z+2R_1)+2},\\
b_1 &= \tfrac{[-m_{1,1}E-(m_{1,0}-m_{1,1})B_1]Z}{m_{1,1}(Z+2R_1)+2},\\
a_3 &= \tfrac{m_{1,0}Z}{m_{1,0}(Z+2R_1)+2},\ a_4 = \tfrac{2m_{1,0}R_1+2}{m_{1,0}(Z+2R_1)+2},\\
b_2 &= \tfrac{-m_{1,0}ZE}{m_{1,0}(Z+2R_1)+2}, b_3 = \tfrac{[-m_{1,1}E+(m_{1,0}-m_{1,1})B_1]Z}{m_{1,1}(Z+2R_1)+2},\\
\delta_1 &= \tfrac{[m_{1,0}(Z+2R_1)+1]B_1}{2},\\
a_5 &= \tfrac{m_{2,1}(Z-R_2)-1}{m_{2,1}(Z+R_2)+1}, b_4 = \tfrac{[-m_{2,1}E-(m_{2,0}-m_{2,1})B_2]Z}{m_{2,1}(Z+R_2)+1},\\
a_6 &= \tfrac{m_{2,0}(Z-R_2)-1}{m_{2,0}(Z+R_2)+1},\ b_5 = \tfrac{-m_{2,0}EZ}{m_{2,0}(Z+R_2)+1},\\
b_6 &= \tfrac{[-m_{2,1}E+(m_{2,0}-m_{2,1})B_2]Z}{m_{2,1}(Z+R_2)+1},\\
\delta_2 &= \tfrac{[m_{2,0}(Z+R_2)+1]B_2}{2}.
\end{aligned}$$

(16), (17) have the forms of (1), (5), respectively. Since this system satisfies Assumption 1 and 2, we can apply Theorem 1. Now, taking a constructive strategy, we show that this PDE system (16), (17) is controllable at some time.

Dividing the first transmission line at the middle point and using the method of the characteristics, we derive the following differential equation with continuous argument.

$$\begin{aligned}
\alpha_1^1(t + l/\nu) &= \alpha_2^2(t) + u_1(t)\\
\alpha_1^2(t + l/\nu) &= \alpha_1^1(t)\\
\alpha_2^1(t + l/\nu) &= f_2(\alpha_1^2(t), \alpha_4^1(t))\\
\alpha_2^2(t + l/\nu) &= \alpha_2^1(t)\\
\alpha_3^1(t + l/\nu) &= f_3(\alpha_1^2(t), \alpha_4^1(t))\\
\alpha_4^1(t + l/\nu) &= f_4(\alpha_3^1(t)) + (Z/(Z+R_2))u_2(t).
\end{aligned}$$

This difference equation is characterized by a discrete-time system below.

$$\begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \\ \xi_5 \\ \xi_6 \end{bmatrix}_{k+1} = \begin{bmatrix} \xi_4 \\ \xi_1 \\ f_2(\xi_2, \xi_6) \\ \xi_3 \\ f_3(\xi_2, \xi_6) \\ f_4(\xi_5) \end{bmatrix}_k + \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & \frac{Z}{Z+R_2} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}_k. \tag{18}$$

Here, for the sake of simplicity, new symbols $\xi_i$ are used. In fact, the discrete-time system (18) is controllable at $k_f = 4$. Therefore, it turns out that the original PDE (16) is controllable at $t = 4l/\nu$.

## V. CONCLUSIONS

In this report, we investigated the controllability and the observability of networked systems of linear hyperbolic PDEs with coupled boundary conditions. To this end, using the method of characteristics, we induced a finite-dimensional discrete-time system on boundaries, which has small dimension, and showed that the PDE system and the discrete-time system are equivalent in the sense of controllability and observability. The effectiveness of our approach was shown by an simple example.

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