2011 50th IEEE Conference on Decision and Control and
European Control Conference (CDC-ECC)
Orlando, FL, USA, December 12-15, 2011

Discrete Time Immersion and Invariance Adaptive Control for Systems in Strict Feedback Form

Yaprak Yalcin and Alessandro Astolfi

Abstract— **This paper presents a new design procedure for the adaptive stabilization (regulation) via state feedback for discrete-time nonlinear systems in parametric strict-feedback form. The algorithm utilizes discrete-time adaptive backstepping in the controller construction together with a new method for the parameter estimator design. This approach provides a recursive construction which guarantees boundness of the closed-loop trajectories and global convergence to the origin of the state of the closed-loop system. The performance of the proposed method are illustrated by simulations.**

I. INTRODUCTION

Adaptive control techniques have been extensively reported in the literature for continuous-time nonlinear systems, see for example [1], [2], [3]. Likewise for discrete-time systems there are several results. Lyapunov-based designs are reported in [4], [5]. A preliminary study proposing a "look-ahead" adaptive backstepping design for a class of discrete-time strict-feedback systems without using Lyapunov functions has been presented in [6], see also [7]. In [8], [9] a recursive design scheme different from standard backstepping has been proposed. Finally, [10] considers direct adaptive control for systems with matched uncertainties and [11] describes a periodic adaptive control approach. Most of the recent works deal with the robust adaptive control problems. The problem is studied for systems in strict-feedback form perturbed by a class of nonlinear uncertainties in [12], where local stability is proved without using Lyapunov functions. This method is improved in [13]. A robust backstepping adaptive controller design for nonlinear discrete-time systems in parametric-strict-feedback form without overparametrization is given [14], [15]. In [16], [17] robust asymptotic and output tracking adaptive control problems are considered for strict feedback SISO systems. Finally, in [18] some results on the robust control of first-order nonlinear systems with both parametric and non-parametric uncertainties are presented, and in [19], [20], [21] the solution to some discrete-time adaptive control problems for a class of strict feedback systems with unknown control directions are developed.

In this paper an adaptive controller design via state-feedback for the adaptive regulation of linearly parametrised discrete-time systems in feedback form is presented. The novelty of the method is that it allows for stable dynamics to be assigned to the parameter estimation error, extending the idea introduced in [3], [22] for continuous-time systems to the discrete-time setting.

By analogy to the continuous-time method, the discrete-time parameter estimation error dynamics include a free function which allows to "shape" the parameter estimation error dynamics, which can be regarded as linear time-varying.

The rest of the paper is organized as follows. In Section II preliminary results on the estimator and adaptive control design for a class of linearly parametrized nonlinear systems are given. The main results are presented in Section III. Finally, the method is applied to the problem of wing rock elimination in high-performance aircrafts and simulation results are presented in Section IV.

In the sequel, the notation $x_i^+(k) = x_i(k+1)$, $x_i^-(k) = x_i(k-1)$, $A^{+m}(x_1,..,x_n) = A(x_1^{+m},..,x_n^{+m})$ and $A^{-m}(x_1,..,x_n) = A(x_1^{-m},..,x_n^{-m})$, where $A(x_1,..,x_n) \in \Re^n \to \Re^p$ is used. Note that when convenient the index k is omitted.

II. PRELIMINARY RESULTS

Consider a class of discrete-time nonlinear systems described by the equation

$$x^+ = f_0(x) + f_1(x)\theta + g(x)u, \tag{1}$$

where $x(k) \in \Re^n$ is the state vector, $u(k) \in \Re^m$ is the control vector, $\theta \in \Re^p$ is a vector of constant parameters and f_0, f_1 and g are mappings of appropriate dimensions, with $f_0(0) = 0$ and $f_1(0) = 0$, and the problem of designing a discrete-time adaptive state feedback control law of the form

$$\hat{\theta}^+ = \alpha(x,\hat{\theta}), \quad u = \upsilon(x,\hat{\theta}), \tag{2}$$

such that all trajectories of the closed-loop system are bounded and

$$\lim_{k\to\infty} x(k) = 0. \tag{3}$$

Additionaly, we may require that the equilibrium $(x,\hat{\theta}) = (0,\theta)$ be stable. To solve this problem define the parameter estimation error $z = \hat{\theta} - \theta + \breve{\beta}(x^-,x)$ with $\breve{\beta}(x^-,x) = \beta(x^-)x$. Note that $z^+ = \hat{\theta}^+ - \theta + \beta(x)x^+$, hence

$$\begin{aligned} z^+ - z &= \hat{\theta}^+ - \hat{\theta} + \beta(x)x^+ - \beta(x^-)x \\ &= \hat{\theta}^+ - \hat{\theta} + \beta(x)f_1(x)\theta + \\ &\quad \beta(x)(f_0(x) + g(x)u) - \beta(x^-)x \qquad (4) \\ &= \hat{\theta}^+ - \hat{\theta} + \beta(x)f_1(x)[\hat{\theta} + \beta(x^-)x - z] + \\ &\quad \beta(x)(f_0(x) + g(x)u) - \beta(x^-)x. \end{aligned}$$

A. Astolfi is with the Dept. of Electrical and Electronic Engineering, Imperial College London, London, SW7 2AZ, UK and DISP, University of Roma "Tor Vergata", Via del Politecnico 1, 00133 Rome, Italy, a.astolfi@ic.ac.uk.
Y. Yalcin is with the Department of Electrical-Electronics Engineering, Istanbul Technical University, Maslak, Istanbul, Turkey yalciny@itu.edu.tr.

Selecting the update law as

$$\begin{aligned}\hat{\theta}^+ &= \hat{\theta} - \beta(x) f_1(x)[\hat{\theta} + \beta(x^-)x] - \\ &\quad \beta(x)(f_0(x) + g(x)u) + \beta(x^-)x,\end{aligned} \quad (5)$$

yields the parameter estimation error dynamics

$$z^+ = [I - \beta(x) f_1(x)] z. \quad (6)$$

Note that if the function $\beta(x)$ is such that

$$\bar{\sigma}[I - \beta(x) f_1(x)] < 1, \quad (7)$$

then the parameter estimation error converges to zero. This observation motivates the following preliminary result.

Proposition 1: Consider the nonlinear system (1). Suppose there exists a control law

$$u = \upsilon(x, \theta), \quad (8)$$

such that the zero equilibrium of the closed-loop system

$$x^+ = f_0(x) + f_1(x)\theta + g(x)\upsilon(x, \theta), \quad (9)$$

is globally asymptotically stable. Let $\beta(x)$ be such that

$$\bar{\sigma}[I - \beta(x) f_1(x)] \leq 1, \quad (10)$$

and such that the trajectories of the system

$$z^+ = [I - \beta(x) f_1(x)] z, \quad (11)$$

satisfy, for all x,

$$\lim_{k\to\infty} g(x)[\upsilon(x, \theta + z) - \upsilon(x, \theta)] = 0. \quad (12)$$

Then all trajectories of the closed-loop system

$$\begin{aligned} x^+ &= f_0(x) + f_1(x)\theta + g(x)\upsilon(x, \theta_{est}) \\ \hat{\theta}^+ &= \hat{\theta} - \beta(x) f_1(x)[\hat{\theta} + \beta(x^-)x] - \\ &\quad \beta(x)(f_0(x) + g(x)u) + \beta(x^-)x,\end{aligned} \quad (13)$$

where $\theta_{est} = \hat{\theta} + \beta(x^-)x = \theta + z$, are bounded and $\lim_{k\to\infty} x(k) = 0$. □

III. MAIN RESULTS

In this section we consider discrete-time systems described by the equations

$$\begin{aligned} x_1^+ &= x_2 + \Phi_1^T(x_1)\theta, \\ x_2^+ &= x_3 + \Phi_2^T(x_1, x_2)\theta, \\ &\vdots \\ x_i^+ &= x_{i+1} + \Phi_i^T(x_1, .., x_i)\theta, \\ &\vdots \\ x_n^+ &= u + \Phi_n^T(x_1, .., x_n)\theta, \end{aligned} \quad (14)$$

where $x_i(k) \in \Re$, $i = 1, .., n$, are the states, $u(k) \in \Re$ is the control input, $\Phi_i \in \Re^i \to \Re^p$, $i = 1, .., n$, are mappings such that $\Phi_i(0) = 0$ and $\theta \in \Re^p$ is a vector of unknown constant parameters.

The objective is to design a discrete-time adaptive controller described by equations of the form (2) such that all closed-loop trajectories are bounded and

$$\lim_{k\to\infty}(x_1(k) - x_1^*(k)) = 0, \quad (15)$$

where $x_1^*(k)$ is a given reference signal. This objective is achieved in two steps, as detailed hereafter.

A. Estimator Design

To begin with we design a stable estimator for the parameter θ. Let, similarly to Section II,

$$z_i = \hat{\theta}_i - \theta + \breve{\beta}_i, \quad i = 1, .., n, \quad (16)$$

where the functions $\breve{\beta}_i$ are defined as

$$\breve{\beta}_i(x_1^-, .., x_i^-, x_i) = \beta_i(x_1^-, .., x_i^-)x_i, \quad (17)$$

hence

$$z_i = \hat{\theta}_i - \theta + \beta_i(x_1^-, .., x_i^-)x_i, \quad (18)$$

and

$$z_i^+ = \hat{\theta}_i^+ - \theta + \beta_i(x_1, .., x_i)x_i^+. \quad (19)$$

As a result

$$\begin{aligned} z_i^+ - z_i &= \hat{\theta}_i^+ - \hat{\theta}_i + \beta_i(x_1, .., x_i)x_i^+ - \beta_i(x_1^-, .., x_i^-)x_i \\ &= \hat{\theta}_i^+ - \hat{\theta}_i + \beta_i(x_1, .., x_i)[x_{i+1} + \Phi_i^T(x_1, .., x_i)\theta] \\ &\quad -\beta_i(x_1^-, .., x_i^-)x_i \\ &= \hat{\theta}_i^+ - \hat{\theta}_i - \beta_i(x_1^-, .., x_i^-)x_i + \beta_i(x_1, .., x_i) \times \\ &\quad [x_{i+1} + \Phi_i^T(x_1, .., x_i)[\hat{\theta}_i + \beta_i(x_1^-, .., x_i^-)x_i - z_i]], \end{aligned} \quad (20)$$

and selecting

$$\begin{aligned} \hat{\theta}_i^+ &= \hat{\theta}_i + \beta_i(x_1^-, .., x_i^-)x_i - \beta_i(x_1, .., x_i) \times \\ &\quad [x_{i+1} + \Phi_i^T(x_1, .., x_i)[\hat{\theta}_i + \beta_i(x_1^-, .., x_i^-)x_i]], \end{aligned} \quad (21)$$

yields

$$z_i^+ = [I - \beta_i(x_1, .., x_i)\Phi_i^T(x_1, .., x_i)]z_i. \quad (22)$$

Let

$$\beta_i = [I + \Phi_i\Phi_i^T]^{-1}\Phi_i, \quad (23)$$

and note that

$$[I - \beta_i(x_1, .., x_i)\Phi_i^T(x_1, .., x_i)] = [I + \Phi_i\Phi_i^T]^{-1}, \quad (24)$$

hence

$$z_i^+ = [I + \Phi_i \Phi_i^T]^{-1} z_i, \quad i = 1, ..., n. \tag{25}$$

Consequently, the following statement holds.

Proposition 2: Consider the system (25). The zero equilibrium of the system is (uniformly) stable and $\Phi_i^T(x_1(k), ..., x_i(k)) z_i(k)$ is a l^2 signal. In addition, if $x_j(k)$, $j = 1, ..., i$, is bounded, then

$$\lim_{k\to\infty} \Phi_i^T(x_1(k), ..., x_i(k)) z_i(k) = 0. \tag{26}$$

□

B. Controller Design

In this section an adaptive controller solving the considered control problem is designed.

Theorem 1 : Consider the system (14), and the control law

$$u = x_1^{*+n} - \sum_{m=1}^{n} (\Phi_{mest}^{+(n-m)})^T \theta_{mest}^{+(n-m)}, \tag{27}$$

where x_1^* is the reference signal to be tracked, $\theta_{iest} = \hat{\theta}_i + \beta_i(x_1^-, .., x_i^-) x_i$, with $\hat{\theta}_i$ obtained using the update law (21) and β_i given by (23); $\Phi_{mest}^{+(i-m-1)} = \Phi_m(x_{1est}^{+(i-m-1)}, .., x_{iest}^{+(i-m-1)})$, $i = 2, .., n$, with $x_{mest} = x_m$ and $x_{mest}^{+(i-m-1)}$, $(i-m-1) > 0$, estimated $(i-m-1)$-step future values of x_m obtained by recursively evaluating and utilizing θ_{mest}^{+k}, $k = 1, .., (i-m-1)$ in equations (14). Then the closed-loop system is such that all trajectories of the closed-loop system are bounded and

$$\lim_{k\to\infty} (x_1(k) - x_1^*(k)) = 0. \tag{28}$$

Moreover the equilibrium

$$\tilde{x}_i = 0, \quad z_i = 0, \quad i = 1, .., n, \tag{29}$$

where

$$\begin{aligned} \tilde{x}_i &= x_i - x_i^*, \quad i = 1, .., n, \\ x_i^* &= x_1^{*+(i-1)} - \sum_{m=1}^{i-1} (\Phi_{mest}^{+(i-m-1)})^T \theta_{mest}^{+(i-m-1)}, \quad i = 2, .., n, \end{aligned}$$

is Lyapunov stable. □

Remark : The control law u given in (27) consists of "look-ahead" values x_{iest}^+ of the state variable x_i. Note that

$$x_i^+ = x_{i+1} + \Phi_i^T(x_1, .., x_i)\theta, \tag{30}$$

$$x_{iest}^+ = x_{i+1} + \Phi_i^T(x_1, .., x_i)\theta_{iest}. \tag{31}$$

Hence

$$x_i^+ - x_{iest}^+ = -\Phi_i^T(x_1, .., x_i) z_i. \tag{32}$$

This, straightforwardly, implies

$$x_{iest}^{+m}|_{\Phi_i^T(x_1,..,x_i)z_i=0} = x_i^{+m}|_{\Phi_i^T(x_1,..,x_i)z_i=0}, \tag{33}$$

since

$$\begin{aligned} x_{iest}^{+m}|_{\Phi_i^T(x_1,..,x_i)z_i=0} &= (x_{iest}^+)^{+(m-1)}|_{\Phi_i^T(x_1,..,x_i)z_i=0}, \\ &= (x_i^+)^{+(m-1)}|_{\Phi_i^T(x_1,..,x_i)z_i=0}, \\ &= x_i^{+m}|_{\Phi_i^T(x_1,..,x_i)z_i=0}. \end{aligned} \tag{34}$$

This analysis reveals that, as long as $\Phi_i^T(x_i) z_i = 0$, the value of the estimated variable x_{iest}^+ is equal to the true value of x_i^+ and this relation is also valid for all i and all future values. □

IV. EXAMPLE

The proposed methodology is applied to the problem considered in [22]: wing rock elimination in high-performance aircrafts. The discrete-time Euler model of the continuous-time system given in [22], which describes the motion of the wing, is given by

$$\begin{aligned} x_1^+ &= x_1 + T x_2, \\ x_2^+ &= x_2 + T x_3 + T \Phi^T(x_1, x_2)\theta, \\ x_3^+ &= x_3 + \frac{T}{\tau}(v - x_3), \end{aligned} \tag{35}$$

where the states $x_1(k)$, $x_2(k)$ and $x_3(k)$ describe the roll angle, the roll rate and the aileron deflection angle, respectively, τ is the aileron time constant, $v(k)$ is the control input, $\theta \in R^5$ is a vector of unknown constant parameters,

$$\Phi(x_1, x_2) = [1, x_1, x_2, x_1|x_2|, x_2|x_2|], \tag{36}$$

and T is the sampling period. Applying the co-ordinates and input transformation

$$\begin{aligned} \hat{x}_1 &= x_1, \\ \hat{x}_2 &= x_1 + T x_2, \\ \hat{x}_3 &= x_1 + 2T x_2 + T^2 x_3, \\ u &= F + \frac{T^3}{\tau} v, \end{aligned} \tag{37}$$

with

$$F = 2\hat{x}_3 - \hat{x}_2 + (1 - \frac{T}{\tau})(\hat{x}_3 + \hat{x}_1 - 2\hat{x}_2), \tag{38}$$

yields the system in strict feedback form

$$\begin{aligned}\hat{x}_1^+ &= \hat{x}_2, \\ \hat{x}_2^+ &= \hat{x}_3 + \Phi_2^T(\hat{x}_1, \hat{x}_2)\theta, \\ \hat{x}_3^+ &= u + \Phi_3^T(\hat{x}_1, \hat{x}_2)\theta.\end{aligned} \tag{39}$$

where

$$\begin{aligned}\Phi_2 &= T^2\Phi(\hat{x}_1, \hat{x}_2), \\ \Phi_3 &= 2\Phi_2 = 2T^2\Phi(\hat{x}_1, \hat{x}_2),\end{aligned} \tag{40}$$

Simulations are carried out for $T = 0.5$ and the parameters $\tau = \frac{1}{15}$, $\theta_0 = [0, -26.72, 0.76485, -2.9225, 0]^T$ as in [1], and for $x_1^* = 0$. The initial conditions for the parameter estimator have been set to $\hat{\theta}(0) = 0$, and the initial state has been selected as $x(0) = [0.4,\ 0,\ 0]^T$. Figure 1 and Figure 2 show the time histories of the parameter estimation errors $z_2(k)$ and $z_3(k)$. Note that, consistently with the theory, $z_2(k)$ and $z_3(k)$ do not converge to zero. Figure 3 and Figure 4 show the time histories of the perturbations $\Phi_2(x)z_2(k)$ and $\Phi_3(x)z_3(k)$, respectively, which converge to zero. Finally, Figure 5 and Figure 6 display the time histories of the states $x_1(k)$, $x_2(k)$, and $x_3(k)$ and of the control signal.

V. CONCLUSIONS AND FUTURE WORKS

In this paper a novel adaptive controller design method for the adaptive regulation of linearly parameterized discrete-time systems in strict-feedback form has been developed. The control law is constructed using a backstepping-type procedure together with a novel parameter update law, which renders the zero equilibrium of the parameter estimation error system stable. This property is obtained, as in the continuous-time immersion and invariance approach, including a function of the state in the definition of the parameter estimation error. The results are illustrated on the Euler model of the system which describes the wing rock oscillation in high-performance aircraft. Further work to develop a stable parameter estimator which does not require overparametrization is under way.

REFERENCES

[1] M. Krstic, I. Kanellakopoulos, and P. Kokotovic, *Nonlinear and Adaptive Control Design*. New York: Wiley-Intrescience, 1995.

[2] R. Marino and P. Tomei, *Nonlinear Control Design: Geomeric, Adaptive and Robust*. Springer, 2008.

[3] A. Astolfi, D. Karagiannis, and R. Ortega, *Nonlinear and Adaptive Control with Applications*. Springer, 2008.

[4] W. Haddad, T. Hayakawa, and A. Leonessa, "Direct adaptive control for discrete-time nonlinear uncertain dynamical systems," in *Proceedings of The American Control Conference*, Anchorage, AK, 2002.

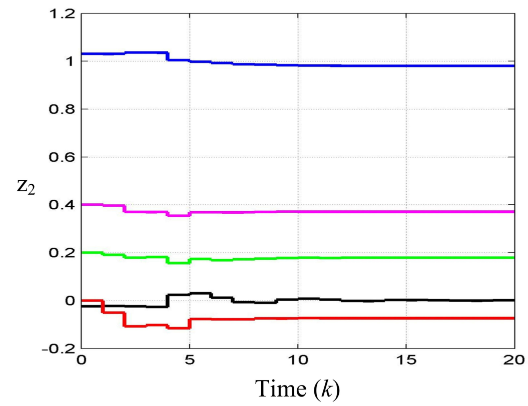

Fig. 1. The time history of the variable $z_2(k)$.

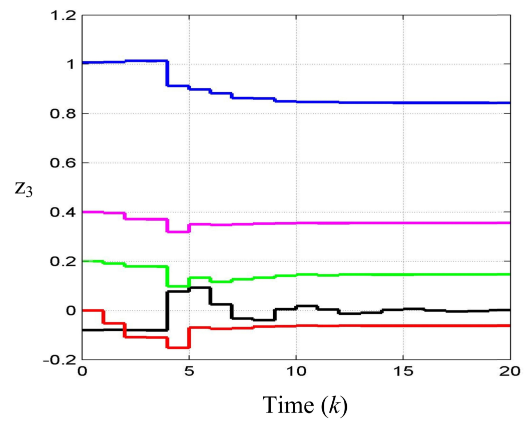

Fig. 2. The time history of the variable $z_3(k)$.

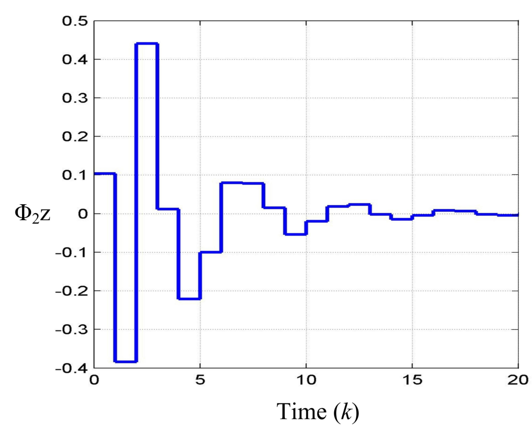

Fig. 3. The time history of the perturbation $\Phi_2(x(k))z_2(k)$.

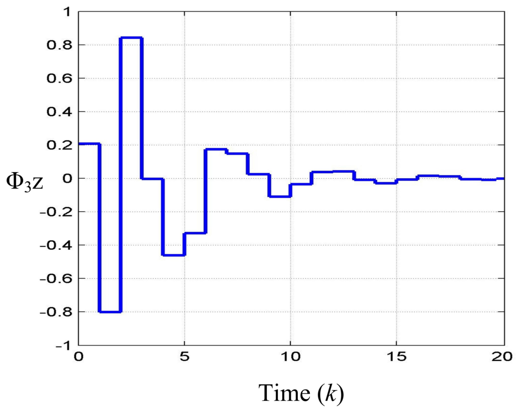

Fig. 4. The time history of the perturbation $\Phi_3(x(k))z_3(k)$.

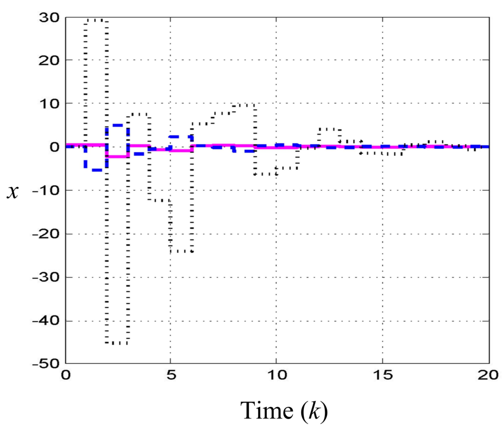

Fig. 5. The time histories of the states $x_1(k)$ (solid), $x_2(k)$ (dashed) and $x_3(k)$ (dotted).

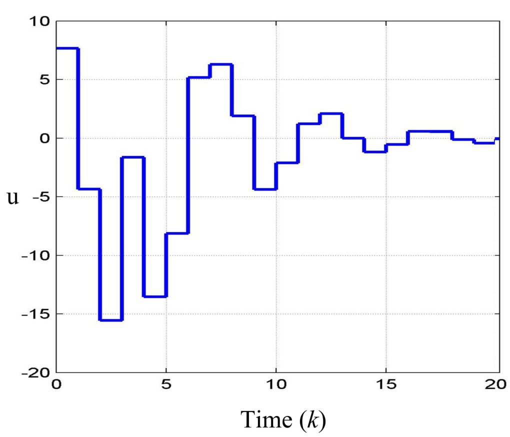

Fig. 6. The time history of the control signal.

[5] T. Hayakawa, W. Haddad, and A. Leonessa, "A lyapunov-based adaptive control framework for discrete-time non-linear systems with exogenous disturbances," *Int. J. Control*, vol. 77, no. 3, pp. 250–263, 2004.

[6] P.-C. Yeh and P. Kokotovic, "Adaptive control of a class of nonlinear discrete-time systems," *International Journal of Control*, vol. 62, pp. 303–324, 1995.

[7] A. Madani, S. Monaco, and D. Normand-Cyrot, "Adaptive control of discrete-time dynamics in parametric strict-feedback form," in *Proceedings of the 35th Conference on Decision and Control*, Kobe, Japan, 1996.

[8] J. Zhao and I. Kanellakopoulos, "Adaptive control of discrete-time strict-feedback nonlinear systems," in *Proceedings of the American Control Conference*, Albuquerque, New Mexico, 1997.

[9] ——, "Discrete-time adaptive control of output-feedback nonlinear systems," in *Proceedings of the 36th IEEE Conference on Decision and Control*, San Diego, CA, 1997, pp. 4326–4331.

[10] S. Fu and C. Cheng, "Direct adaptive control designs for nonlinear discrete-time systems with matched uncertainties," in *Proceedings of the 2005 IEEE International Conference on Mechatronics*, Taipei, Taiwan, 2005.

[11] D. Huang, J. Xu, and Z. Hou, "A discrete-time periodic adaptive control approach for parametric-strict-feedback systems," in *Joint 48th IEEE Conference on Decision and Control and 28th Chinese Control Conference Shanghai*, P.R.China, 2009.

[12] Y. Zhang, C. Wen, and Y. Soh, "Robust adaptive control of uncertain discrete-time systems," *Automatica*, vol. 35, pp. 321–329, 1999.

[13] ——, "Discrete-time robust backstepping adaptive control for nonlinear time-varying systems," *IEEE Transactions on Automatic Control*, vol. 45, no. 9, pp. 1749–1755, 2000.

[14] Y. Zhang, W. H. Chen, and Y. Soh, "Improved robust backstepping adaptive control for nonlinear discrete-time systems without overparameterization," *Automatica*, vol. 33, pp. 864–867, 2008.

[15] Y. Zhang, C. Wen, and Y. Soh, "Robust adaptive control of nonlinear discrete-time systems by backstepping without overparameterization," *Automatica*, vol. 37, pp. 551–558, 2001.

[16] C. Yang, S. Dai, S. Ge, and T. Lee, "Adaptive asymptotic tracking control of a class of discrete-time nonlinear systems with parametric and nonparametric uncertainties," in *Proceedings of the American Control Conference*, Hyatt Regency Riverfront, St. Louis, MO, USA, 2009.

[17] S. Ge, C. Yang, S. Dai, Z. Jiao, and T. Lee, "Robust adaptive control of a class of nonlinear strict-feedback discrete-time systems with exact output tracking," *Automatica*, vol. 45, pp. 2537–2545, 2009.

[18] H. Ma, K. Lum, and S. Ge, "Adaptive control for a discrete-time first-order nonlinear system with both parametric and non-parametric uncertainties," in *Proceedings of the 46th IEEE Conference on Decision and Control*, New Orleans, LA, USA, 2007.

[19] S. Ge, C. Yang, S. Dai, and T. Lee, "Adaptive control of a class of strict-feedback discrete-time nonlinear systems with unknown control gains and preceded by hysteresis," in *Proceedings of the American Control Conference*, Hyatt Regency Riverfront, St. Louis, MO, USA, 2009.

[20] S. Ge, C. Yang, and T. Lee, "Adaptive robust control of a class of nonlinear strict-feedback discrete-time systems with unknown control directions," *Systems and Control Letters*, vol. 57, pp. 888–895, 2008.

[21] ——, "Output feedback adaptive control of a class of nonlinear discrete-time systems with unknown control directions," *Automatica*, vol. 45, pp. 270–276, 2009.

[22] D. Karagiannis and A. Astolfi, "Nonlinear adaptive control of systems in feedback form:an alternative to adaptive backstepping," *Systems and Control Letters*, vol. 57, pp. 773–739, 2008.