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Adjustable Navigation Functions for Unknown Sphere Worlds

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Abstract— **This paper introduces an algorithm for automatically tuning analytic navigation functions for sphere worlds. The tuning parameter must satisfy a lower bound to ensure collision avoidance and convergence. Until now analytic navigation functions have been manually tuned, although existence of a lower bound had been proved. A theoretical improvement on this lower bound is provided and the method is extended to unbounded manifolds. Then the required formulas are derived and algorithm described. So the lower bound is here evaluated in terms of sphere world centers and radii. Automated tuning enables completely unattended solution of any navigation problem in unknown sphere worlds and a priori known worlds which belong to the sphere world diffeomorphism class.**

I. INTRODUCTION

A fundamental problem of robotics is motion planning. A great variety of manifestations exists and equally numerous different solution approaches. Among them we may mention sampling-based, combinatorial and feedback methods for continuous spaces.

A motion planning problem is defined over continuous space as finding a path for an agent leading it to the desired destination while avoiding collisions with obstacles. It can be abstracted from the geometric to a topological viewpoint, remaining in the same connected component. The path can be first generated in a convenient space which captures the problem's topological structure. As a second step, geometric detail is introduced, diffeomorphically transforming the path from the geometrically convenient topological to real space. This procedure enables integration of trajectory generation and trajectory tracking, leading to closed-loop feedback motion planning.

Artificial Potential Fields were introduced by Khatib [1] as a closed-loop obstacle avoidance navigation method. A scalar potential is constructed over the workspace. Obstacles repel the agent, while the goal configuration attracts it. Following the potential negative gradient, the agent is safely led to its destination. Arisal of local minima in certain obstacle arrangements can prevent convergence.

Navigation Functions (NF) have been proposed by Rimon and Koditschek [2] to overcome the problem of local minima. After showing that complete disappearance of stationary points is unobtainable, they defined an almost globally asymptotically stable potential field. Subject to conditions, only a subset of zero measure traps the agent in a finite set of remaining saddle points, which are unstable equilibria. The NF potential is defined on a sphere world and diffeomorphically mapped to real space.

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Global knowledge is needed in the original navigation function formulation. This requirement is relaxed in [3], [4] by defining polynomial NFs and in [5] by implementing C^2 switches for multi-agent systems with finite sensing radii.

Tuning hinders implementation. The NF field is shaped by a parameter. As proved in [6] there exists a lower bound on this tuning parameter which clears the field of local minima other than the destination. They become saddles and the potential a NF. In addition to existence, calculation of this lower bound is outlined, but no explicit formula is provided. In consequence, using NFs until now required manual adjustment of the tuning parameter, also for multi-agent systems.

We provide an algorithm to calculate the tuning parameter for theoretically guaranteed navigation. The lower bound used is improved compared to the original formulation. The improvement is offered by cancellation of terms with equivalent effects. Direct substitution of sphere centers and radii suffices to find the desired bound.

The above algorithm enables safe tuning globally. We have rearranged computation to efficiently update for discovered obstacles. Initializing constraints for a new obstacle has time computational complexity $\Theta(M_z)$, where M_z the number of until then known obstacles. Updating those constraints related to already known obstacles upon discovering new obstacles can as well be arranged to require $\Theta(M_z)$. Moreover, there is the option to apply these calculated constraints only when necessary. This is also implemented here and allows for provably correct locally oriented tuning, for finite number of obstacles.

The rest of this paper is organized as following: the problem is defined in § II and NFs in § III, the new lower bound is proved in § IV, the algorithm for updatable tuning is presented in § V and simulation case studies follow in § VI to support our results. Concluding remarks are summarized in § VII where future research is considered.

II. PROBLEM DEFINITION

A compact connected subset of n-dimensional Euclidean space E^n whose boundary is formed by the disjoint union of a finite number of $(n-1)$-dimensional spheres is defined as a *sphere world*. Let $M+1, M \in \mathbb{N}$ be their number.

The whole sphere world is bounded by a sphere defining the workspace $\mathscr{W} \triangleq \{q \in E^n : \|q\|^2 \leq \rho_0^2\}, \rho_0 > 0$. Spheres bound the internal obstacles $\mathscr{O}_j \triangleq \{q \in E^n : \|q - q_j\|^2 < \rho_j^2\}, \rho_j > 0, j \in I_1 \triangleq \{1, 2, \ldots, M\}$. The zero$^{\text{th}}$ obstacle is defined as $\mathscr{O}_0 \triangleq E^n \setminus \mathscr{W}$ and the free space $\mathscr{F} \triangleq \mathscr{W} \setminus$

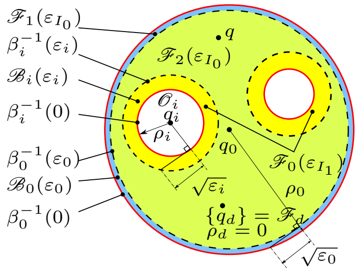

Fig. 1. Sets defined on a sphere world and sample NF for $k = 2$.

$\bigcup_{j \in I_1} \mathscr{O}_j$ remains after removing all internal obstacles from $\mathscr{W}$. Let $I_0 \triangleq \{0, 1, \dots, M\}$.

We consider a holonomic agent whose state x is governed by the control law $\dot{x} = -\nabla\varphi$ where φ is a NF on $\mathscr{F}$ as defined later. As proved in [6] this solves the motion planning problem in $\mathscr{F}$.

We are interested in an algorithm to tune the analytic potential field φ to make it a NF while exploring unknown sphere worlds. It is also desirable to reduce the effect on φ of obstacles distant to the agent, in a provably correct way. This scheme should be also applicable to a priori known worlds diffeomorphic [6], [4], [7] to sphere worlds.

III. NAVIGATION FUNCTIONS

A. Definition

A NF on a compact connected analytic manifold with boundary $\mathscr{F} \subset E^n$ is defined as a map $\varphi : \mathscr{F} \to [0,1]$ which is ($\varphi \in C^2[\mathscr{F}, [0,1]]$ suffices in 1)

1) Analytic on $\mathscr{F}$: locally convergent power series exists,
2) Polar on $\mathscr{F}$: unique minimum at $q_d \in \mathscr{F}$,
3) Morse on $\mathscr{F}$: all critical points are non-degenerate,
4) Admissible on $\mathscr{F}$: uniformly maximal on $\partial\mathscr{F}$,

and can solve the motion planning problem for $\mathscr{F}$ [6].

When $\mathscr{F}$ is a valid sphere world there exists an N such that $\forall k \geq N$ the following is a NF on $\mathscr{F}$

$$\varphi \triangleq \sigma_d \circ \sigma \circ \hat{\varphi} = \frac{\gamma_d}{\left(\gamma_d^k + \beta\right)^{\frac{1}{k}}} \tag{1}$$

where $\sigma_d(x) \triangleq x^{\frac{1}{k}}, \sigma(x) \triangleq \frac{x}{x+1}, \hat{\varphi} \triangleq \frac{\gamma}{\beta}$ and $\gamma, \beta : \mathscr{F} \to [0, \infty)$ are defined as $\gamma \triangleq \gamma_d^k, k \in \mathbb{N} \setminus \{0, 1\}, \gamma_d \triangleq \|q - q_d\|^2, \beta \triangleq \prod_{j \in I_0} \beta_j, \bar{\beta}_i \triangleq \prod_{j \in I_0 \setminus i} \beta_j \geq 0, \beta_0 \triangleq \rho_0^2 - \|q\|^2, \beta_j \triangleq \|q - q_j\|^2 - \rho_j^2, j \in I_1, q \in \mathscr{F}$.

B. Sphere World subsets

The following sets are used and illustrated in Fig. 1:

1) Destination point $\mathscr{F}_d \triangleq \{q_d\}$,
2) Free space boundary $\partial\mathscr{F} \triangleq \beta^{-1}(0) = \bigcup_{i \in I_0} \beta_i^{-1}(0)$
3) "Near" internal obstacles $\mathscr{F}_0(\varepsilon_{I_1}) \triangleq \bigcup_{i \in I_1} \mathscr{B}_i(\varepsilon_i) \setminus \{q_d\}, \mathscr{B}_i(\varepsilon_i) \triangleq \{q \in E^n : 0 < \beta_i < \varepsilon_i\}$,
4) Set "near" workspace boundary $\mathscr{F}_1(\varepsilon_{I_0}) \triangleq \mathscr{B}_0(\varepsilon_0) \setminus (\{q_d\} \cup \mathscr{F}_0(\varepsilon_{I_1}))$,
5) Set "away" from obstacles $\mathscr{F}_2(\varepsilon_{I_0}) \triangleq \mathscr{F} \setminus (\{q_d\} \cup \partial\mathscr{F} \cup \mathscr{F}_0(\varepsilon_{I_1}) \cup \mathscr{F}_1(\varepsilon_{I_0}))$,

where $\varepsilon_{I_0} \triangleq \{\varepsilon_i\}_{i \in I_0}, \varepsilon_{I_1} \triangleq \{\varepsilon_i\}_{i \in I_1}$. We define $\varepsilon_i, \varepsilon_{iu}, i \in I_0, \varepsilon'_{i0}, \varepsilon''_{i0}, \varepsilon'_{i2}, \varepsilon''_{i2}, \varepsilon_{i3j}, \varepsilon_{i3} \triangleq \min_{j \in I_0 \setminus i}\{\varepsilon_{i3j}\}, i \in I_1$ as

$$0 < \varepsilon_i < \varepsilon_{iu} \triangleq \begin{cases} \varepsilon_{0u}, & i = 0 \\ \min\{\varepsilon'_{i0}, \varepsilon''_{i0}, \varepsilon'_{i2}, \varepsilon''_{i2}, \varepsilon_{i3}\}, & i \in I_1 \end{cases}.$$

With this notation ε_i applies to annulus $\mathscr{B}_i$ of obstacle $\mathscr{O}_i$. These differ from [6] where ε_1 applies to $\mathscr{B}_0(\varepsilon_1)$ and where sets are functions of a single parameter $\varepsilon \triangleq \min_{i \in I_0}\{\varepsilon_i\}$. Here the sets are functions of $M{+}1$ parameters ε_{I_0} defined as $\mathscr{B}_i(\varepsilon_i), i \in I_0, \mathscr{F}_0(\varepsilon_{I_1}), \mathscr{F}_1(\varepsilon_{I_0}), \mathscr{F}_2(\varepsilon_{I_0})$. The condition $\mathscr{F}_0(\varepsilon_{I_1}) \subset \mathscr{F}$, since $q_0 = 0 \in E^n$, is equivalent to

$$\varepsilon_i < (\|q_i - q_j\| - \rho_j)^2 - \rho_i^2 \triangleq \varepsilon_{i3j}, \forall j \in I_0 \setminus i, \forall i \in I_1.$$

Hereafter sets $\mathscr{F}_i$ are denoted omitting their arguments.

IV. NAVIGATION FUNCTION TUNING

A. Proof overview

Quite informally the proof can be summarized as following. Show that k can be linked to obstacle neighbourhood widths ε_{I_0} so that changing ε_{I_0} no critical points escape "away" from obstacles. Any critical points are now trapped near obstacles. Then shrink ε_{I_0} until the obstacle neighbourhoods are so tight around them that no *minima* or *degenerate points* arise.

B. Tuning parameter lower bound

Proposition 3.4 [6]: For every set ε_{I_0} there exists a positive integer $N(\varepsilon_{I_0})$ such that if $k \geq N(\varepsilon_{I_0})$ then there are no critical points of $\hat{\varphi}$ in $\mathscr{F}_2$. A sufficient inequality for this to be true is $\frac{1}{2}\frac{\sqrt{\gamma_d}\|\nabla\beta\|}{\beta} < k, \forall q \in \mathscr{F}_2$ and an upper bound on the left side is $\frac{1}{2}\sqrt{\gamma_d}\sum_{i \in I_0}\left(\frac{\|\nabla\beta_i\|}{\beta_i}\right)$. This has been originally bounded by $\frac{1}{2}\frac{1}{\varepsilon}\max_{\mathscr{W}}\left\{\sqrt{\gamma_d}\right\}\sum_{i \in I_0}\max_{\mathscr{W}}\{\|\nabla\beta_i\|\}$, since $\beta_i \geq \varepsilon_i, \forall i \in I_0, \forall q \in \mathscr{F}_2$. Here a new improved bound is proposed, taking into consideration that β_i appears also in $\|\nabla\beta_0\| = 2\sqrt{\rho_0^2 - \beta_0}$ and $\|\nabla\beta_i\| = 2\sqrt{\beta_i + \rho_i^2}$, so that

$$\sum_{i \in I_0}\left(\frac{\|\nabla\beta_i\|}{\beta_i}\right) = 2\sum_{i \in I_0} Q_i(\beta_i) \leq 2\sum_{i \in I_0} Q_{ii}$$

where $Q_0(x) \triangleq \sqrt{\frac{\rho_0^2}{x^2} - \frac{1}{x}}, Q_{00} \triangleq \sqrt{\frac{\rho_0^2}{\varepsilon_0^2} - \frac{1}{\rho_0^2}}, Q_i(x) \triangleq \sqrt{\frac{\rho_i^2}{x^2} + \frac{1}{x}}, Q_{ii} \triangleq Q_i(\varepsilon_i), i \in I_1$. Then $Q_i(\beta_i(q)) \leq Q_{ii}, \forall q \in \mathscr{F}_2, \forall i \in I_0$ since $0 < \varepsilon_i \leq \beta_i, \forall q \in \mathscr{F}_2$ and $0 < \varepsilon_0 \leq \beta_0, \forall q \in \mathscr{F}_2$ and also $\mathscr{F}_2 \subset \mathscr{W} \implies \max_{\mathscr{F}_2}\{\beta_0\} \leq \max_{\mathscr{W}}\{\beta_0\} = \rho_0^2$. Let $N(\varepsilon_{I_0}) \triangleq (\rho_0 + \|q_d\|)\sum_{i \in I_0} Q_{ii}$. Noting that $\mathscr{F}_2 \subset \mathscr{W} \setminus \{\partial\mathscr{O}_0 \cup \mathscr{B}_0(\varepsilon_0)\} \implies \max_{\mathscr{F}_2}\left\{\sqrt{\gamma_d}\right\} < \max_{\mathscr{W}}\left\{\sqrt{\gamma_d}\right\} = \rho_0 + \|q_d\|$ leads to $\frac{1}{2}\sqrt{\gamma_d}\frac{\|\nabla\beta\|}{\beta} < N(\varepsilon_{I_0})$, so by setting $k \geq N(\varepsilon_{I_0})$ we ensure that all critical points are "pushed" to the set "near" obstacles $(\{q_d\} \cup \partial\mathscr{F} \cup \mathscr{F}_0 \cup \mathscr{F}_1)$. The proof that q_d is a non-degenerate local minimum and no critical points exist on boundary $\partial\mathscr{F}$ remains same with the original, enabling us to work with $\hat{\varphi}$ since $\sigma_d \circ \sigma$ is a diffeomorphism.

C. Upper bounds on lower bound

1) No critical points near world boundary: The squared "width" ε_0 of $\overline{\mathscr{B}_0(\varepsilon_0)}$ will be determined to clear the 0th obstacle neighbourhood $\mathscr{B}_0(\varepsilon_0)$ of critical points.

This is proved by Proposition 3.7 [6]: If $k \geq N(\varepsilon_{I_0})$, then there exists an ε_{0u} such that $\hat{\varphi}$ has no critical points on $\mathscr{F}_1$, as long as $\varepsilon_0 < \varepsilon_{0u}$. This Proposition still holds for the changed lower bound $N(\varepsilon_{I_0})$ on k since for $\varepsilon_{0u} \triangleq \rho_0^2 - \|q_d\|^2$

$$\frac{1}{4}\frac{\nabla\bar{\beta}_0 \cdot \nabla\gamma_d}{\bar{\beta}_0} \leq \frac{1}{2}\frac{\left\|\nabla\bar{\beta}_0\right\|\sqrt{\gamma_d}}{\bar{\beta}_0} \leq \frac{1}{2}\sqrt{\gamma_d}\sum_{i\in I_1}\left(\frac{\bar{\beta}_i}{\beta}\|\nabla\beta_i\|\right)$$
$$\leq (\rho_0 + \|q_d\|)\sum_{i\in I_1} Q_{ii} < N(\varepsilon_{I_0}) \leq k, \forall q \in \mathscr{F}_1$$

because $\varepsilon_i \leq \beta_i(q), \forall q \in \mathscr{F}_1, \forall i \in I_1$.

2) No degenerate points near internal obstacles: Non-degenracy of critical points is ensured by Proposition 3.9 [6]. its proof requires existence of at least one negative eigenvalue § IV-C.3 combined with a positive eigenvalue due to positive definiteness along $\nabla\beta_i$. The latter is assured by bounds $\varepsilon'_{i2}, \varepsilon''_{i2}$ on ε_i equal to

$$\frac{1}{2}\rho_i^2, \quad \frac{1}{4}\frac{\min_{\overline{\mathscr{B}_i(\min\{\varepsilon'_{i2},\varepsilon_{i3}\})}}\left\{\sqrt{\bar{\beta}_i}\,\|\nabla\beta_i\|\right\}}{\max_{\overline{\mathscr{B}_i(\min\{\varepsilon'_{i2},\varepsilon_{i3}\})}}\left\{\sqrt{\left|\hat{v}^{\mathrm{T}}D^2\bar{\beta}_i\hat{v}\right|}\right\}}$$

respectively, according to the original formulation, where $\|\hat{v}\| = 1$. This corresponds to non-degeneracy of critical points in $\mathscr{F}_0$ (near internal obstacles), which are the only remaining critical points of $\hat{\varphi}$. Let $\varepsilon_{i23} \triangleq \min\{\varepsilon'_{i2}, \varepsilon_{i3}\}$. Note that $\varepsilon_i < \varepsilon'_{i2}$ should be satisfied to derive ε''_{i2} and $\varepsilon_i < \varepsilon_{i3}$ allows for available expressions of extrema for β_i, γ_d to be substituted.

Observe that $\bar{\beta}_i$ and $D^2\bar{\beta}_i$ arise in nominator and denominator, respectively. This leads to the same problem as in § IV-B. Terms $\bar{\beta}_i, D^2\bar{\beta}_i$ both contain $\beta_j, j \neq i$. So we have the *same* β_j in *both* nominator and denominator. After manipulation we end up dividing $\min_{\overline{\mathscr{B}_i(\varepsilon_{i23})}}\{\beta_j\}$ by $\max_{\overline{\mathscr{B}_i(\varepsilon_{i23})}}\{\beta_j\}$ which results in an ill valued constraint. Here improved bounds are derived. A detailed derivation can be found in [8].

We can cancel β_j by dividing both numerator and denominator by $\bar{\beta}_i$. But this should be done *before* applying $\min\{\}$ and $\max\{\}$. So we return to a previous step of the original proof requiring that

$$\left(1-\frac{1}{k}\right)\bar{\beta}_i\|\nabla\beta_i\|^2 - \beta_i^2\left|\hat{r}_i^{\mathrm{T}}D^2\bar{\beta}_i\hat{r}_i\right| - 2\beta_i\bar{\beta}_i > 0$$
$$\iff \left[\frac{1}{2}\left(1-\frac{1}{k}\right)\bar{\beta}_i\|\nabla\beta_i\|^2 - 2\beta_i\bar{\beta}_i\right]$$
$$+\left[\frac{1}{2}\left(1-\frac{1}{k}\right)\bar{\beta}_i\|\nabla\beta_i\|^2 - \beta_i^2\left|\hat{r}_i^{\mathrm{T}}D^2\bar{\beta}_i\hat{r}_i\right|\right] > 0$$

where $\hat{r}_i \triangleq \frac{\nabla\beta_i}{\|\nabla\beta_i\|}$. We refer to the first term $(*)$ and second $(**)$. If we require that $k \geq 2$ ($k = 1 \implies \det(D^2\varphi)(q_d)) = 0$) then $\frac{1}{2} \leq 1 - \frac{1}{k} < 1$ therefore the term $(*)$ is greater than $\frac{1}{2}\left(\frac{1}{2}\right)\bar{\beta}_i\|\nabla\beta_i\|^2 - 2\beta_i\bar{\beta}_i$. A sufficient condition to ensure that term $(*)$ be positive is $0 < \bar{\beta}_i\left[\frac{1}{4}\|\nabla\beta_i\|^2 - 2\beta_i\right] \overset{\bar{\beta}_i>0,\forall q\in\mathscr{B}_i(\varepsilon_i)}{\iff} \beta_i < \rho_i^2$ so here we select $\varepsilon'_{i2} \triangleq \rho_i^2, i \in I_1$ which is slightly improved.

Let $\bar{\beta}_{j_1 j_2 \dots j_r} \triangleq \prod_{i\in I_0\setminus\{j_1,j_2,\dots,j_r\}}\beta_i$.

Let us now examine term $(**)$ which is greater than $\frac{1}{4}\bar{\beta}_i\|\nabla\beta_i\|^2 - \beta_i^2 2R_i$ for $k \geq 2$, where

$$R_i \triangleq \sum_{j\in I_0\setminus i}\left(\bar{\beta}_{ij} + \sum_{l\in I_0\setminus\{i,j\}}\left(\bar{\beta}_{ijl}\|\nabla\beta_j\|\|\nabla\beta_l\|\right)\right)$$

The inequality $\left|\hat{v}^{\mathrm{T}}D^2\bar{\beta}_i\hat{v}\right| \leq R_i$ is proved in Appendix B.2 [6] for any unit vector $\hat{v}$, hence holds also for $\hat{r}_i$. A sufficient condition for term $(**)$ to be positive is $0 < \frac{1}{4}\bar{\beta}_i\|\nabla\beta_i\|^2 - 2\beta_i^2 R_i, \forall q \in \mathscr{B}_i(\varepsilon_i)$. Now we divide both nominator and denominator by $\bar{\beta}_i$, noting that $\|\nabla\beta_i\|^2 = \left(\beta_i + \rho_i^2\right)$

$$\frac{1}{4}\bar{\beta}_i\|\nabla\beta_i\|^2 - 2\beta_i^2 R_i = \bar{\beta}_i\Bigg[\beta_i + \rho_i^2 -$$
$$- 2\beta_i^2\sum_{j\in I_0\setminus i}\left(\frac{1}{\beta_j} + 4Q_j(\beta_j)\sum_{l\in I_0\setminus\{i,j\}}Q_l(\beta_l)\right)\Bigg].$$

To make this expression positive

$$\sqrt{\frac{\beta_i + \rho_i^2}{2\sum_{j\in I_0\setminus i}\left(\frac{1}{\beta_j} + 4Q_j(\beta_j)\sum_{l\in I_0\setminus\{i,j\}}Q_l(\beta_l)\right)}} > \beta_i,$$

$\forall q \in \mathscr{B}_i(\varepsilon_i)$. Let $\beta_{ji}^{\min} \triangleq \min_{\overline{\mathscr{B}_i(\varepsilon_{i23})}}\{\beta_j\}, Q_{0i} \triangleq \sqrt{\frac{\rho_0^2}{(\beta_{0i}^{\min})^2} - \frac{1}{(\beta_{0i}^{\max})}}, Q_{ji} \triangleq Q_j(\beta_{ji}^{\min})$ and since $\beta_{ii}^{\min} = 0$, then a sufficient condition on ε_i for the inequality to hold is

$$\varepsilon''_{i2} \triangleq \frac{\rho_i}{\sqrt{2\sum_{j\in I_0\setminus i}\left(\frac{1}{\beta_{ji}^{\min}} + 4Q_{ji}\sum_{l\in I_0\setminus\{i,j\}}Q_{li}\right)}}$$
$$> \varepsilon_i > \beta_i, \forall q \in \mathscr{B}_i(\varepsilon_i), \forall i \in I_1.$$

3) No local minima near internal obstacles: This is ensured by $\varepsilon'_{i0}, \varepsilon''_{i0}$, originally $\|q_d - q_i\|^2 - \rho_i^2$ and

$$\frac{\min_{\overline{\mathscr{B}_i(\varepsilon'_{i0})}}\left\{2|\nu_i(q)|\bar{\beta}_i^2\right\}}{\max_{\overline{\mathscr{B}_i(\varepsilon'_{i0})}}\left\{\begin{array}{c}\frac{1}{2}\bar{\beta}_i\nabla\bar{\beta}_i\cdot\nabla\gamma_d\\ +\gamma_d\hat{v}^{\mathrm{T}}\left[\left(1-\frac{1}{k}\right)\nabla\bar{\beta}_i\nabla\bar{\beta}_i^{\mathrm{T}} - \bar{\beta}_i D^2\bar{\beta}_i\right]\hat{v}\end{array}\right\}}$$

respectively, where $\nu_i(q) \triangleq \frac{1}{4}\nabla\beta_i \cdot \nabla\gamma_d - \gamma_d = -\frac{1}{4}\left[2(q_i - q_d)\right]\cdot\left[2(q - q_d)\right] = -\frac{1}{4}\nabla\gamma_d(q_i)\cdot\nabla\gamma_d(q)$. For detailed derivation see [8]. There are two primary concerns.

Firstly in the original formulation $\min_{\overline{\mathscr{B}_i(\varepsilon'_{i0})}}\{|\nu_i(q)|\} = 0$ because ε'_{i0} places q_d on the boundary of $\overline{\mathscr{B}_i(\varepsilon'_{i0})}$. This can be avoided by redefining $\varepsilon'_{i0} \triangleq \lambda'_{i0}\left(\|q_d - q_i\|^2 - \rho_i^2\right)$ where $\lambda'_{i0} \in (0,1)$ is a scaling factor of our choice. Selecting λ'_{i0} close to 1 leads to $|\nu_i(q)| \to 0^+ \implies \varepsilon_i \to 0^+$, an undesired behaviour. If λ'_{i0} is selected too small the same.

The second relates to division of minima by maxima of β_j within $\overline{\mathscr{B}_i(\varepsilon_i)}$, as was the case also in § IV-B and § IV-C.2. Here we divide both nominator and denominator of ε''_{i0} by $\gamma_d\bar{\beta}_i^2$ before applying the $\min, \max$ operators. The new nominator is

$$-2\frac{\nu_i(q)}{\gamma_d(q)} = 2\frac{\nabla\gamma_d(q_i)}{\|\nabla\gamma_d(q)\|}\cdot\frac{\nabla\gamma_d(q)}{\|\nabla\gamma_d(q)\|} = 2\frac{r_i}{r}\cos\theta \triangleq f(r,\theta)$$

where $r \triangleq \|q-q_d\|, r_i = \|q_i - q_d\|, \theta \triangleq (\widehat{q_i - q_d, q - q_d})$. Finding a lower bound on the nominator reduces to a 2-dimensional problem due to sphere symmetry. Using Lagrange multipliers the nominator minimum over the annulus centered at q_i with inner radius ρ_i and outer radius $\rho'_{i0} < r_i$ can be found to be $\min_{\overline{\mathscr{B}_i(\varepsilon_{i03})}} \left\{-2\frac{\nu_i(q)}{\gamma_d(q)}\right\} = f(r_i + \rho'_{i0}, 0) = 2\frac{r_i}{r_i+\rho'_{i0}} > 1$ since $\rho'_{i0} < r_i$, see [8].

The new denominator will be

$$\frac{1}{2}\frac{\nabla\bar{\beta}_i}{\bar{\beta}_i}\cdot\frac{\nabla\gamma_d}{\gamma_d} + \hat{t}_i^{\mathrm{T}}\left[\left(1-\frac{1}{k}\right)\frac{\nabla\bar{\beta}_i}{\bar{\beta}_i}\frac{\nabla\bar{\beta}_i^{\mathrm{T}}}{\bar{\beta}_i} - \frac{D^2\bar{\beta}_i}{\bar{\beta}_i}\right]\hat{t}_i \quad (2)$$

where $\hat{t}_i \triangleq \frac{\nabla\beta_i^{\perp}}{\|\nabla\beta_i\|}$. Since $D^2\beta_i = 2I$, $\hat{t}_i^{\mathrm{T}}\frac{D^2\bar{\beta}_i}{\bar{\beta}_i}\hat{t}_i$ equals

$$\frac{1}{\bar{\beta}_i}\hat{t}_i^{\mathrm{T}}\sum_{j\in I_0\setminus i}\left(2\frac{\bar{\beta}_i}{\beta_j}I + \frac{\nabla\beta_j}{\beta_j}\sum_{l\in I_0\setminus\{i,j\}}\left(\bar{\beta}_i\frac{\nabla\beta_l^{\mathrm{T}}}{\beta_l}\right)\right)\hat{t}_i$$
$$= \sum_{j\in I_0\setminus i}\left(\frac{2}{\beta_j} + \frac{1}{\beta_j}\sum_{l\in I_0\setminus\{i,j\}}\frac{\hat{t}_i^{\mathrm{T}}\nabla\beta_j\nabla\beta_l^{\mathrm{T}}\hat{t}_i}{\beta_l}\right).$$

Substitution in (2) and grouping of terms leads to

$$\left[\begin{array}{c}\frac{1}{2}\frac{\nabla\bar{\beta}_i}{\bar{\beta}_i}\cdot\frac{\nabla\gamma_d}{\gamma_d} + \left(1-\frac{1}{k}\right)\hat{t}_i^{\mathrm{T}}\left[\frac{\nabla\bar{\beta}_i}{\bar{\beta}_i}\frac{\nabla\bar{\beta}_i^{\mathrm{T}}}{\bar{\beta}_i}\right]\hat{t}_i \\ -\sum_{j\in I_0\setminus i}\left(\frac{1}{\beta_j}\sum_{l\in I_0\setminus\{i,j\}}\frac{\hat{t}_i^{\mathrm{T}}\nabla\beta_j\nabla\beta_l^{\mathrm{T}}\hat{t}_i}{\beta_l}\right)\end{array}\right] - 2\sum_{j\in I_0\setminus i}\frac{1}{\beta_j}$$

The first term above is referred as A_i, the second $B-i$. We want to find $\max_{\overline{\mathscr{B}_i(\varepsilon_{i03})}}\{A_i - B_i\} \overset{B_i\geq 0}{=} \max_{\overline{\mathscr{B}_i(\varepsilon_{i03})}}\{A_i\} - \min_{\overline{\mathscr{B}_i(\varepsilon_{i03})}}\{B_i\}$. Let $G_i \triangleq A_i - B_i$.

We need not impose the constraint $\varepsilon_i < \varepsilon''_{i0}$ if $G_i < 0$. Instead of checking G_i's sign we can check *some* cases when $G_i < 0$. If $|A_i| < B_i$ then $G_i < 0$. In such a case it suffices *for that particular obstacle* i, to require just $\varepsilon_i < \varepsilon'_{i0}$ and not also $\varepsilon_i < \varepsilon''_{i0}$. This reduces unnecessary constraints.

Continuing with the case $|A_i| > B_i$, for which we still do not know the sign of G_i, we seek an upper bound on $\max_{\overline{\mathscr{B}_i(\varepsilon_{i03})}}\{A_i\}$ and proceed by bounding $|A_i|$ from above using the triangular inequality

$$|A_i| \leq \left|\frac{1}{2}\frac{\nabla\bar{\beta}_i}{\bar{\beta}_i}\cdot\frac{\nabla\gamma_d}{\gamma_d}\right| + \left|\left(1-\frac{1}{k}\right)\hat{t}_i^{\mathrm{T}}\left[\frac{\nabla\bar{\beta}_i}{\bar{\beta}_i}\frac{\nabla\bar{\beta}_i^{\mathrm{T}}}{\bar{\beta}_i}\right]\hat{t}_i\right|$$
$$+\left|\sum_{j\in I_0\setminus i}\left(\frac{1}{\beta_j}\sum_{l\in I_0\setminus\{i,j\}}\frac{\hat{t}_i^{\mathrm{T}}\nabla\beta_j\nabla\beta_l^{\mathrm{T}}\hat{t}_i}{\beta_l}\right)\right|.$$

Individual upper bound for the three terms are now found. By the triangular and Schwarz inequalities

$$\left|\frac{1}{2}\frac{\nabla\bar{\beta}_i}{\bar{\beta}_i}\cdot\frac{\nabla\gamma_d}{\gamma_d}\right| = \frac{1}{2}\left|\sum_{j\in I_0\setminus i}\left(\frac{\bar{\beta}_i}{\beta_j\bar{\beta}_i}\nabla\beta_j\cdot\frac{\nabla\gamma_d}{\gamma_d}\right)\right|$$
$$\leq \frac{1}{2}\sum_{j\in I_0\setminus i}\left(\frac{\|\nabla\beta_j\|}{\beta_j}\frac{\|\nabla\gamma_d\|}{\gamma_d}\right) = 2\frac{1}{\sqrt{\gamma_d}}\sum_{j\in I_0\setminus i}Q_j(\beta_j).$$

By $\left|\hat{v}^{\mathrm{T}}ab^{\mathrm{T}}\hat{v}\right| \leq \|a\|\|b\|$ proved using Schwarz inequality

$$\left|\left(1-\frac{1}{k}\right)\hat{t}_i^{\mathrm{T}}\left[\frac{\nabla\bar{\beta}_i}{\bar{\beta}_i}\frac{\nabla\bar{\beta}_i^{\mathrm{T}}}{\bar{\beta}_i}\right]\hat{t}_i\right| \leq \frac{\|\nabla\bar{\beta}_i\|^2}{\bar{\beta}_i^2}$$
$$= \left(2\sum_{j\in I_0\setminus i}Q_j(\beta_j)\right)^2$$

since $1-\frac{1}{k}\leq 1$. Also

$$\frac{1}{\beta_j}\sum_{l\in I_0\setminus\{i,j\}}\frac{\left|\hat{t}_i^{\mathrm{T}}\nabla\beta_j\nabla\beta_l^{\mathrm{T}}\hat{t}_i\right|}{\beta_l} \leq \frac{\|\nabla\beta_j\|}{\beta_j}\sum_{l\in I_0\setminus\{i,j\}}\frac{\|\nabla\beta_l\|}{\beta_l}$$
$$\Longrightarrow \left|\sum_{j\in I_0\setminus i}\left(\frac{1}{\beta_j}\sum_{l\in I_0\setminus\{i,j\}}\frac{\hat{t}_i^{\mathrm{T}}\nabla\beta_j\nabla\beta_l^{\mathrm{T}}\hat{t}_i}{\beta_l}\right)\right|$$
$$\leq 4\sum_{j\in I_0\setminus i}\left(Q_j(\beta_j)\sum_{l\in I_0\setminus\{i,j\}}Q_l(\beta_l)\right).$$

Let $\beta_{ji}^{\min} \triangleq \min_{\overline{\mathscr{B}_i(\varepsilon_{i03})}}\{\beta_j\}$ for Q_{0i}, Q_{ji}, $\gamma_{di}^{\min} \triangleq \min_{\overline{\mathscr{B}_i(\varepsilon_{i03})}}\{\gamma_d\}$, substitution leads to upper bound

$$\varepsilon''_{i0} \triangleq \frac{1}{\begin{array}{c}2\frac{1}{\sqrt{\gamma_{di}^{\min}}}\sum_{j\in I_0\setminus i}Q_{ji} + \left(2\sum_{j\in I_0\setminus i}Q_{ji}\right)^2 + \\ +4\sum_{j\in I_0\setminus i}\left(Q_{ji}\sum_{l\in I_0\setminus\{i,j\}}Q_{li}\right) - 2\sum_{j\in I_0\setminus i}\frac{1}{\beta_{ji}^{\max}}\end{array}}$$
$$> \varepsilon_i > \beta_i, \forall q\in\mathscr{B}_i(\varepsilon_i), \forall i\in I_1.$$

D. Case of unknown zeroth obstacle $\mathscr{O}_0$

So far automatically tuning a NF for a sphere world with internal obstacles was analyzed, but initially no obstacle is known. Any unknown obstacles must be disjoint spheres. An internal $\mathscr{O}_i, i\neq 0$ may be discovered before $\mathscr{O}_0$, so meanwhile the workspace is unbounded $\mathscr{O}_0 = \emptyset \implies \mathscr{W} = E^n$. This is not covered by the original NF formulation. We now extend NFs to unbounded worlds with internal spheres.

Propositions 2.7, 3.2, 3.3 [6] still hold, so we can work with $\hat{\varphi}$. In case of a single internal obstacle $\mathscr{O}_i, i\neq 0$ Propositions 3.6 and 3.9 [6] hold for $\varepsilon_i < \min\{\varepsilon'_{i0}, \varepsilon'_{i2}\}$ ($\varepsilon''_{i0}, \varepsilon''_{i2}$ not needed, $\varepsilon_{i3}, \varepsilon_{0u}$ undefined). If more internal obstacles are only present $\varepsilon_i < \min\{\varepsilon_{i03}, \varepsilon''_{i0}, \varepsilon_{i23}, \varepsilon''_{i2}\}$ applies (undefined ε_{0u}).

When at least one new obstacle is discovered at $t_m, m\in\mathbb{N}\setminus 0 = \mathbb{N}^*$ the NF is updated, increasing the number $M_z\in\mathbb{N}^*, z\in\mathbb{N}^*$ of currently known internal obstacles. Note that no or several new obstacles may be sensed at t_m so $m\neq z$. Let $i_{\min} = 1$ if $\mathscr{O}_0$ remains unknown, $i_{\min} = 0$ otherwise, $I_z \triangleq \{i_{\min},\ldots,M_z\}, I_{1z} \triangleq I_z\setminus 0, {}^z\beta \triangleq \prod_{i_{\min}}^{M_z}\beta_i$.

Let $\alpha_z \triangleq \varphi_z(x(t_m))$ be the updated NF potential at the agent's position $x(t_m)$ after the update. For all new obstacles discovered at t_m their ε_{iu} are calculated and ε_{iu} of already known obstacles recursively updated, then k_z is updated.

If updating occurs only when the agent is within $\mathscr{F}\setminus\partial\mathscr{F}$ then $x(t_m)\in\mathscr{F}\setminus\partial\mathscr{F}$ ($\mu(\partial\mathscr{F}) = 0$) then $\alpha_z < 1$. Since x is a gradient system it cannot overcome α_z before the next NF update at t_{m+1}. Let the closed $\mathscr{P}_z \triangleq \{q\in\mathscr{F} : \varphi_z(q)\leq\alpha_z\}$ which is positive invariant until the next NF update (convergence is guaranteed by the finite total number of unknown obstacles by Theorem 1). By definition $x(t_m)\in\mathscr{P}_z, \varphi_z(q_d) = 0\leq\alpha_z \implies q_d\in\mathscr{P}_z$.

To ensure $\mathscr{P}_z$ is bounded select $k_z > M_z \iff \lim_{\|q\|\to\infty}\varphi_z(q) = 1$. Hence there exists a sphere $\mathscr{Q}_z(\rho_b) \triangleq \{q\in E^n : \|q-q_d\|\leq\rho_b\}$ such that $\varphi_z(q) > \alpha_z, \forall q\notin\mathscr{Q}_z$ implying $\mathscr{P}_z\subseteq\mathscr{Q}_z$. The closed and bounded $\mathscr{P}_z$ is compact. The following provides an upper bound on $\sqrt{\gamma_d}$ in reachable $\mathscr{P}_z$

Proposition 1: If $\sqrt{\gamma_d(q)} > \max_i \{\|q_i - q_d\|\}$ and $\sqrt{\gamma_d(q)} > a_1^{m_1} a_2^{m_2}$ where $a_1 \triangleq \frac{4^{M_z}}{{}^z\beta(x(t_m))}, a_2 \triangleq \gamma_d(x(t_m))$ $m_1 \triangleq \begin{cases} 0, a_1 \leq 1 \\ \frac{1}{2}, a_1 > 1 \end{cases}, m_2 \triangleq \begin{cases} \frac{1}{2}, a_2 \leq 1 \\ \frac{M_z+1}{2}, a_2 > 1 \end{cases}$ then $q \notin \mathscr{P}_z$.

Proof: Since $k_z > M_z \iff k_z \geq M_z + 1$ it follows that $\frac{1}{2(k_z - M_z)} \in \left(0, \frac{1}{2}\right] \implies a_1^{m_1} \geq a_1^{\frac{1}{2(k_z - M_z)}}$ and $\frac{k_z}{2(k_z - M_z)} \in \left(\frac{1}{2}, \frac{M_z+1}{2}\right] \implies a_2^{m_2} \geq a_2^{\frac{k_z}{2(k_z - M_z)}}, \forall k_z > M_z$.

Hence $\sqrt{\gamma_d(q)} > a_1^{m_1} a_2^{m_2} \geq a_1^{\frac{1}{2(k_z - M_z)}} a_2^{\frac{k_z}{2(k_z - M_z)}} \implies \gamma_d^{k_z - M_z}(q) > \frac{\gamma_d^{k_z}(x(t_m))}{{}^z\beta(x(t_m))} 4^{M_z} \implies \gamma_d^{k_z}(q) > \hat{\varphi}(x(t_m)) \left(4\gamma_d(q)\right)^{M_z}$. The triangular inequality yields $\sqrt{\beta_i(q)} = \|q - q_i\| \leq \|q - q_d\| + \|q_d - q_i\| = \sqrt{\gamma_d(q)} + \|q_i - q_d\|$ and for $\sqrt{\gamma_d(q)} > \max_i \{\|q_i - q_d\|\}$, as required by hypothesis, it follows that $\beta_i(q) < 4\gamma_d(q), \forall i \in I_{1z} \implies {}^z\beta(q) < (4\gamma_d(q))^{M_z}$. Substituting $\gamma_d^{k_z}(q) > \hat{\varphi}(x(t_m))^z\beta(q)$ and since $\partial\mathscr{F} \cap \mathscr{P}_z = \emptyset$ we can examine only the interior $\mathscr{F} \setminus \partial\mathscr{F}$ where ${}^z\beta(q) > 0$ and there the previous is equivalent to $\hat{\varphi}(q) > \hat{\varphi}(x(t_m))$. Since $\sigma_d \circ \sigma$ is strictly increasing this implies $\varphi_z(q) > \varphi_z(x(t_m)) \implies q \notin \mathscr{P}_z$. ■

Let $q \in \mathscr{P}_z$ and suppose $\sqrt{\gamma_d(q)} > \rho_a \triangleq \max\{\max_i \{\|q_i - q_d\|\}, a_1^{m_1} a_2^{m_2}\}$. By Prop. 1 $q \notin \mathscr{P}_z$, a contradiction, hence $\sqrt{\gamma_d(q)} \leq \rho_a, \forall q \in \mathscr{P}_z$.

Let $\mathscr{R}_z(\rho_a) \triangleq \{q \in E^n : \|q - q_d\| \leq \rho_a\}$ so $\mathscr{P}_z \subseteq \mathscr{R}_z$. Substituting the upper bound on $\sqrt{\gamma_d}$ in $\mathscr{R}_z$ in the inequality $\max_{\mathscr{R}_z}\{\sqrt{\gamma_d}\} \sum_{I_{1z}} Q_{ii} < k_z$ to find a lower bound for k_z within the positive invariant set $\mathscr{P}_z$ leads to $\rho_a \sum_{I_{1z}} Q_{ii} < k_z$. This prevents critical points arising "away" from obstacles in $\mathscr{P}_z$, where the agent is confined. For unknown $\mathscr{O}_0$

$$k_z > \max\left\{\rho_a \sum_{i \in I_{1z}} Q_{ii}, \quad M_z\right\} \triangleq N(\varepsilon_{I_{1z}}) \tag{3}$$

Local minima may arise outside $\mathscr{R}_z \supseteq \mathscr{P}_z$ but the agent cannot reach them before either it converges to q_d or the NF is again updated at time t_{m+1}. A detailed proof can be found in [8].

V. TUNING UPDATING

A. Algorithm description

Let $\mathscr{S}(t)$ the agent's open sensing set at time t. Sensing occurs in discrete time $t_{m+1} = t_m + T_s$. Provided $\mathscr{S}(t_m) \cap x(t_{m+1}) \neq \emptyset$ the agent does not venture into unknown territory, ensured by a small enough T_s. To ensure constraints remain valid, k_z is nondecreasing. Initially no obstacle is known, so $I_{z=0} = \emptyset, \beta = 1, k_{z=0} = 2$ and $V = \varphi(x(t)) = \sigma_d \circ \sigma \circ \frac{\gamma_d^k}{1}$ does not contain any obstacles.

Next two alternatives exist. Either the system converges to q_d without sensing any obstacles, or an obstacle is discovered, either $\mathscr{O}_0$ or $\mathscr{O}_1$. If only a single internal obstacle is known, $\varepsilon_i < \min\{\varepsilon'_{i0}, \varepsilon'_{i2}\}$ in (3). If more internal obstacles are only known $\varepsilon_i < \min\{\varepsilon_{i03}, \varepsilon''_{i0}, \varepsilon_{i23}, \varepsilon''_{i2}\}$ in (3).

When $\mathscr{O}_0$ is discovered previous ε_i constraints are updated as described later, and $N(\varepsilon_{I_z}) \leq k_z$ as defined in § IV-B instead of (3). If only $\mathscr{O}_0$ is known $\varepsilon_0 < \varepsilon_{0u}$ in $N(\varepsilon_{I_z})$. When any new internal obstacle $\mathscr{O}_i$ is discovered calculation of $\varepsilon'_{i0}, \varepsilon''_{i0}, \varepsilon'_{i2}, \varepsilon''_{i2}, \varepsilon_{i3}$ can be performed in time $\Theta(M_z)$, § V-B. A high level overview of the updating scheme follows.

```
procedure NEW z + 1^TH DISCOVERED 𝒪_n
    if n ≠ 0 then
        if M_z == 0 and i_min == 1 then
            ε_1u ← min{ε'_10, ε'_12}, NEW ε_1
        else if M_z == 1 and i_min == 1 then
            NEW ε_iu, ε'_i0, ε''_i0, ε'_i2, ε''_i2, ε_i3, ε_i, ∀i ∈ {1, 2}
        else
            NEW ε_nu, ε'_n0, ε''_n0, ε'_n2, ε''_n2, ε_n3, ε_n
            UPDATE ε_iu, ε''_i0, ε''_i2, ε_i3, i ≠ n
        end if
    else
        ε_0u ← ρ_0^2 − ‖q_d‖^2, NEW ε_0
        if M_z > 0 then
            UPDATE ε_iu, i ≠ 0
        else if M_z == 0 then
            NEW ε_1u, ε'_10, ε''_10, ε'_12, ε''_12, ε_13, ε_1
        end if
    end if
    k_{z+1} ← UPDATE k_z
end procedure
```

```
procedure UPDATE k_z
    for i = 1 : M_{z+1} do
        if ε_iu < ε_i then    ▷ See §V-C for "and β_i < ε_i"
            ε_i^old ← ε_i,    ε_i ← λε_iu, λ ∈ (0, 1)
            Q_ii^old ← f(ε_i^old),    Q_ii^new ← f(ε_i)
            ΔQ_ii ← Q_ii^new − Q_ii^old
            ∑Q_ii ← ∑Q_ii + ΔQ_ii
        end if
    end for
    if i_min == 0 then
        k_lb ← (ρ_0 + ‖q_d‖) ∑Q_ii
    else
        k_lb ← 1 + max{ρ_a ∑_{I_1z} Q_ii,   M_z}
    end if
    k_{z+1} ← max{2, k_z, k_lb}
end procedure
```

B. Computational complexity

When a new $\mathscr{O}_n$ is discovered $\varepsilon'_{n0}, \varepsilon''_{n0}, \varepsilon'_{n2}, \varepsilon''_{n2}, \varepsilon_{n3}$ are initialized. Also $\varepsilon_{i03}, \varepsilon''_{i0}, \varepsilon_{i23}, \varepsilon''_{i2}$ of M_z already known obstacles are affected and need update.

For initialization, only term $Q_{jn} \sum_{l \in I_{z+1} \setminus \{n,j\}} Q_{ln}$ appears not to be linear in M_{z+1}, but if arranged as $\left(Q_{jn} \sum_{l \in I_{z+1} \setminus n} Q_{ln}\right) - \left(Q_{jn}\right)^2$ it becomes linear by computing first $\sum_{l \in I_{z+1} \setminus n} Q_{ln}$ and then multiplying by Q_{ji} and subtracting Q_{ji}^2 for M_z obstacles in the outer summation.

The update can be performed in $\Theta(M_z)$ because of two reasons. Firstly, $\varepsilon'_{i0}, \varepsilon'_{i2}, i \neq n$ of previous obstacles remain unchanged. Changes in ε_{i3}^{new} can only be caused if the new $\varepsilon_{i3n} < \varepsilon_{i3}^{old}$. So ε_{i3} can only decrease. Hence $\mathscr{B}_i\left(\varepsilon_{i03}^{\mathrm{new}}\right) \subseteq \mathscr{B}_i\left(\varepsilon_{i03}^{\mathrm{old}}\right)$ (same for ε_{i23}), therefore $\frac{1}{\beta_{ji}^{\mathrm{min,new}}} \leq \frac{1}{\beta_{ji}^{\mathrm{min,old}}}, \frac{1}{\beta_{ji}^{\mathrm{max,new}}} \geq \frac{1}{\beta_{ji}^{\mathrm{max,old}}}, Q_{ji}^{\mathrm{new}} \leq Q_{ji}^{\mathrm{old}}$, so previous bounds remain valid and need not be recalculated. Secondly, updating $Q_{ji} \sum_{l \in I_z \setminus \{i,j\}} Q_{li}$ can be arranged recursively as

$$\sum_{j \in I_{z+1} \setminus i} \left(Q_{ji}^{\text{new}} \sum_{l \in I_{z+1} \setminus \{i,j\}} Q_{li}^{\text{new}} \right) \leq 2 Q_{ni}^{\text{new}} \sum_{j \in I_z \setminus i} \left(Q_{ji}^{\text{old}} \right) + \sum_{j \in I_z \setminus i} \left(Q_{ji}^{\text{old}} \sum_{l \in I_z \setminus \{i,j\}} Q_{li}^{\text{old}} \right),$$

similarly for other updated quantities.

C. Locally oriented tuning of analytic Navigation Functions

Not all constraints need to become effective for provably correct navigation. When an obstacle is discovered, an ε_i can be arbitrarily selected. If used in $N(\varepsilon_{I_z})$, then critical points remain only within $\mathscr{B}_i(\varepsilon_i)$. As long as the agent does not enter $\mathscr{B}_i(\varepsilon_i)$, although updated, $\varepsilon'_{i0}, \varepsilon''_{i0}, \varepsilon'_{i2}, \varepsilon''_{i2}, \varepsilon_{i3}$ need not be applied. This is equivalent to adding "and $\beta_i < \varepsilon_i$" to line 3 of UPDATE k_z.

If for arbitrary ε_i local minima remain within $\mathscr{B}_i(\varepsilon_i)$ and attract the agent, it will eventually enter $\mathscr{B}_i(\varepsilon_i)$. We check this entrance and then apply the calculated constraint $\varepsilon_i < \varepsilon_{ui}$, ensuring those local minima within $\mathscr{B}_i(\varepsilon_i)$ become saddles.

D. Convergence

Theorem 1: Let $\mathscr{M}$ be a valid sphere world with initially unknown $\mathscr{O}_i$. Let $\mathscr{S}(t_m)$ the agent's sensing set at time t_m and assume T_s small enough for the agent to remain in $\bigcup_m \mathscr{S}(t_m)$. If a NF can be found for each intermediate space as obstacles are discovered, then the agent converges to the destination q_d.

The proof can be found in [8] and relies on the finite number of unknown obstacles, ensuring finite many switches, and that each NF leads almost any initial condition to q_d.

VI. Simulation Results

In Fig. 2 navigation in an unknown 2d sphere world with automatic k_z is compared to manually selected constant $k = 2$ (top) and $k = 10$ (middle). As $\mathscr{O}_0$ and internal obstacles are gradually discovered, the NF is updated. While a constant k leads to failure, updating k_z results in a shorter path, but with high k_z which repels only close to obstacles. A 3d unknown sphere world Fig. 3 illustrates applicability to any dimension, a strong advantage of NF. Due to high numerical values of k we use the normalized gradient to avoid exponentiation. Gradient trajectories (integral lines) remain unaffected, Lemma 7 [9]. Because $\nabla\varphi = \left(\gamma_d^k + \beta\right)^{-\frac{1}{k}-1}\left(\beta\nabla\gamma_d - \frac{\gamma_d}{k}\nabla\beta\right)$, exact cancellation of k powers is possible $\frac{\nabla\varphi}{\|\nabla\varphi\|} = \frac{\beta\nabla\gamma_d - \frac{\gamma_d}{k}\nabla\beta}{\left\|\beta\nabla\gamma_d - \frac{\gamma_d}{k}\nabla\beta\right\|}$. The simulation integration step is selected less than minimum distance to any obstacle and sensing radius.

VII. Conclusions and Future Work

An algorithm for automated, constantly updating NF tuning for provably correct collision avoidance and convergence to destination has been provided for worlds comprised of unknown disjoint spheres. Provided an updatable diffeomorphism links the sphere world with real world, the present method can be extended to general unknown worlds.

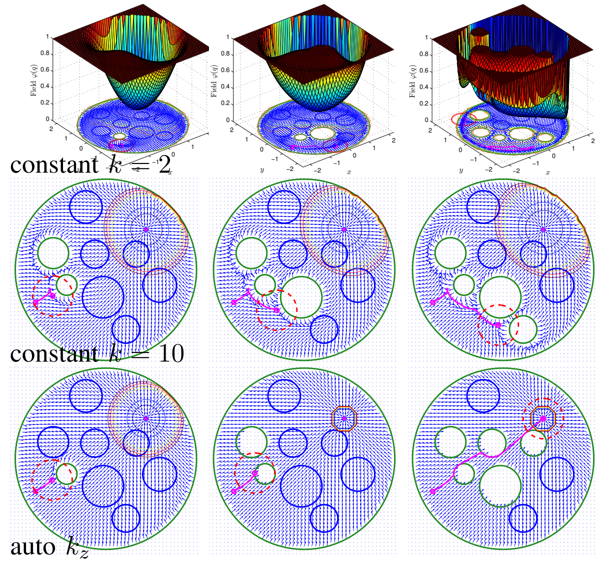

Fig. 2. Navigation in unknown 2-dimensional sphere world succeeds using automatic k_z (bottom), but fails with constant $k = 2, 10$ (top, middle). Green: sensed, Blue: unsensed obstacles, Red: sensing set.

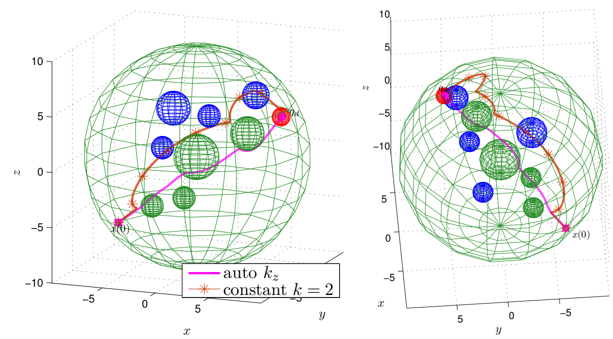

Fig. 3. Adjustable exploration of unknown 3-dimensional sphere world (2 viewpoints) with constant $k = 2$ and updating k_z. Green: discovered obstacles (auto k_z), Blue: unsensed obstacles (auto k_z), Red: sensing set.

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