Noise Analysis in Satellite Attitude Estimation using Angular Rate and a Single Vector Measurement

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Abstract— **This paper investigates the effect of noisy measurements of the angular rate in a nonlinear attitude estimator for satellites. The attitude estimator uses measurement of a single attitude sensor such as sun, earth horizon, star tracker or magnetometer together with a rate gyro, and guarantees exponential convergence of the attitude estimation error to zero under no noise condition. This paper presents stochastic and deterministic upper bounds for the attitude estimation error affected by the noise in gyro. A realistic simulation is presented to illustrate the results.**

I. INTRODUCTION

Measurement noise has a significant effect on performance of any attitude estimation algorithm and can even cause the algorithm to diverge. For example, recent proliferation of Micro-Electro-Mechanical Systems (MEMS) components has led to development of a range of low cost and light weight inertial measurements units (IMU). The signal output of low cost IMU systems, however, is characterized by low-resolution signals subject to high noise levels [2].

The Extended Kalman Filter (EKF), which was developed to account for nonlinearity in systems, has been applied in a number of spacecraft attitude determination systems. However, EKF is based on linearization of system dynamics and can diverge due to modeling uncertainty and measurement noise. Moreover, assumptions on noise characteristics might not be satisfied in real situation. Implementation of the EKF is also computationally complex [3], [13] and [14]. Computational burden were alleviated in [3] by integration of the adaptive methods with the EKF by resorting to *multiple* vector measurements from sun sensors, earth sensors and magnetometers. A dual Unscented Kalman Filter (UKF) was proposed in [7] which used the first and the second order terms of a nonlinear system such that the resulting UKF had superior convergence properties and improved accuracy in presence of measurement noise grace to using *two noisy* vector measurements and noisy angular velocity measurement. The left-invariant extended Kalman filter investigated in [4], took into account the particular geometry and symmetries of the system model on the Lie group. The measurement noise was defined in a way to preserve the symmetries; however, no concrete convergence properties were demonstrated.

In [5], [2] and [6] nonlinear estimators were proposed and stability of the attitude estimation error was proven in presence of noisy measurements. However, the observers were derived assuming that perfect attitude was already available by batch preprocessing of multiple vector measurements. It happens that in a realistic situation the statistical properties of noise are not exactly known. Therefore, an alternative approach is to specify a deterministic (hard) bound on the measurement noise that is usually available in many applications. In [8], an attitude estimation algorithm was proposed that globally minimized the attitude estimation error by assuming that the initial attitude together with the initial angular velocity and measurement noise stayed within some given ellipsoidal bounds. Ultimate boundedness of the attitude attitude and the angular velocity estimation errors within the ellipsoidal sets was shown assuming that *multiple* vector measurements were available. In [1] and [12] nonlinear observers were designed for satellite attitude determination using a 3-axis gyro and only *a single* vector measurement. Exponential and asymptotic convergence of the estimation error to zero were respectively proven in the *absence* of noise.

Therefore, noise analysis for asymptotically convergent nonlinear observers using *a single* vector measurement and gyro remains open in literature. In this paper we present stochastic and deterministic noise analysis for the observer of [1]. The proposed observer ensures exponential convergence of the estimation error to zero in no noise condition. In presence of noise in gyro measurements, deterministic and stochastic upper bounds for the estimation error are presented. Computation of the upper bounds are based on solution of delayed integral equations. The analysis presents also a method for optimal tuning of the observer gain that minimizes the effect of noise in the attitude estimate. The paper is organized as follows. Section II is devoted to the problem formulation and definitions. In section III the observer dynamics for the attitude estimation is presented. The estimation error dynamics in presence of sensor noise, is derived in section IV. Stochastic and deterministic analysis are presented in sections V and VI, respectively. Finally, section VII presents the simulation results.

II. DEFINITIONS

We denote the smallest and the largest singular value of the matrix A by $\underline{\sigma}(A)$ and $\bar{\sigma}(A)$, respectively. Also for any $a \in \mathbb{R}^3$, $\|a\|$ denotes the euclidian norm of the vector.

Definition 1: For any vector $a = [a_1 \ a_2 \ a_3]^T \in \mathbb{R}^3$, we define the skew operator by

$$S(a) = \begin{pmatrix} 0 & -a_3 & a_2 \\ a_3 & 0 & -a_1 \\ -a_2 & a_1 & 0 \end{pmatrix} \qquad (1)$$

The unskew operator is defined such that $S^{-1}(S(a)) = a$.

Property 1: $S(a)S(b) = ba^T - a^T b I$

The kinematic equation of the satellite, considered as a rigid body, is given by

$$\dot{R} = RS(\omega) \qquad (2)$$

where $R \in SO(3)$ denotes the attitude matrix of the satellite which determines the rotation of the satellite body frame with respect to the inertial frame. $\omega \in \mathbb{R}^3$ is the angular velocity of the body frame with respect to the inertial frame and expressed in the body frame. It is assumed that the angular velocity is measured by a 3-axis rate gyroscope whose output is contaminated by additive noise, and is given by

$$\omega_g(t) = \omega(t) + n(t) \qquad (3)$$

where $\omega_g(t)$ denotes the gyro measurement, $\omega(t)$ is the true satellite angular velocity and $n(t)$ is the gyro noise. Moreover, it is assumed that *only one* attitude sensor such as magnetometer or sun sensor is available for attitude estimation. Depending on type of the used attitude sensor, we define a reference vector $v_r \in \mathbb{R}^3$ which is expressed in the inertial frame and depends on satellite position in orbit [1]. Examples of such vectors are a vector with its origin at Sun location and its end at satellite position in case of using sun sensor, or Earth's magnetic field vector evaluated at the satellite position in orbit, in case of using a magnetometer. We denote $v_b \in \mathbb{R}^3$ as the representation of the reference vector in the satellite body frame. We assume that v_b is measured by means of an attitude sensor. By virtue of definition of R, the reference vector and the measured vector are related by

$$v_b = R^T v_r \qquad (4)$$

Since the magnitudes of v_r and v_b have no information on satellite attitude, we assume that $\|v_r\| = \|v_b\| = 1$.

III. OBSERVER

Considering the measurement noise in gyro, the observer of [1] is given by

$$\dot{\hat{R}} = \hat{R}S(\hat{\omega}) \qquad (5)$$

$$\hat{\omega} = \omega_g - k_\omega \gamma_\omega \qquad (6)$$

$$\gamma_\omega = \hat{R}^T S^{-1}(\hat{R}v_b v_r^T - v_r v_b^T \hat{R}^T) \qquad (7)$$

where $\hat{R}$ denotes the estimate of the attitude matrix and k_ω is the observer gain which is a constant positive scalar. The attitude estimation error matrix is defined by

$$\tilde{R} = \hat{R}R^T \qquad (8)$$

Evidently, $\hat{R}$ converges to R if and only if $\tilde{R}$ converges to the identity matrix. Note that the corresponding rotation matrix $\tilde{R}$ can be represented using the Rodrigues formula[16]

$$\tilde{R} = \text{rot}(\theta, \lambda) = I + \sin\theta S(\lambda) + (1 - \cos\theta)S^2(\lambda) \qquad (9)$$

where $\theta \in \mathbb{R}$, $0 \leq \theta \leq \pi$ denotes the angle of rotation. The vector $\lambda \in \mathbb{R}^3$ represents the axis of rotation such that $\|\lambda\| = 1$.

IV. ERROR DYNAMICS

As shown in [1], the angle-axis representation of the estimation error is given by

$$\dot{\theta} = \lambda^T R(\hat{\omega} - \omega) \qquad (10)$$

$$\dot{\lambda} = \frac{1}{2}\big(I - \frac{\sin\theta}{1 - \cos\theta}S(\lambda)\big)S(\lambda)R(\hat{\omega} - \omega) \qquad (11)$$

Substituting for $\hat{\omega}$ from (6) into (10) and (11) leads to

$$\dot{\theta} = -k_\omega \sin\theta \lambda^T P\lambda + \lambda^T Rn \qquad (12)$$

$$\dot{\lambda} = k_\omega S^2(\lambda)P\lambda + \frac{1}{2}\big(I - \frac{\sin\theta}{1 - \cos\theta}S(\lambda)\big)S(\lambda)Rn \qquad (13)$$

where

$$P = \text{tr}(v_r v_r^T)I - v_r v_r^T = I - v_r v_r^T \qquad (14)$$

Here, tr represents the trace of a matrix. The matrix $P \geq 0$ was shown to be a positive semi-definite matrix with rank two but its integral over some interval of time is a full rank matrix.

V. STOCHASTIC ANALYSIS

Considering the Lyapunov function

$$W = \frac{1}{4}\text{tr}\big((I - \tilde{R})^T(I - \tilde{R})\big) = 1 - \cos\theta \qquad (15)$$

and assuming that all the sensors are noise free, the almost globally exponential convergence of W to zero has been proved in [1] provided that the following condition is satisfied

$$\frac{1}{T}\int_t^{t+T} P(\tau)d\tau \geq \beta I \qquad (16)$$

where $I \in \mathbb{R}^{3\times 3}$ is the identity matrix and β, T are constant scalars. Note that the convergence of W to zero is equivalent to convergence of $\tilde{R}$ to I, i.e. , $\hat{R}$ converges to R, exponentially.

Remark 1: A practical way to compute T and β satisfying the inequality (16), is to compute β as a function of T by

$$\beta(T) = \min_{0 \leq t \leq T_0} \underline{\sigma}\big(\frac{1}{T}\int_t^{t+T} P(\tau)d\tau\big) \qquad (17)$$

where T_0 is the period of signal v_r. $\qquad\square$

In this section we assume that the gyro measurement contains additive noise $n(t)$ with the following stochastic properties

$$E[n(t)] = \bar{n} = 0 \qquad (18)$$

$$\text{var}[n(t)] = E[(n(t) - \bar{n})^T(n(t) - \bar{n})] = \sigma_n^2 \qquad (19)$$

where $E[.]$ denotes the statistical expectation. Note that by (15), $E[W(t)]$ is closely related to variance of the attitude estimation error.

Theorem 1: Under the condition (16), $E[W(t)]$ satisfies the delayed integral inequality given by

$$E[W(t + T)] \leq$$

$$(1 - \frac{k_\omega \beta T}{1 + k_\omega^2 T^2})E[W(t)] + \frac{8k_\omega \beta T + k_\omega T^3 k_\psi^2 \sigma_n^2}{4(1 + k_\omega^2 T^2)}$$

$$+ \sigma_n \int_t^{t+T} \sqrt{2E[W(\tau)] - E[W(\tau)]^2}d\tau \qquad (20)$$

where $k_\psi^2 = 4\sqrt{2}$ is calculated in Appendix IX-B. □

Remark 2: The delayed integral inequality (20) when transformed into an equality can be solved numerically provided that an initial condition $E[W(t_0)]$ is known. The resulting solution, denoted by $\bar{E}(t)$, is an upper bound for any solution of the integral inequality. □

Remark 3: In practical situations, it is important to investigate the steady state effect of noise on the estimation error. The steady state value of $\bar{E}(t)$, denoted by $E_{ss} = \lim_{t\to\infty} \bar{E}(t)$, satisfies the following algebraic equality

$$E_{ss} = (1 - \frac{k_\omega \beta T}{1 + k_\omega^2 T^2})E_{ss} + \frac{8k_w\beta T + k_\omega T^3 k_\psi^2 \sigma_n^2}{4(1 + k_\omega^2 T^2)}$$
$$+ \sigma_n T\sqrt{2E_{ss} - E_{ss}^2} \tag{21}$$

Note that E_{ss} depends on β and T that are themselves related to the properties of reference vector v_r, the observer gain k_w and variance of measurement noise, σ_n. Therefore, for a given reference signal v_r and in light of (16)-(17), an optimal estimator can be defined as a one that minimizes $E_{ss}(k_\omega, T)$ by

$$C_E = \min_{k_\omega, T} E_{ss}(k_\omega, T)$$

□

Proof of Theorem 1: Differentiating W with respect to time and substituting from (12), yields

$$\dot{W} = -k_\omega \sin^2 \theta \lambda^T P\lambda + \sin\theta \lambda^T Rn \tag{22}$$

Now, taking the expectation from both sides of (22), yields

$$E[\dot{W}] = -k_\omega E[\sin^2 \theta \lambda^T P\lambda] + E[\sin\theta \lambda^T Rn] \tag{23}$$

According to the Cauchy-Schwartz inequality [10], we have

$$E[X^T Y] \le \sqrt{E[X^T X]}\sqrt{E[Y^T Y]}$$

Choosing $X = \sin\theta R^T \lambda$ and $Y = n$, yields

$$E[\sin\theta \lambda^T Rn] \le \sqrt{E[\sin^2 \theta]}\sqrt{E[n^T n]} \tag{24}$$

where

$$\sin^2 \theta = 1 - \cos^2 \theta = 2W - W^2 \tag{25}$$

Therefore, (24) simplifies to

$$E[\sin\theta \lambda^T Rn] \le \sqrt{E[2W - W^2]}\sigma_n \tag{26}$$

Combining (23) and (26) leads to

$$E[\dot{W}] \le -k_\omega E[\sin^2 \theta \lambda^T P\lambda] + \sqrt{E[2W - W^2]}\sigma_n \tag{27}$$

Integrating both sides of (27) in the interval $[t, t+T]$, yields

$$E[W(t+T)] - E[W(t)] \le$$
$$- k_\omega E\Big[\int_t^{t+T} \sin^2 \theta(\tau)\lambda(\tau)^T P(\tau)\lambda(\tau)d\tau\Big]$$
$$+ \sigma_n \int_t^{t+T} \sqrt{E[2W(\tau) - W^2(\tau)]}d\tau \tag{28}$$

Next, we find an upper bound for the first term in the RHS of (28) in terms of $E[W(t)]$. Similar to [1], we can write

$$\int_t^{t+T} \sin^2 \theta(\tau)\lambda(\tau)^T P(\tau)\lambda(\tau) \ge$$
$$\frac{1}{2}\int_t^{t+T} \sin^2 \theta(t)\lambda(t)^T P(\tau)\lambda(t)d\tau$$
$$- \int_t^{t+T} \big[\sin\theta(\tau)\lambda(\tau)^T - \sin\theta(t)\lambda(t)^T\big]P(\tau)\times$$
$$\big[\sin\theta(\tau)\lambda(\tau) - \sin\theta(t)\lambda(t)\big]d\tau \tag{29}$$

A lower bound for the first term in RHS of (29) can be found by left multiplying (16) by $\sin\theta(t)\lambda(t)$ and right multiplying it by $\sin\theta(t)\lambda(t)^T$ and substituting from (25) as follows:

$$\int_t^{t+T} \sin^2 \theta(t)\lambda(t)^T P(\tau)\lambda(t)d\tau \ge \beta T\big(2W(t) - W(t)^2\big) \tag{30}$$

Also, the second term in the RHS of (29) can be bounded by the use of the next Lemma.

Lemma 1: For any $t > 0$ we have

$$\int_t^{t+T} \big[\sin\theta(\tau)\lambda(\tau)^T - \sin\theta(t)\lambda(t)^T\big]P(\tau)\times$$
$$\big[\sin\theta(\tau)\lambda(\tau) - \sin\theta(t)\lambda(t)\big]d\tau$$
$$\le k_\omega^2 T^2 \int_t^{t+T} \sin^2 \theta(\tau)\lambda(\tau)^T P(\tau)\lambda(\tau)d\tau$$
$$+ \frac{T^2 k_\psi^2}{4}\int_t^{t+T} n(\tau)^T n(\tau)d\tau \tag{31}$$

Proof: See Appendix IX-A.

Introducing (30) and (31) into (29), yields

$$\int_t^{t+T} \sin^2 \theta(\tau)\lambda(\tau)^T P(\tau)\lambda(\tau)d\tau \ge \frac{\beta T}{2(1 + k_\omega^2 T^2)}\times$$
$$\big(2W(t) - W(t)^2\big) - \frac{T^2 k_\psi^2}{4(1 + k_\omega^2 T^2)}\int_t^{t+T} n(\tau)^T n(\tau)d\tau \tag{32}$$

Taking the expectation from both sides of (32), implies

$$E\Big[\int_t^{t+T} \sin^2 \theta(\tau)\lambda(\tau)^T P(\tau)\lambda(\tau)d\tau\Big] \ge$$
$$\frac{\beta T}{2(1 + k_\omega^2 T^2)}\big(2E[W(t)] - E[W(t)^2]\big) - \frac{T^3 k_\psi^2}{4(1 + k_\omega^2 T^2)}\sigma_n^2 \tag{33}$$

Applying (33), we can rewrite (28) by

$$E[W(t+T)] \le (1 - \frac{k_\omega \beta T}{1 + k_\omega^2 T^2})E[W(t)]$$
$$+ \frac{k_w\beta T}{2(1 + k_\omega^2 T^2)}E[W(t)^2] + \frac{k_\omega T^3 k_\psi^2 \sigma_n^2}{4(1 + k_\omega^2 T^2)}$$
$$+ \sigma_n \int_t^{t+T} \sqrt{2E[W(\tau)] - E[W(\tau)^2]}d\tau \tag{34}$$

By Jensen's inequality [10], if $\varphi : \mathbb{R} \to \mathbb{R}$ is a convex function such that the random variables X and $\varphi(X)$ have finite expectations, then

$$\varphi[E(X)] \le E[\varphi(X)] \tag{35}$$

Choosing $\varphi(W) = W^2$ implies

$$E[W^2] \geq E[W]^2$$

Therefore,

$$\sqrt{2E[W] - E[W^2]} \leq \sqrt{2E[W] - E[W]^2} \quad (36)$$

It is evident that $E[W(t)^2] \leq 4$ due to the fact that $W(t) = 1 - \cos\theta \leq 2$. Considering this fact and (36), (20) will be concluded from (34). ∎

VI. DETERMINISTIC ANALYSIS

Assume that a deterministic upper bound for noise is known as

$$\sup_{t \geq 0} \|n(t)\| \leq n_{max}$$

We aim to compute an upper bound for the estimation error, $W(t)$.

Theorem 2: Under condition (16), the estimation error $W(t)$ satisfies the following difference inequality

$$W(t+T) \leq (1 - \frac{k_\omega \beta T}{1 + k_\omega^2 T^2})W(t) + \frac{k_\omega \beta T}{2(1 + k_\omega^2 T^2)}W(t)^2$$
$$+ \frac{T^3 k_\psi^2 k_\omega}{4(1 + k_\omega^2 T^2)}n_{max}^2 + T n_{max} \quad (37)$$

Proof: First integrate (22) and substitute from (32).

$$W(t+T) \leq (1 - \frac{k_\omega \beta T}{1 + k_\omega^2 T^2})W(t) + \frac{k_\omega \beta T}{2(1 + k_\omega^2 T^2)}W(t)^2$$
$$+ \frac{T^2 k_\psi^2 k_\omega}{4(1 + k_\omega^2 T^2)} \int_t^{t+T} n(\tau)^T n(\tau)d\tau$$
$$+ \int_t^{t+T} \sin\theta(\tau)\lambda(\tau)^T R(\tau)n(\tau)d\tau \quad (38)$$

According to inner product definition of two vectors, we have

$$< X, Y > = X^T Y \leq \|X\|\|Y\| \quad (39)$$

where $< X, Y >$ denotes the inner product of X and Y. Choosing $X = \sin\theta R^T \lambda$, $Y = n$ and using (39), yields

$$\sin\theta(\tau)\lambda(\tau)^T R(\tau)n(\tau) \leq |\sin\theta|n_{max} \leq n_{max} \quad (40)$$

Also, note that

$$n(\tau)^T n(\tau) = \|n(\tau)\|^2 \leq n_{max}^2 \quad (41)$$

Applying (40) and (41) to (38), the inequality (37) results. ∎

Remark 4: By knowing $W(t_0)$, (37) can be solved numerically in the equality case. The obtained solution is an upper bound for all solutions of (37). □

The RHS of (37) is a second order polynomial with respect to $W(t)$. The roots of this polynomial are

$$W_{min} = 1 - \sqrt{1 - c\gamma^{-1}}, \ W_{max} = 1 + \sqrt{1 - c\gamma^{-1}} \quad (42)$$

where

$$c = \frac{T^3 k_\psi^2 k_\omega}{4(1 + k_\omega^2 T^2)}n_{max}^2 + T n_{max}, \ \gamma = \frac{k_\omega \beta T}{2(1 + k_\omega^2 T^2)}$$

Since the RHS of (37) is negative for $W_{min} < W(t) < W_{max}$ and positive elsewhere, the following results can be inferred

- if $W(0) > W_{max}$ then $W(\infty) < 2$.
- if $W(0) < W_{max}$ then $W(\infty) \leq W_{min}$.

Based on the previous discussions it is desirable to have the smallest W_{min} and the largest W_{max}. Given β, T satisfying (16), it can be shown that the optimizing value of the observer gain is $k_\omega^* = \frac{1}{T}$

VII. SIMULATION

In this section simulation results for the deterministic approach is presented. We consider a satellite equipped with a 3-axis gyro and a magnetometer, and moving in a circular sun-synchronous orbit with the inclination $98.7°$, right ascension of ascending node equal to $200°$ and altitude of 800 Km, yielding an orbital period of 100 minutes. Rotation dynamics of the satellite is given by $I_s\dot{\omega} + \omega \times I_s\omega = \tau$, where $I_s \in \mathbb{R}^{3 \times 3}$ is the moment of inertia matrix of the satellite and τ is the external torque exerted to satellite. We assume that the satellite is equipped with a gravity boom and no other external torque except the gravitational torque is exerted to the satellite. Similar to the Purdue Imaging Satellite (PI-SAT), the constant moment of inertia matrix of the satellite is given by $I_s = diag([13.654, 13.555, 0.765])$Kgm^2 (with the extended gravity boom) and the length and the tip mass of the gravity boom are considered as $l_{boom} = 3$m and $m_{boom} = 1$Kg, respectively [1]. The gravitational torque is expressed by $\tau = 3\sqrt{\mu_e/R_o^3}S(c_3)I_s c_3$ [17] where μ_e is the Earth gravitational constant, R_o is the radius of the orbit and c_3 is the third column of the rotation matrix from the orbit frame to satellite body frame which is denoted by R_o^b. Computation of R_o^i is based on the Keplerian elements of the satellite orbit and using the Kepler's laws [1]. To obtain the reference vector v_r for the magnetometer, we used an orbit estimator to obtain R_o^i and then used the International Geomagnetic Reference Field (IGRF) model [15] to compute the magnetic field vector of the earth in the position of the satellite. The maximum amplitude of noise is assumed to be $n_{max} = 0.0034 \frac{deg}{hour}$. In order to preserve the evolution of the estimated matrix $\hat{R}$ on the SO(3), the observer has been implemented in discrete time by the method proposed in [9]. As discussed in the previous section, for a fixed T, the maximum value of $W_2(k_\omega, T)$ and the minimum value of $W_1(k_\omega, T)$ are achieved by choosing $k_\omega^* = \frac{1}{T}$. Subsequently, it is also possible to find T that minimizes $W_1(k_\omega^*, T)$ and maximizes $W_2(k_\omega^*, T)$. For $T^* = 45$, the final optimized values are given by $W_1(k_\omega^*, T^*) = 0.0017$ and $W_2(k_\omega^*, T^*) = 1.9983$.

For optimal values of $T^* = 45$ and $k_\omega^* = 0.023$, (37) is simulated in equality case and the resultant upper bound for $W(t)$ is depicted in Fig. 1.

To see how the optimal choice of observer gain affects the true estimation error, which is evaluated by (15), W(t) is compared for the optimal gain ($k_\omega^* = 0.023$) and non-optimal gain ($k_\omega = 0.08$) in Fig. 2. Also, the velocity error ($\tilde{\omega} = \hat{\omega} - \omega$) in the optimal and non-optimal case are shown
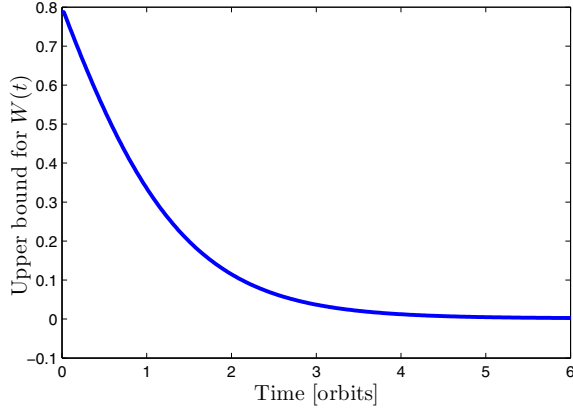
Fig. 1. Upper bound for the true attitude estimation error in optimal case (for the solid curve in Fig. 2), $k_\omega = 0.023$. One orbit equals 100 mins.
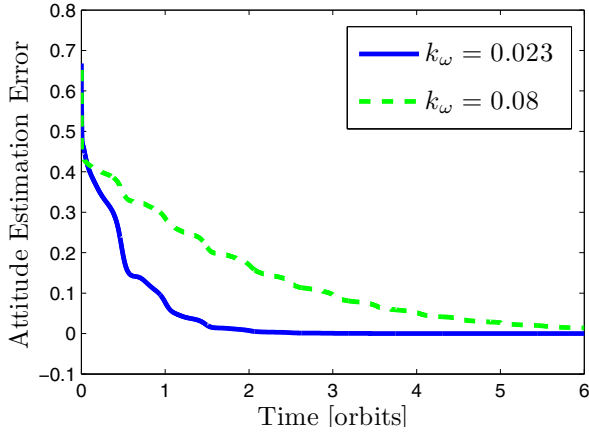

Fig. 2. Comparison between the true attitude estimation error in optimal (solid) and non-optimal (dash) case. One orbit equals 100mins.

in Fig. 3 illustrating the impact of optimal tuning of observer gain in noise reduction.

VIII. CONCLUSIONS

The observer (5)-(7) guarantees exponential convergence of the estimation error to zero under no-noise condition. In presence of noise in gyro measurements, stochastic and deterministic upper bounds for the estimation error are presented. Computation of the upper bounds are based on solution of some delayed integral equations. Computation of the upper bounds in steady-state period amounts to solving algebraic equations.

IX. APPENDIX

A. Proof of Lemma 1

It is evident that

$$\sin\theta(\tau)\lambda(\tau) - \sin\theta(t)\lambda(t) = \int_t^\tau \frac{d}{d\delta}\{sin\theta(\delta)\lambda(\delta)\}d\delta$$
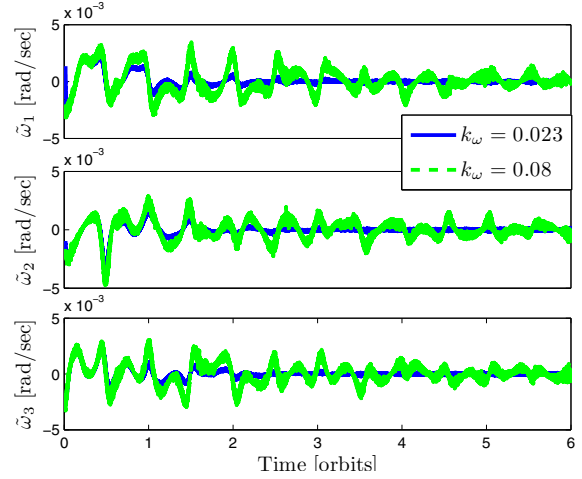
(43)

Fig. 3. Comparison between the error in estimation of angular velocity ($\tilde{\omega}$) in optimal (solid) and non-optimal case (dash).

Using (12) and (13), yields

$$\frac{d}{d\delta}\{\sin\theta(\delta)\lambda(\delta)\} = \cos\theta(\delta)\dot{\theta}(\delta)\lambda(\delta) + \sin\theta(\delta)\dot{\lambda}(\delta)$$

$$= -k_\omega \sin\theta\big(I - (1 - \cos\theta)\lambda\lambda^T\big)P\lambda + \cos\theta\lambda^T Rn\lambda$$

$$+ \frac{1}{2}\sin\theta\big(I - \frac{\sin\theta}{1 - \cos\theta}S(\lambda)\big)S(\lambda)Rn$$

Substituting $S^2(\lambda) = \lambda\lambda^T - I$ and $\sin^2\theta = 1 - \cos^2\theta$ together with straightforward algebraic manipulations, gives

$$\frac{d}{d\delta}\{\sin\theta(\delta)\lambda(\delta)\} = -k_\omega \sin\theta\big(I - (1 - \cos\theta)\lambda\lambda^T\big)P\lambda$$

$$+ \frac{1}{2}\big((1 + \cos\theta)I + \sin\theta S(\lambda) - (1 - \cos\theta)\lambda\lambda^T\big)Rn$$

Let define

$$q(\delta) = \sin\theta(\delta)\lambda(\delta)$$
$$M(\delta) = I - (1 - \cos\theta)\lambda\lambda^T$$
$$\psi(\delta) = (1 + \cos\theta)I + \sin\theta S(\lambda) - (1 - \cos\theta)\lambda\lambda^T \quad (44)$$

Then, (43) can be rewritten as

$$q(\tau) - q(t) = \int_t^\tau \big[\frac{1}{2}\psi(\delta)R(\delta)n(\delta) - k_\omega M(\delta)P(\delta)q(\delta)\big]d\delta$$

Therefore,

$$\int_t^{t+T}\big[q(\tau)^T - q(t)^T\big]P(\tau)\big[q(\tau) - q(t)\big]d\tau =$$

$$\int_t^{t+T}\Big[\int_t^\tau\big(\frac{1}{2}n(\delta)^T R(\delta)^T \psi(\delta)^T -$$

$$k_\omega q(\delta)^T P(\delta)^T M(\delta)^T\big)d\delta\Big]P(\delta)\times$$

$$\Big[\int_t^\tau\big(\frac{1}{2}\psi(\delta)R(\delta)n(\delta) - k_\omega M(\delta)P(\delta)q(\delta)\big)d\delta\Big]d\tau$$

(45)

By the Fact 1 in [1] we have $\bar{\sigma}(P(\tau)) = 1$. Applying this fact and Fact 2 in [1] for $a = t$, $b = \tau$ and $x = \frac{1}{2}\psi(\delta)R(\delta)n(\delta) -$

$k_\omega M(\delta) P(\delta) q(\delta)$ into (45), we have

$$\int_t^{t+T} \left[q(\tau)^T - q(t)^T \right] P(\tau) \left[q(\tau) - q(t) \right] d\tau \leq$$

$$\int_t^{t+T} \int_t^\tau (\tau - t) \times$$

$$\left\| \frac{1}{2} \psi(\delta) R(\delta) n(\delta) - k_\omega M(\delta) P(\delta) q(\delta) \right\|^2 d\delta d\tau \tag{46}$$

According to Parallelogram law in inner product space [11], we have

$$\|X + Y\|^2 \leq 2\|X\|^2 + 2\|Y\|^2 \tag{47}$$

Hence, (47) can be rewritten as

$$\int_t^{t+T} \left[q(\tau)^T - q(t)^T \right] P(\tau) \left[q(\tau) - q(t) \right] d\tau \leq$$

$$\int_t^{t+T} \int_t^\tau (\tau - t) \left(2 \left\| \frac{1}{2} \psi(\delta) R(\delta) n(\delta) \right\|^2 + \right.$$

$$\left. 2 \left\| k_\omega M(\delta) P(\delta) q(\delta) \right\|^2 \right) d\delta d\tau \tag{48}$$

Noting that $0 < 1 - \cos\theta < 2$, by the Fact 1 in [1] we can find that $\bar{\sigma}(M(\delta)) = 1$. Also, noting that $P^T P = P$ and changing the sequence of integration we have

$$\int_t^{t+T} \left[q(\tau)^T - q(t)^T \right] P(\tau) \left[q(\tau) - q(t) \right] d\tau \leq$$

$$\int_t^{t+T} \left(\frac{1}{2} \left\| \psi(\delta) R(\delta) n(\delta) \right\|^2 + 2k_\omega^2 \left\| M(\delta) P(\delta) q(\delta) \right\|^2 \right) \times$$

$$\int_\delta^{t+T} (\tau - t) d\tau d\delta \leq$$

$$\int_t^{t+T} \left(\frac{1}{2} \left\| \psi(\delta) R(\delta) n(\delta) \right\|^2 + \right.$$

$$\left. 2k_\omega^2 \left\| M(\delta) P(\delta) q(\delta) \right\|^2 \right) \left\{ \frac{T^2 - (\delta - t)^2}{2} \right\} d\delta \leq$$

$$k_\omega^2 T^2 \int_t^{t+T} q(\delta)^T P(\delta) q(\delta) d\delta +$$

$$\frac{T^2}{4} \int_t^{t+T} \left\| \psi(\delta) R(\delta) n(\delta) \right\|^2 d\delta \tag{49}$$

Note that

$$\left\| \psi(\delta) R(\delta) n(\delta) \right\|^2 \leq k_\psi^2 n(t)^T n(t) \tag{50}$$

where $k_\psi^2 = 4\sqrt{2}$ is the upper bound for $\psi^T \psi$ calculated in Appendix IX-B. Then, (49) can be rewritten by

$$\int_t^{t+T} \left[q(\tau)^T - q(t)^T \right] P(\tau) \left[q(\tau) - q(t) \right] d\tau \leq$$

$$k_\omega^2 T^2 \int_t^{t+T} q(\delta)^T P(\delta) q(\delta) d\delta + \frac{T^2 k_\psi^2}{4} \int_t^{t+T} n(\delta)^T n(\delta) d\delta \tag{51}$$

Finally, equation (31) will be proven by replacing $q(t)$ with $\sin\theta\lambda$ in (51). ∎

B. Calculation of the upper bound k_ψ for $\psi(t)$:

According to (44) and by some straightforward algebraic operations, we have

$$\psi(\delta)^T \psi(\delta) = 2(1 + \cos\theta)(I - \lambda\lambda^T) + 4\cos^2\theta\lambda\lambda^T \tag{52}$$

Note that the first and the second terms on the RHS of (52) are perpendicular and the maximum value for both of them is 4. Also, $\|\lambda\| = 1$. Therefore,

$$\psi(\delta)^T \psi(\delta) \leq \sqrt{16 + 16} I = 4\sqrt{2} I \tag{53}$$

which implies $k_\psi^2 = 4\sqrt{2}$.

X. ACKNOWLEDGMENT

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