2011 50th IEEE Conference on Decision and Control and
European Control Conference (CDC-ECC)
Orlando, FL, USA, December 12-15, 2011

Synchronization of networked piecewise smooth systems

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***Abstract*— This paper is concerned with the synchronization of networks of switching dynamical systems. In particular, conditions are derived for all nodes in a network of such systems to converge towards a common synchronous evolution. The key assumption is for the vector field to be in a suitable form that we call *QUAD affine*. Under this assumption, we show that the network of interest synchronizes even if the vector field is discontinuous and sliding motion is possible. The theoretical results are complemented by numerical simulations on a testbed example.**

I. Introduction

Switching dynamical systems are commonly used in control to model systems or devices of interest and design appropriate hybrid control laws, e.g. [1], [2]. It is therefore of fundamental importance in applications to investigate the emergence of coordinated motion in networked switching systems. Examples include the coordinated motion of mechanical oscillators with friction [3], [4], [5], switching power devices [6], [7] and all those networks whose nodes are affected by switchings on a macroscopic timescale.

While in the case of networks of smooth dynamical systems results on synchronizability abound in the literature (see, for instance, [8], [9], [10], [11], [12]), when switching node dynamics are considered, the analysis becomes much more cumbersome and only few results are available [13], [14]. The aim of this paper is to discuss this pressing open problem and derive conditions that guarantee the emergence of a synchronous evolution in networks of systems with discontinuous vector fields. In particular, we study the case where the nodes' own switching dynamics can be expressed as the sum of two terms: the first is common to each node in the network, while the second is different from node to node. We show that, if the coupling configuration is appropriately selected, then the network asymptotically achieves bounded synchronization. The analysis is based on the use of appropriate Lyapunov functions and, as shown in the paper, includes the case where sliding motion [15] is present.

The rest of the paper can be outlined as follows. In Sec. II, we introduce the concept of bounded synchronization and give some useful definitions and lemmas. In Sec. III we recall some important concepts related to piecewise-smooth systems. These notions are then used in Sec. IV to derive a novel set of conditions guaranteeing synchronization of a generic network of switching dynamical systems. The theoretical results are illustrated by numerical simulations on a network of chaotic relay systems in Sec. V. Conclusions are drawn in Sec. VI

II. Preliminaries

A generic network of diffusively coupled dynamical systems can be described by

$$\dot{x}_i = f_i(x_i,t) - c\sum_{j=1}^{N}\ell_{ij}\Gamma x_j, \qquad \forall i = 1,\dots,N, \quad (1)$$

where $x_i \in \mathbb{R}^n$ is the state of the i-th node, f_i is the (possibly discontinuous) vector field describing the own dynamics of the i-th node, c is the coupling gain, $\Gamma \in \mathbb{R}^{n\times n}$ is the inner coupling matrix, and ℓ_{ij} is the element (i,j) of the Laplacian matrix $\mathcal{L}$, defined as

$$\ell_{ij} = \begin{cases} -1, & \text{if } i \neq j \text{ and } (i,j) \in \mathcal{E} \\ 0, & \text{if } i \neq j \text{ and } (i,j) \notin \mathcal{E} \\ -\sum_{\substack{k=1 \\ k\neq i}}^{N} \ell_{ik}, & \text{if } i = j \end{cases},$$

where $\mathcal{E}$ is the set of all the network edges. Here and in what follows we will suppose the graph to be connected (i.e. for every pair of nodes (i,j) there exists a path from node i to node j.).

In the literature on complex networks (see for instance [16], [17]), a common way of defining the synchronization error $e_i = [e_{i1},\dots,e_{in}]^T$ at a generic node i is to set

$$e_i(t) = x_i(t) - \bar{x}(t), \quad (2)$$

where $\bar{x}(t) = \frac{1}{N}\sum_{j=1}^{N} x_i(t)$ is the *average trajectory* of the network. According to (2), we give the following definition of bounded synchronization:

Definition 1: Network (1) is said to be *bounded synchronized* if one of the two following conditions hold:

(a) There exists a constant $\epsilon_1 > 0$ such that

$$\lim_{t\to\infty} ||e(t)|| \leq \epsilon_1, \quad (3)$$

where $e = [e_1^T,\dots,e_N^T]^T$.

(b) There exists a constant $\epsilon_2 > 0$ such that

$$\lim_{t\to+\infty} ||x_i(t) - x_j(t)|| \leq \epsilon_2, \qquad \forall\, i,j = 1,\dots,N, \quad (4)$$

Furthermore, if (3) is satisfied for a given $\epsilon_1 = \epsilon$, then network (1) is said to be ϵ-*bounded synchronized*.

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Here, we report the following lemma to state the equivalence of (a) and (b).

Lemma 1: Conditions (3) and (4) are equivalent.

Proof: (a)⇒(b). Let us consider a generic couple of nodes (i, j). Adding and subtracting $\bar{x}(t)$ we can write

$$||x_i(t) - x_j(t) + \bar{x}(t) - \bar{x}(t)|| \leq$$
$$||x_i(t) - \bar{x}(t)|| + ||x_j(t) - \bar{x}(t)|| \,.$$

Since e is bounded by hypothesis, then its component e_i is also bounded, $\forall i = 1, \ldots, N$. This implies the boundedness of $||x_i(t) - x_j(t)||$.

(b)⇒(a) Since (4) holds, then $\forall \bar{\epsilon} > \epsilon_2$ there exists a positive scalar $\bar{t}$ such that $||x_i(t) - x_j(t)|| < \bar{\epsilon}$ for all $t > \bar{t}$. Now, let us introduce the ball $\mathcal{B}_{\bar{\epsilon}}(x_i(t)) = \{z \in \mathbb{R}^n : ||z - x_i(t)|| \leq \bar{\epsilon}\}$ and the set $\mathcal{U}(t) = \mathcal{B}_{\bar{\epsilon}}(x_1(t)) \cup \mathcal{B}_{\bar{\epsilon}}(x_2(t)) \cup \cdots \cup \mathcal{B}_{\bar{\epsilon}}(x_N(t))$, $t > \bar{t}$. Clearly, as (4) holds, then $\mathcal{B}_{\bar{\epsilon}}(x_i(t)) \cap \mathcal{B}_{\bar{\epsilon}}(x_j(t)) \neq \varnothing$, for all $(i, j) \in \mathcal{E}$. Since $\bar{x}(t) \in \mathcal{U}(t)$, then we can conclude that

$$\lim_{t\to\infty} ||x_i(t) - \bar{x}(t)|| < 2N\bar{\epsilon},$$

for all $(i, j) \in \mathcal{E}$. ∎

Definition 2: Similarly to what stated in [18], [19], we say that a vector field $f : \mathbb{R}^n \times \mathbb{R}^+ \mapsto \mathbb{R}^n$ is QUAD if and only if, for any $x, y \in \mathbb{R}^n$, $t \in \mathbb{R}^+$:

$$(x - y)^T[f(x, t) - f(y, t)] \leq (x - y)^T W(x - y),$$

where $W = diag\{w_1, \ldots, w_n\}$ is an arbitrary diagonal matrix of order n.
Note that this property is also known as one-sided Lipschitz condition when $W = wI_n$ [20]. Furthermore, the QUAD condition is also related to some other relevant properties of a given vector field, such as contraction and the classical Lipschitz condition (see [19] and references therein for further details).

Lemma 2: [21]

1) The Laplacian matrix $\mathcal{L}$ in a connected undirected network is positive semi-definite. Moreover, it has a simple eigenvalue at 0 and all the other eigenvalues are positive.
2) the smallest nonzero eigenvalue $\lambda_2(\mathcal{L})$ of the Laplacian matrix satisfies

$$\lambda_2(\mathcal{L}) = \min_{z^T 1_N, z\neq 0} \frac{z^T L z}{z^T z}.$$

III. Piecewise-smooth dynamical systems

Following [1] p.73, we now give the definition of a piecewise-smooth dynamical system.

Definition 3: Given a finite collection of disjoint, open and non empty sets $\mathcal{S}_1, \ldots \mathcal{S}_p$ such that $\mathcal{D} = \bigcup_{i=1}^p \bar{\mathcal{S}}_k \subseteq \mathbb{R}^n$ is a connected set, a dynamical system $\dot{x} = f(x, t)$ is called a *piecewise-smooth dynamical system* (PWS system) when it is defined by a finite set of vector fields, i.e.

$$f(x, t) = F_k(x, t), \qquad x \in \mathcal{S}_k. \tag{5}$$

The intersection $\Sigma_{kh} := \bar{\mathcal{S}}_k \cap \bar{\mathcal{S}}_h$ is either a lower dimensional manifold or it is the empty set. Each vector field $F_k(x, t)$ is smooth in both the state x and time t, for any $x \in \mathcal{S}_k$. Furthermore, it is continuously extended on the boundary $\partial\mathcal{S}_k$.

For a PWS sytem defined as above, many different solutions can be considered (see [20] and references therein). In this paper we focus on so called *Filippov solutions* [15]. Such solutions are absolutely continuous curves $x(t) : \mathbb{R} \mapsto \mathbb{R}^n$ that satisfy the differential inclusion:

$$\dot{x}(t) \in \mathcal{F}[f](x, t), \tag{6}$$

where $\mathcal{F}[f](x, t)$ is the *Filippov set-valued function* $\mathcal{F}[f] : \mathbb{R}^n \times \mathbb{R} \mapsto \mathcal{P}(\mathbb{R}^n)$, with $\mathcal{P}(\mathbb{R}^n)$ being the collection of all subsets in $\mathbb{R}^n$, defined as

$$\mathcal{F}[f](x, t) = \bigcap_{\delta>0} \bigcap_{m(\mathcal{S})=0} \bar{co}\left\{f(\mathcal{B}_\delta(x)\backslash\mathcal{S}, t)\right\},$$

where $\mathcal{S}$ is any set of zero Lebesgue measure $m(\cdot)$ and $\mathcal{B}_\delta$ is a ball of radius δ.

As reported in [20], computing the Filippov set-valued function can be a daunting task. We now report two useful rules we will apply in what follows:

Consistency: If $f : \mathbb{R}^n \times \mathbb{R} \mapsto \mathbb{R}^n$ is continuous at $x \in \mathbb{R}^n$, then

$$\mathcal{F}[f](x, t) = \{f(x, t)\}\,.$$

Sum: If $h, g : \mathbb{R}^n \times \mathbb{R} \mapsto \mathbb{R}^n$ are locally bounded at $x \in \mathbb{R}^n$, then

$$\mathcal{F}[h + g](x, t) \subseteq \mathcal{F}[h](x, t) + \mathcal{F}[g](x, t).$$

Moreover, if either h or g is continuous at x, then equality holds.

IV. Network of PWS systems

We now study the problem of giving conditions to guarantee bounded synchronization of a network of PWS systems. To this aim, we firstly introduce the following class of dynamical systems.

Definition 4: A PWS system is said to be *QUAD affine* if its vector field can be written in the form:

$$f(x, t) = h(x, t) + g(x, t), \tag{7}$$

where:

1) $h(x, t)$ is a QUAD function continuous in both the arguments.
2) $g(x, t)$ is a PWS function that can be expressed as $g(x, t) = G_k(x, t)$ if $x \in \mathcal{S}_k$, $k \in \{1, \ldots, p\}$. $g(x, t)$ is such that there exist positive scalars $M_k < +\infty$ such that

$$||G_k(x, t)|| < M_k, \qquad \forall x \in \mathbb{R}^n, \forall t > 0,$$

for all $k \in \{1, \ldots, p\}$.

Notice that QUAD affine systems can exhibit sliding mode and chaotic solutions.

Applying the consistency and sum rules reported in Sec.III, it is easy to check that the differential inclusion (6) describing a QUAD affine system can be written in the form:

$$\dot{x} = h(x, t) + \xi, \tag{8}$$

with $\xi \in \mathcal{F}[g](x,t)$. Furthermore, notice that

$$||\xi|| < M, \qquad \forall x \in \mathbb{R}^n, t > 0,$$

where $M = \max_k M_k$.

Now, we are ready to state the following result.

Theorem 1: Given a connected network of identical QUAD affine systems of the form

$$\dot{x}_i = f(x_i,t) - c\sum_{j=1}^{N} \ell_{ij}\Gamma x_j, \quad x_i \in \mathbb{R}^n, \forall i = 1,\ldots,N, \tag{9}$$

if the inner coupling matrix $\Gamma \in \mathbb{R}^{n\times n}$ is positive definite, then there exists a constant $\bar{c} < +\infty$ such that for any coupling gain $c > \bar{c}$ the network achieves ϵ-bounded synchronization. Furthermore,

1) An upper bound for the critical value of the coupling gain $\bar{c}$ is given by

$$\hat{c} = \frac{\lambda_{max}(W)}{\lambda_2(\mathcal{L}\otimes\Gamma)}. \tag{10}$$

2) For any $c \geq \hat{c}$, an upper bound on ϵ can be given as:

$$\hat{\epsilon} \leq \frac{\sqrt{N}M}{c\lambda_2(\mathcal{L}\otimes\Gamma) - \lambda_{max}(W)}. \tag{11}$$

Proof: To study the stability of a network of QUAD affine nodes, we need to rewrite the network model (9) in terms of the synchronization error defined in (2):

$$\dot{e}_i = f(x_i,t) - \frac{1}{N}\sum_{j=1}^{N} f(x_j,t) - c\sum_{j=1}^{N}\ell_{ij}\Gamma e_j, \tag{12}$$

for all $i = 1,\ldots,N$.

Now, let us consider

$$V(e) = \frac{1}{2}\sum_{i=1}^{N} e_i^T e_i \tag{13}$$

as a candidate Lyapunov function for the error system.

The derivative of V along the trajectory of the error system (12), namely $\dot{V}(e) = \sum_{i=1}^{N} e_i^T \dot{e}_i$, can be written as

$$\dot{V} = \sum_{i=1}^{N} e_i^T \left(f(x_i,t) - \frac{1}{N}\sum_{j=1}^{N} f(x_j,t) - c\sum_{j=1}^{N}\ell_{ij}\Gamma e_j \right). \tag{14}$$

From (8) and summing and subtracting $\sum_{i=1}^{N} e_i^T h(\bar{x},t)$, we then have

$$\begin{aligned}\dot{V} \;=\; & \sum_{i=1}^{N} e_i^T (h(x,t) - h(\bar{x},t)) + \\ & + \sum_{i=1}^{N} e_i^T \left[\left(h(\bar{x},t) - \frac{1}{N}\sum_{j=1}^{N}[h(x_j,t)+\xi_j]\right)\right] + \\ & + \sum_{i=1}^{N} e_i^T \xi_i - c\sum_{i=1}^{N}\sum_{j=1}^{N} e_i^T \ell_{ij}\Gamma e_j.\end{aligned}$$

From (2), follows that $\sum_{i=1}^{N} e_i^T = 0$, and therefore we have $\sum_{i=1}^{N} e_i^T \left[\left(h(\bar{x},t) - \frac{1}{N}\sum_{j=1}^{N}[h(x_j,t)+\xi_j]\right)\right] = 0$. Hence, the time derivative of the Lyapunov function along the trajectory of the error can be rewritten in compact form as:

$$\dot{V} = e^T\Phi(x) + e^T\Xi - ce^T(\mathcal{L}\otimes\Gamma)e, \tag{15}$$

where $\Phi(x,t) = [(h(x_1,t) - h(\bar{x},t))^T, \ldots, (h(x_N,t) - h(\bar{x},t))^T]^T$ and $\Xi = [\xi_1^T, \ldots, \xi_N^T]^T$.

Considering that, from Definition 4, the function h is QUAD, we have

$$\dot{V} \leq e^T\left[I_N\otimes W - c\mathcal{L}\otimes\Gamma\right]e + \sqrt{N}\,\|e\|\,M. \tag{16}$$

Let us rewrite the synchronization error as $e = a\hat{e}$, with $\hat{e} = \frac{e}{\|e\|}$. Then, we have

$$\dot{V} \leq a^2\hat{e}^T[I_N\otimes W - c\mathcal{L}\otimes\Gamma]\hat{e} + a\sqrt{N}M \tag{17}$$

Considering Lemma 2, simple algebraic manipulations yield

$$\dot{V} \leq [\lambda_{max}(W) - c\lambda_2(\Pi)]a^2 + a\sqrt{N}M, \tag{18}$$

where $\Pi = \mathcal{L}\otimes\Gamma$, $\lambda_2(\Pi)$ is the smallest nonzero eigenvalue of matrix Π and $\lambda_{max}(W)$ is the maximum eigenvalue of matrix W. Notice that, being $\lambda_2(\Pi)$ positive, we can choose $c > \lambda_{max}(W)/\lambda_2(\Pi)$ such that $\lambda_{max}(W) - c\lambda_2(\Pi) < 0$. Then, we have that $a > \sqrt{N}M/\left[c\lambda_2(\Pi) - \lambda_{max}(W)\right]$ implies $\dot{V} < 0$. Hence, network (9) is ϵ-bounded synchronized.

The upper bounds (10) and (11) on $\bar{c}$ and ϵ, respectively, follow trivially from (18). ■

Remark 1: Notice that, if the coupling strenght c is a control parameter, then it can be used to make the bound (11) arbitrarily small.

A simple class of piecewise-smooth dynamical systems are *bounded switching systems*, defined below.

Definition 5: Let us consider a switching dynamical system

$$\dot{x} = g(x,t). \tag{19}$$

where the PWS function $g(x,t) = G_k(x,t)$ if $x \in \mathcal{S}_k$. We say that (19) is a *bounded switching system* (BSS) if and only if the PWS function g is such that there exists a finite positive scalar such that

$$||G_k(x,t)|| \leq M_k \qquad \forall x \in \mathbb{R}^N,\ \forall t \geq 0,$$

for all $k \in \{1,\ldots,p\}$.

It is worth noting that a straightforward corollary for bounded synchronization of networks of BSSs can be derived from Theorem 1.

Corollary 1: Consider a connected network of BSSs of the form

$$\dot{x}_i = g(x_i,t) - c\sum_{j=1}^{N}\ell_{ij}\Gamma x_j, \quad x_i \in \mathbb{R}^n, \forall i = 1,\ldots,N, \tag{20}$$

If the inner coupling matrix $\Gamma \in \mathbb{R}^{n\times n}$ is positive definite, then there exists a constant $\bar{c} < +\infty$ such that for any coupling gain $c > \bar{c}$ the network achieves ϵ-bounded synchronization. Furthermore, increasing c, the bound ϵ can be made arbitrarily small.

Proof: Since each vector field $g_i(x_i,t)$ can be written as $f_i(x_i,t) = f(x_i,t) + g_i(x_i,t)$, with $f(x_i,t) = 0$ being the trivial QUAD component, from Theorem 1 follows the thesis. ■

V. Numerical Examples

Several examples of discontinuous dynamical systems satisfying the assumptions of Theorem 1 can be made. In particular, any QUAD system with a discontinuous feedback nonlinearity such relay, saturation and hysteresis is also a QUAD affine system. Here, we consider a network of five classical relay systems, e.g. [22], whose dynamics are described by:

$$\dot{x}_i = Ax_i + Br_i + u_i, \quad y_i = Cx_i, \quad r_i = -sgn(y_i),$$

where

$$A = \begin{bmatrix} 1.35 & 1 & 0 \\ -99.93 & 0 & 1 \\ -5 & 0 & 0 \end{bmatrix},$$

$$B = [1, -2,\ 1]^T, \quad C = [1,\ 0,\ 0],$$

where u is the coupling protocol, namely:

$$u = -c \sum_{j=1}^{N} \ell_{ij} \Gamma x_j.$$

As shown in [1], [23], with this choice of parameter values each relay exhibits both sliding motion and chaotic behaviour.

The Laplacian matrix describing the network topology is

$$\mathcal{L} = \begin{bmatrix} 3 & -1 & 0 & -1 & -1 \\ -1 & 4 & -1 & -1 & -1 \\ 0 & -1 & 2 & -1 & 0 \\ -1 & -1 & -1 & 4 & -1 \\ -1 & -1 & 0 & -1 & 3 \end{bmatrix},$$

while the inner coupling matrix is $\Gamma = I_3$, that is, the nodes are coupled through all state variables. In our simulation, we set the coupling gain $c = 50$, while the initial conditions are chosen randomly.

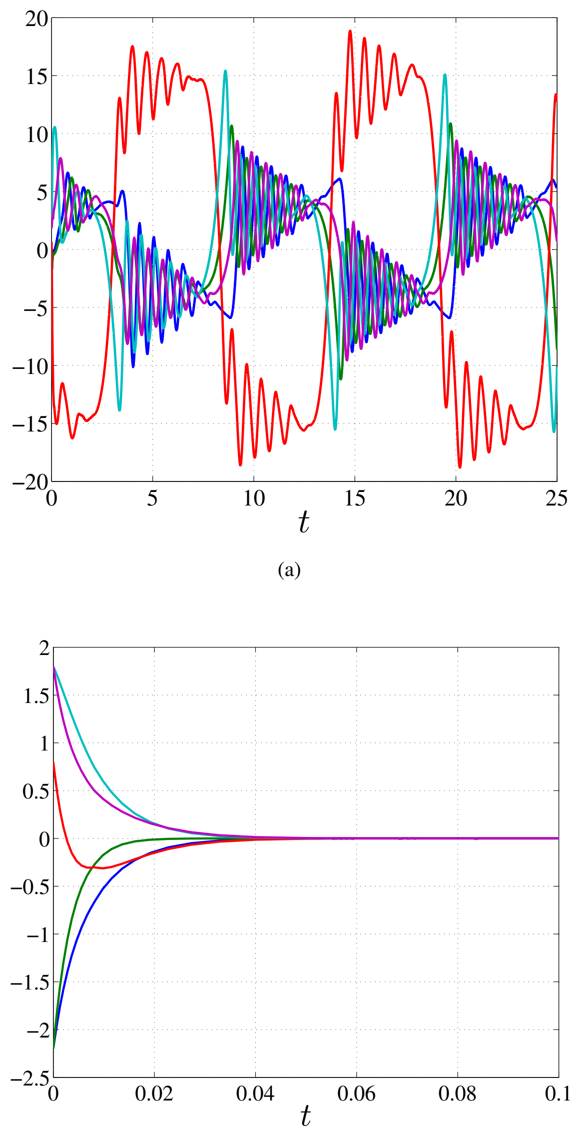
As illustrated in Figs. 1, 2, 3, we compare the behavior of the coupled network with the case of disconnected nodes. In particular, Figs. 1, 2 show the time evolution of the second component of the synchronization error for each node, for both the uncoupled and coupled case, while Fig. 3 shows the evolution in the state space.

Despite the presence of sliding motion, we observe the coupling to be effective in causing all nodes to synchronize.

Notice that this numerical result is consistent with Theorem 1. In fact, the node dynamics can be expressed in terms of (7), where $h(x, t) = Ax$ is the QUAD term and where $g(x, t) = Br$ is the bounded switching term. Hence, we can use Theorem 1 to get an upper bound on the minimum coupling gain guaranteeing ϵ-bounded synchronization. Notice that

$$(x-y)^T (h(x) - h(y)) = (x-y)^T A (x-y) =$$
$$(x-y)^T A_{sym} (x-y) \leq \lambda_{max}(A_{sym}) (x-y)^T (x-y),$$

where $A_{sym} = \frac{1}{2}(A + A^T)$. For the considered example, we have $\lambda_{max}(A_{sym}) = 50$, while $\lambda_2(\Pi) = \lambda_2(\mathcal{L} \otimes I) = 2$. Therefore, from (10), the lower bound $\hat{c}$ is 25. Since we selected $c = 50 > \hat{c}$, we can use Theorem 1 to derive a conservative estimate of the bound ϵ. To this aim, we need to compute the bound M on the affine term. Notice that it can be easily evaluated by computing the Euclidean norm of the vector $[B^T, B^T, ..., B^T]^T$. In our example, we have $M = 5.5$. Therefore, using (11), we can conclude that an upper bound for the norm of the stack error vector e is $\hat{\epsilon} \leq 0.25$. This result is consistent with what is observed in Fig. 1(b).



(a)

(b)

Fig. 1. Time evolution of error components $e_i^{(2)}(t)$ for the network of chaotic relays: (a) uncoupled case; (b) coupled case.

VI. Conclusions

In this paper, we have derived sufficient conditions for ϵ-bounded synchronization of networked piecewise-smooth systems. In particular, we have shown that a class of

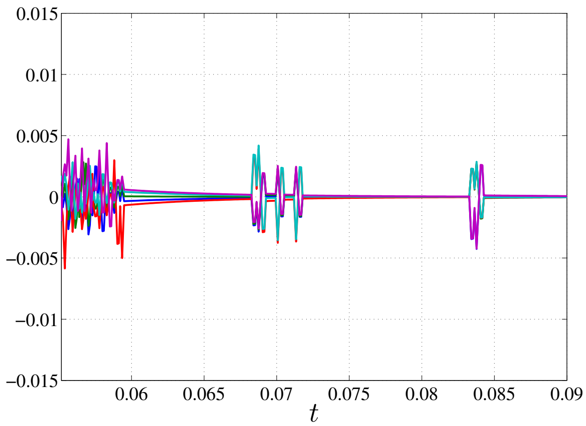
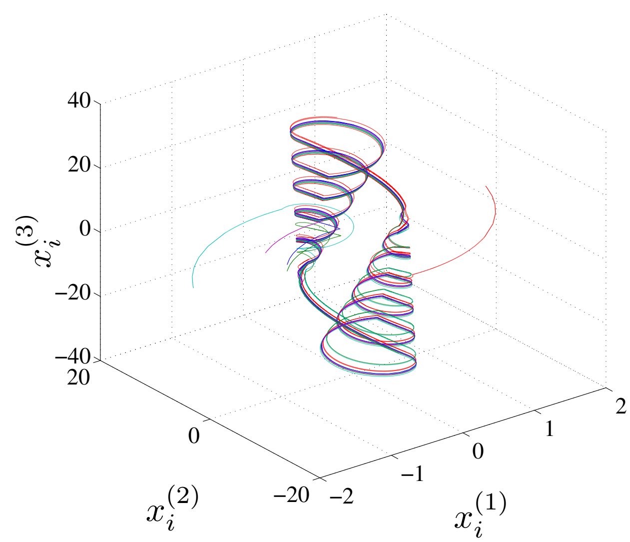
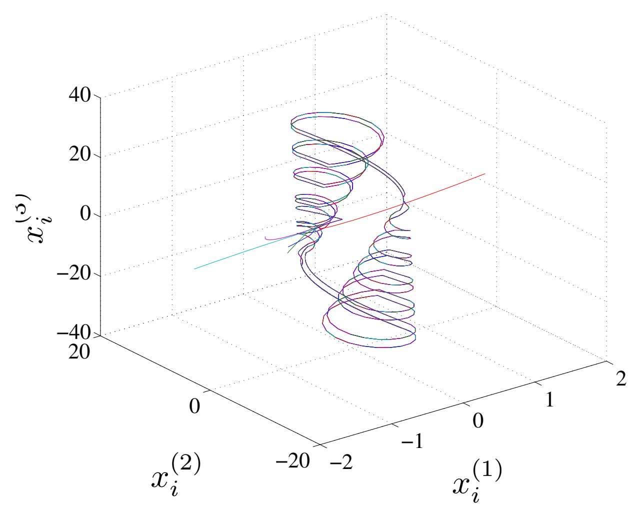


Fig. 2. Zoom of the evolution of the synchronization error for $t > 0.055s$ showing bounded convergence.



(a)



(b)

Fig. 3. State space evolution for the network of chaotic relays: (a) uncoupled case; (b) coupled case.

piecewise-smooth systems, that we called QUAD affine, can achieve ϵ-bounded synchronization. The vector field characterizing this class of systems is characterized by two additive terms: a QUAD continuous function and a bounded piecewise-smooth function.

Furthermore, we have given a conservative estimation on the minimum coupling gain guaranteeing ϵ-bounded synchronization, and an upper bound for ϵ. Analytical results are complemented by numerical simulations on a network of chaotic relay systems. The proposed example shows how bounded synchronization can be achieved even in presence of sliding motion and chaotic behaviour of the network.

We wish to emphasize that, despite the simplicity of our approach which assumes boundedness of the PWS term, no other generic result is currently available on networks of PWS systems exhibiting sliding. Thus, we believe our approach can offer a simple yet powerful alternative to give conditions for the synchronization of networks of PWS systems.

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