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On Infinity Norms as Lyapunov Functions for Continuous-time Dynamical Systems

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Abstract— **This paper considers the synthesis of polyhedral Lyapunov functions for continuous-time dynamical systems. A proper conic partition of the state-space is employed to construct a finite set of linear inequalities in the elements of the Lyapunov weight matrix. For dynamics described by linear and polytopic differential inclusions, it is proven that the feasibility of the derived set of linear inequalities is necessary and sufficient for the existence of an infinity norm Lyapunov function. Furthermore, it is shown that the developed solution naturally applies to relevant classes of continuous-time nonlinear systems. An extension to non-symmetric polyhedral Lyapunov functions is also presented.**

I. Introduction

Polyhedral Lyapunov functions (or shortly, LFs) have attracted an increased interest in the recent years, as they are less conservative compared to quadratic Lyapunov functions. Also, in many control problems constraints are expressed by linear inequalities, which makes polyhedral LFs more suitable for constructing an estimate of the domain of attraction. For an excellent exposition of related results see [1].

One of the most popular issues related to polyhedral LFs is the existence and synthesis of a Lyapunov function defined using a weighted infinity norm. In that respect, it is well known [2] that existence of an infinity norm LF is equivalent with existence of a 0-symmetric polyhedral contractive (invariant) set. For asymptotically stable systems described by a linear polytopic differential or difference inclusion it is known [3]–[6] that existence of an infinity norm LF is a necessary condition. Available methods for constructing an infinity norm LF for linear systems or polytopic differential inclusions (see [1] for a detailed overview) stem from the necessary and sufficient conditions proposed in [3], which are also mentioned in [4], where polyhedral Lyapunov Functions are embedded in the theory of vector norms as Lyapunov functions. However, these conditions, although non-conservative, require the solution of a rather difficult non-convex problem, which includes a bilinear matrix equality constraint.

Recently, a novel, geometric approach to the synthesis of infinity norm LFs was proposed for discrete-time dynamical systems in [7]. Therein, a proper conic partition of the state-space was employed to obtain an alternative set of necessary and sufficient conditions for the existence of an infinity norm Lyapunov function. This approach retains computational tractability even when applied to certain relevant classes of nonlinear dynamics, which is a unique property. The aim of this paper is to apply the approach of [7] to systems described by linear differential equations or linear polytopic differential inclusions. This requires different tools and already provides a useful relaxation, i.e., an eigen value restriction (decomposition) or the strict diagonal dominance property [6] is no longer required. Furthermore, similarly to the discrete-time case, a natural extension to several relevant classes of continuous-time nonlinear systems is obtained. In particular, a useful result is obtained for continuous-time quadratic nonlinear systems, in the form of sufficient conditions for the existence of a local polyhedral LF given by a set of linear inequalities. A generalization to non-symmetric polyhedral Lyapunov functions is also provided.

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II. Preliminaries

A. Basic notation and definitions

Let $\mathbb{R}$, $\mathbb{R}_+$, $\mathbb{Z}$ and $\mathbb{Z}_+$ denote the field of real numbers, the set of non-negative reals, the set of integer numbers and the set of non-negative integers, respectively. For every $c \in \mathbb{R}$ and $\Pi \subseteq \mathbb{R}$ define $\Pi_{\geq c} := \{k \in \Pi \mid k \geq c\}$ and similarly $\Pi_{\leq c}$, $\mathbb{R}_\Pi := \Pi$ and $\mathbb{Z}_\Pi := \mathbb{Z} \cap \Pi$. For a set $\mathcal{S} \subseteq \mathbb{R}^n$, $\operatorname{int}(\mathcal{S})$ denotes the interior and $\partial\mathcal{S}$ denotes the boundary of $\mathcal{S}$. A set $\mathcal{S}$ is called 0-symmetric (or shortly, symmetric) if for all $x \in \mathcal{S}$ it holds that $-x \in \mathcal{S}$. For any $Z \in \mathbb{R}^{l\times n}$ and $\mathcal{S} \subseteq \mathbb{R}^n$, $-\mathcal{S} := \{-x \mid x \in \mathcal{S}\}$ and $Z\mathcal{S} := \{Zx \mid x \in \mathcal{S}\}$. For a vector $x \in \mathbb{R}^n$, $[x]_i$ denotes the i-th element of x and $\|x\| := \|x\|_\infty = \max_{i=1,\ldots,n} |[x]_i|$ denotes the infinity norm of x, where $|\cdot|$ denotes the absolute value. For a matrix $Z \in \mathbb{R}^{l\times n}$, $[Z]_{ij} \in \mathbb{R}$ denotes the element in the i-th row and j-th column of Z and $[Z]_{i\bullet} \in \mathbb{R}^{1\times n}$ denotes the i-th row of Z. $I_n \in \mathbb{R}^{n\times n}$ denotes the n-th dimensional identity matrix. For a matrix $Z \in \mathbb{R}^{l\times n}$ let $\|Z\| := \sup_{x\neq 0} \frac{\|Zx\|}{\|x\|}$ denote its induced matrix infinity norm and for a matrix $Q \in \mathbb{R}^{n\times n}$ let $\mu(Q) := \limsup_{h\to 0} \frac{\|I_n + hQ\| - 1}{h}$ denote its induced logarithmic norm. It is well known that $\|Z\| = \max_{i\in\mathbb{Z}_{[1,l]}} \sum_{j=1}^n |[Z]_{ij}|$. For a symmetric matrix $Z \in \mathbb{R}^{n\times n}$ let $Z \succ 0 (Z \succeq 0)$ denote that Z is positive definite (semi-definite). For any $x, y \in \mathbb{R}^n$, $c \in \mathbb{R}_+$, $x \leq y$, $x < y$, $x \geq y$ and $x > y$ denote the corresponding set of component-wise inequalities and $\pm x \leq c$ denotes the inequalities $-c \leq x \leq c$. A subset $\mathcal{C}$ of $\mathbb{R}^n$ is a convex cone if and only if $c_1\mathcal{C} \oplus c_2\mathcal{C} = \mathcal{C}$ for any $c_1, c_2 \in \mathbb{R}_+$. A convex cone $\mathcal{C}$ is salient if and only if $\mathcal{C} \cap -\mathcal{C} = \{0\}$. A convex cone $\mathcal{C}$ is pointed if $0 \in \mathcal{C}$. A n-th dimensional cone $\mathcal{C}$ in $\mathbb{R}^n$ is called a proper cone if it is convex, salient, pointed and $\operatorname{int}(\mathcal{C}) \neq \emptyset$. Given a cone $\mathcal{C}$ in $\mathbb{R}^n$, its dual cone, denoted by $\mathcal{C}^\perp$, is defined as $\mathcal{C}^\perp := \{y \in \mathbb{R}^n \mid y^\top x \geq 0, \forall x \in \mathcal{C}\}$. Furthermore, if $\mathcal{C}$ is a proper cone, then so it is $\mathcal{C}^\perp$. For any point $y \in \mathbb{R}^n$, $y \neq 0$, the set $r(y) := \{x \in \mathbb{R}^n \mid x = cy, c \in \mathbb{R}_+\}$ is called a ray.

B. Stability definitions and results

A function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *belongs to class* $\mathcal{K}$ if it is continuous, strictly increasing and $\varphi(0) = 0$. A function $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ *belongs to class* $\mathcal{K}_\infty$ if $\varphi \in \mathcal{K}$ and $\lim_{s\to\infty} \varphi(s) = \infty$. Consider the continuous-time system

$$\dot{x}(t) = \Phi(x(t)), \quad t \in \mathbb{R}_+, \tag{1}$$

where $x : \mathbb{R}_+ \to \mathbb{R}^n$ is the state trajectory and $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ is an arbitrary map, with Φ sufficiently smooth and $\Phi(0) = 0$. For simplicity of exposition, x can also denote a point in $\mathbb{R}^n$. Let $V : \mathbb{R}^n \to \mathbb{R}$ be locally Lipschitz and define the upper right Dini derivative $\mathscr{D}^+V(x(t))$ of V at $x(t)$, for some $t \in \mathbb{R}_+$, as follows:

$$\mathscr{D}^+V(x(t)) := \limsup_{h\to 0^+} \frac{V(x(t+h)) - V(x(t))}{h}. \tag{2}$$

Definition II.1 A set $\mathcal{P} \subseteq \mathbb{R}^n$ is called *positively invariant (PI)* for system (1) if for all $x(0) \in \mathcal{P}$ it holds that $x(t) \in \mathcal{P}$ for all $t \in \mathbb{R}_+$.

Definition II.2 Let $\mathbb{X}$ with $0 \in \text{int}(\mathbb{X})$ be a subset of $\mathbb{R}^n$. System (1) is *Lyapunov stable* if for all $\varepsilon \in \mathbb{R}_{>0}$ there exists a $\delta(\varepsilon) \in \mathbb{R}_{>0}$ such that $\|x(0)\| \leq \delta(\varepsilon)$ implies $\|x(t)\| \leq \varepsilon$ for all $t \in \mathbb{R}_+$. The origin of (1) is *attractive in* $\mathbb{X}$ if for any $x(0) \in \mathbb{X}$ it holds that $\lim_{t\to\infty} \|x(t)\| = 0$. System (1) is *asymptotically stable in* $\mathbb{X}$ if it is Lyapunov stable and attractive in $\mathbb{X}$. System (1) is *exponentially stable in* $\mathbb{X}$ if for any $x(0) \in \mathbb{X}$ it holds that $\|x(t)\| \leq \theta\|x(0)\|e^{-\nu t}$ for some $\theta \in \mathbb{R}_+$, $\nu \in \mathbb{R}_{>0}$. System (1) is globally asymptotically (exponentially) stable (GAS (GES)) if it is asymptotically (exponentially) stable in $\mathbb{R}^n$.

Theorem II.3 *Let* $\mathbb{X} \subseteq \mathbb{R}^n$ *be a PI set for* (1) *with* $0 \in \text{int}(\mathbb{X})$. *Furthermore, let* $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$, $\rho \in \mathbb{R}_{>0}$ *and let* $V : \mathbb{R}^n \to \mathbb{R}_+$ *be a smooth function such that:*

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|), \quad \forall x \in \mathbb{X}, \tag{3a}$$

$$\mathscr{D}^+V(x(t)) \leq -\rho V(x(t)), \quad \forall x(t) \in \mathbb{X}, t \in \mathbb{R}_+. \tag{3b}$$

Then system (1) *is asymptotically stable in* $\mathbb{X}$. *If the above inequalities hold with* $\alpha_1(s) := c_1 s^\nu$, $\alpha_2(s) := c_2 s^\nu$ *for some* $c_1, c_2, \nu \in \mathbb{R}_{>0}$, *then system* (1) *is exponentially stable in* $\mathbb{X}$.

A proof of the above theorem can be found in [1].

Definition II.4 A function V that satisfies (3) is called a *Lyapunov function in* $\mathbb{X}$. A Lyapunov function in $\mathbb{R}^n$ is called a *global Lyapunov function*.

As indicated in [1], [2], for polyhedral functions of the form $V(x) = \|Px\|$, with $P \in \mathbb{R}^{l\times n}$, $l \in \mathbb{Z}_{\geq n}$, observing that $V(x) = \max_i [\tilde{P}]_{i\bullet} x$ with $\tilde{P} := \begin{pmatrix} P & -P \end{pmatrix}^\top$ one obtains that the derivative of V along the trajectories of (1) is

$$\mathscr{D}^+V(x) = \max_{i\in I(x)} \{[\tilde{P}]_{i\bullet}\Phi(x)\}, \tag{4}$$

where $I(x) := \{i \in \mathbb{Z}_{[1,2l]} \mid [\tilde{P}]_{i\bullet} x = V(x)\}$.

Next, consider a linear continuous-time system, i.e.,

$$\dot{x}(t) = Ax(t), \quad t \in \mathbb{R}_+, \tag{5}$$

where $A \in \mathbb{R}^{n\times n}$. Let $V_q(x) := x^\top P_q x$ with $P_q \in \mathbb{R}^{n\times n}$ and $V(x) := \|Px\|$ with $P \in \mathbb{R}^{l\times n}$, $l \in \mathbb{Z}_{\geq n}$, be a quadratic and infinity norm Lyapunov function candidate, respectively.

Theorem II.5 *(i) System* (5) *is GES if and only if there exists a matrix* $P_q \succ 0$ *such that, for any* $Q \succeq 0$, $A^\top P_q + P_q A \preceq -Q$. *(ii) System* (5) *is GES if and only if there exists* $\eta \in \mathbb{R}_{<0}$, *a number* $l \in \mathbb{Z}_{\geq n}$, *a matrix* $P \in \mathbb{R}^{l\times n}$ *with* $\text{rank}(P) = n$ *and a matrix* $Q \in \mathbb{R}^{l\times l}$ *such that* $PA = QP$ *and* $\mu(Q) \leq \eta$.

For the proof see [3]–[5]. In [4], a direct relation between P_q and P was indicated as well. For infinity norm Lyapunov functions, in [4] it was indicated that $V(x) = \|Px\|$ is positive definite and radially unbounded, which is sufficient for GAS, but not for GES. In what follows, a fact established in [7] is recalled, which exposes a direct relation with GES. For any $n \in \mathbb{Z}_{\geq 1}$ let $l \in \mathbb{Z}_{\geq n}$ and $P \in \mathbb{R}^{l\times n}$.

Fact II.6 *The following statements are equivalent.*

(i) $\text{rank}(P) = n$.

(ii) The function $V(x) = \|Px\|$ *satisfies* (3a) *with* $\alpha_1(s) := cs$ *for some* $c \in \mathbb{R}_{>0}$ *and* $\alpha_2(s) := \|P\|s$.

Although the conditions specified by Theorem II.5-*(ii)* are non-conservative, finding a solution that satisfies these conditions is challenging, due to the rank constraint and the bilinear equality constraint. Several attempts were made to design a tractable algorithm that solves this problem, see, e.g., [6], [8], [9]. In the next section, an alternative set of necessary and sufficient conditions for the existence of an infinity norm Lyapunov function is proposed, based on the idea introduced in [7] for discrete-time dynamics. The additional difficulty particular to continuous-time dynamics is related to the expression of the derivative provided by (4), which requires a different treatment (see the proof of Theorem III.6).

III. CONTINUOUS-TIME LINEAR DYNAMICS

The idea is to start directly from the Lyapunov conditions (3) and transform them into a finite set of convex conditions on a specific conic partition of the state-space. To this end, let us define a proper conic partition of $\mathbb{R}^n$.

Definition III.1 Let $l \in \mathbb{Z}_{\geq n}$ and let $\mathcal{L} := \mathbb{Z}_{[1,l]}$. A finite set of cones $\{\mathcal{C}_i\}_{i\in\mathcal{L}}$ is called *a proper l-conic partition of* $\mathbb{R}^n$ if $\cup_{i\in\mathcal{L}}\{\mathcal{C}_i \cup -\mathcal{C}_i\} = \mathbb{R}^n$, $\mathcal{C}_i$ is a proper cone for all $i \in \mathcal{L}$ and $\text{int}(\mathcal{C}_i) \cap \text{int}(\mathcal{C}_j) = \emptyset$ for all $(i, j) \in \mathcal{L} \times \mathcal{L}$ with $i \neq j$.

Fact III.2 *Let* $x \in \mathbb{R}^n$, $\Gamma \in \mathbb{R}_+$ *and* $P \in \mathbb{R}^{l\times n}$. *The following statements are equivalent.*

(i) $\|Px\| \leq \Gamma$.

(ii) $\pm[P]_{i\bullet} x \leq \Gamma$ *for all* $i \in \mathcal{L}$.

The equivalence follows directly from the definition of the infinity norm.

Fact III.3 *Let $x \in \mathbb{R}^n$, $i \in \mathcal{L}$ and $P \in \mathbb{R}^{l\times n}$. The following statements are equivalent.*

(i) $\|Px\| = [P]_{i\bullet}x$.

(ii) $([P]_{i\bullet} \pm [P]_{j\bullet})x \geq 0$ *for all* $j \in \mathcal{L} \setminus \{i\}$.

The equivalence follows from Fact III.2 and the definition of the infinity norm.

Fact III.4 *Let $\mathcal{C}$ be a proper cone in $\mathbb{R}^n$ and let E be a real matrix of suitable dimensions such that*

$$\mathcal{C}^{\perp} = \{x \in \mathbb{R}^n \mid Ex \geq 0\}. \tag{6}$$

Then for any $x \in \mathbb{R}^n$ the following statements are equivalent.

(i) $x^{\top}y \geq 0, \quad \forall y \in \mathcal{C}$.

(ii) $Ex \geq 0$.

The equivalence follows from the definition of a dual cone.

Fact III.5 [7] *Let $P \in \mathbb{R}^{l\times n}$. The following statements are equivalent.*

(i) $\operatorname{rank}(P) = n$.

(ii) There exists a $c \in \mathbb{R}_{>0}$ such that $\|Px\| \geq c\|x\|$ for all $x \in \mathbb{R}^n$.

The main result for linear dynamics is stated next.

Theorem III.6 *Let $l \in \mathbb{Z}_{\geq n}$, $\mathcal{N} := \mathbb{Z}_{[1,n]}$ and $P \in \mathbb{R}^{l\times n}$. The following statements are equivalent.*

(i) The function $V(x) = ||Px||$ is a global Lyapunov function for system (5).

(ii) There exist $c \in \mathbb{R}_{>0}$, $\rho \in \mathbb{R}_{>0}$ and a proper l-conic partition of $\mathbb{R}^n$, i.e., $\{\mathcal{C}_i\}_{i\in\mathcal{L}}$, with the set of matching dual cones $\{\mathcal{C}_i^{\perp}\}_{i\in\mathcal{L}}$ and corresponding matrices $\{E_i\}_{i\in\mathcal{L}}$, such that the following inequalities hold for all $i \in \mathcal{L}$*:*

$$E_i([P]_{i\bullet} \pm c[I_n]_{j\bullet})^{\top} \geq 0, \quad \forall j \in \mathcal{N}, \tag{7a}$$

$$E_i([P]_{i\bullet} \pm [P]_{j\bullet})^{\top} \geq 0, \quad \forall j \in \mathcal{L} \setminus \{i\}, \tag{7b}$$

$$E_i(-\rho[P]_{i\bullet} - [P]_{i\bullet}A)^{\top} \geq 0. \tag{7c}$$

Proof: Let us proceed with the proof of *(ii)* $\Rightarrow$ *(i)*. Fact III.4 and (7a) yield

$$([P]_{i\bullet} \pm c[I_n]_{j\bullet})x \geq 0, \quad \forall j \in \mathcal{N}, \quad \forall x \in \mathcal{C}_i.$$

Letting $\Gamma := [P]_{i\bullet}x$, the above inequality and Fact III.2 yield $[P]_{i\bullet}x \geq c\|x\|$ for all $x \in \mathcal{C}_i$. Similarly, (7b) and Fact III.4 imply

$$([P]_{i\bullet} \pm [P]_{j\bullet})x \geq 0, \quad \forall j \in \mathcal{L} \setminus \{i\}, \quad \forall x \in \mathcal{C}_i.$$

Then, from Fact III.3 it follows that $\|Px\| = [P]_{i\bullet}x$ for all $x \in \mathcal{C}_i$. This further yields that $\|Px\| = -[P]_{i\bullet}x$ for all $x \in -\mathcal{C}_i$. As such, since (7) holds for all $i \in \mathcal{L}$ and $\{\mathcal{C}_i\}_{i\in\mathcal{L}}$ is a proper l-conic partition of $\mathbb{R}^n$, we have that $\|Px\| \geq c\|x\|$ for all $x \in \mathbb{R}^n$. Using Fact III.5 yields that $\operatorname{rank}(P) = n$ and hence, by Fact II.6 $V(x) = \|Px\|$ satisfies (3a) for all $x \in \mathbb{R}^n$. Using a similar reasoning, from (7c) one obtains that $[P]_{i\bullet}Ax \leq -\rho[P]_{i\bullet}x$ for all $x \in \mathcal{C}_i$. The set of constraints (7b) together with Fact III.4 and Fact III.3 yield that $V(x) = [P]_{i\bullet}x$ for all $x \in \mathcal{C}_i$. Next, let us distinguish between two cases: $x \in \operatorname{int}(\mathcal{C}_i)$ and $x \in \partial\mathcal{C}_i$. The first case can be reduced to $([P]_{i\bullet} \pm [P]_{j\bullet})x > 0$ for all $x \in \operatorname{int}(\mathcal{C}_i)$, i.e., if this is not the case one of the hyperplanes is redundant and the property is obtained for a larger cone. This and (4) imply that $\mathscr{D}^{+}V(x) = [P]_{i\bullet}Ax$, for all $x \in \operatorname{int}(\mathcal{C}_i)$. The second case can be reduced to $x \in \partial\mathcal{C}_i \cap \partial\mathcal{C}_j$, for some arbitrary $j \in \mathcal{L}$, $j \neq i$. Then, as $x \in \mathcal{C}_i$ and $x \in \mathcal{C}_j$, the corresponding inequalities (7b) and Fact III.4 yield $[P]_{i\bullet}x = [P]_{j\bullet}x = V(x)$. Hence, from inequalities (7c) and Fact III.4 one obtains that both $(-\rho[P]_{i\bullet} - [P]_{i\bullet}A)x \geq 0$ and $(-\rho[P]_{i\bullet} - [P]_{j\bullet}A)x \geq 0$ hold. This, together with (4) and the result obtained for $x \in \operatorname{int}(\mathcal{C}_i)$ imply that $\mathscr{D}^{+}V(x) \leq -\rho[P]_{i\bullet}x = -\rho V(x)$ for all $x \in \mathcal{C}_i$. Hence, as the same reasoning can be applied for all $i \in \mathcal{L}$ and $\{\mathcal{C}_i\}_{l\in\mathcal{L}}$ is a proper l-conic partition of $\mathbb{R}^n$, it follows that inequality (3b) is satisfied for the considered candidate Lyapunov function.

To prove *(i)* $\Rightarrow$ *(ii)* observe that if $V(x) = \|Px\|$ with $P \in \mathbb{R}^{l\times n}$ is a global Lyapunov function for system (5), then it induces [10] a family of 0-symmetric polytopic λ-contractive sets (with $\lambda = \rho$) for system (5), i.e.,

$$\{\mathcal{P}_{\Gamma}\}_{\Gamma\in\mathbb{R}_{>0}}, \quad \mathcal{P}_{\Gamma} := \{x \in \mathbb{R}^n \mid \|Px\| \leq \Gamma\}.$$

Furthermore, as shown in [7], a proper l-conic partition of $\mathbb{R}^n$ can then always be constructed using any of the sets $\mathcal{P}_{\Gamma}$. As such, let $\Gamma \in \mathbb{R}_{>0}$, consider a proper l-conic partition of $\mathbb{R}^n$ induced by $\mathcal{P}_{\Gamma}$ and take l arbitrary cones $\mathcal{C}_i$ that belong to this partition. Then, let $\{E_i\}_{i\in\mathcal{L}}$ be a set of matrices such that $\mathcal{C}_i^{\perp} = \{x \in \mathbb{R}^n \mid E_i x \geq 0\}$, $i \in \mathcal{L}$, are the corresponding dual cones. Using a similar reasoning as in [7], it follows that either $\|Px\| = -[P]_{i\bullet}x$ or $\|Px\| = [P]_{i\bullet}x$ for all $x \in \mathcal{C}_i$ and all $i \in \mathcal{L}$. Thus, from (3a) together with Fact III.5, Fact III.2, Fact III.3 and Fact III.4 one obtains that $[P]_{i\bullet}$ or $[-P]_{i\bullet}$, respectively, satisfies (7a) and (7b). This together with (3b), (4) and Fact III.4 further yields that (7c) holds for $[P]_{i\bullet}$ or $[-P]_{i\bullet}$, respectively, which completes the proof. ■

Several remarks about the complexity of testing the conditions of Theorem III.6 are in order. Assuming for simplicity that $E_i \in \mathbb{R}^{F\times n}$ for all $i \in \mathcal{L}$ and some $F \in \mathbb{Z}_{\geq n}$, for each $i \in \mathcal{L}$, condition (7a) yields $2nF$, condition (7b) yields $2(l-1)F$ and condition (7c) yields F *linear inequalities* in c and the elements of P, respectively. So, testing (7) for a fixed $l \in \mathbb{Z}_{\geq n}$ and $F \in \mathbb{Z}_{\geq n}$ amounts to solving a single linear program with $ln+1$ variables and $lF(2l+2n-1)$ inequalities. This is a tractable problem for $x \in \mathbb{R}^n$ with n reasonable large, as the number of inequalities and variables has a worst case cubic and square, respectively, dependency on the system dimension. Choosing the number l amounts to the classical problem of finding an upper bound on the number of rows of the matrix P. A solution to this problem can be found in [11]. Also, the results therein can be employed to choose the right number and position of ray directions that define each cone $\mathcal{C}_i$.

IV. Continuous-time nonlinear dynamics

A. Polytopic differential inclusions

Consider systems of the form

$$\dot{x}(t) \in \Phi(x(t)), \quad t \in \mathbb{R}_+, \tag{8}$$

where $\Phi : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$,

$$\Phi(x) := \{Ax \mid A \in \mathrm{Co}(\{A_w\}_{w \in \mathcal{W}})\},$$

$A_w \in \mathbb{R}^{n \times n}$ for all $w \in \mathcal{W} := \mathbb{Z}_{[1,W]}$, $W \in \mathbb{Z}_{\geq 1}$. Notice that finding an infinity norm Lyapunov function for system (8) is equivalent with solving the same problem for a continuous-time switched linear system under arbitrary switching.As shown in [3], the derivative of the function $V(x) = \|Px\|$ along the solutions of the differential inclusion (8) is obtained as

$$\mathscr{D}^+V(x) = \max_{w \in \mathcal{W}} \max_{i \in I(x)} \{[\tilde{P}]_{i\bullet} A_w x\}, \tag{9}$$

where $I(x) = \{i \in \mathbb{Z}_{[1,2l]} \mid [\tilde{P}]_{i\bullet} x = V(x)\}$.

Theorem IV.1 *Let $l \in \mathbb{Z}_{\geq n}$, $\mathcal{N} := \mathbb{Z}_{[1,n]}$ and $P \in \mathbb{R}^{l \times n}$. The following statements are equivalent.*

(i) The function $V(x) = ||Px||$ is a global Lyapunov function for system (5).

(ii) There exist $c \in \mathbb{R}_{>0}$, $\rho \in \mathbb{R}_{>0}$ and a proper l-conic partition of $\mathbb{R}^n$, i.e., $\{\mathcal{C}_i\}_{i \in \mathcal{L}}$, with the set of matching dual cones $\{\mathcal{C}_i^{\perp}\}_{i \in \mathcal{L}}$ and corresponding matrices $\{E_i\}_{i \in \mathcal{L}}$, such that the following inequalities hold for all $i \in \mathcal{L}$:

$$E_i([P]_{i\bullet} \pm c[I_n]_{j\bullet})^\top \geq 0, \quad \forall j \in \mathcal{N}, \tag{10a}$$

$$E_i([P]_{i\bullet} \pm [P]_{j\bullet})^\top \geq 0, \quad \forall j \in \mathcal{L} \setminus \{i\}, \tag{10b}$$

$$E_i(-\rho[P]_{i\bullet} - [P]_{i\bullet} A_w)^\top \geq 0, \quad \forall w \in \mathcal{W}. \tag{10c}$$

Proof: For the proof of *(ii)* $\Rightarrow$ *(i)* let us first notice that condition (10c) represents a set of affine inequalities in A_w. Thus, as $A \in \mathrm{Co}(\{A_w\}_{w \in \mathcal{W}})$, (10c) implies that

$$E_i(-\rho[P]_{i\bullet} - [P]_{i\bullet} A)^\top \geq 0, \quad \forall A \in \mathrm{Co}(\{A_w\}_{w \in \mathcal{W}}).$$

With this result, the remainder of the proof of *(ii)* $\Rightarrow$ *(i)* simply follows the corresponding part of the proof of Theorem III.6. For the converse part of the proof, suppose that V is a global common Lyapunov function for system (8). Then, a proper l-conic partition of $\mathbb{R}^n$ can be constructed as indicated in the proof of Theorem III.6, which yields that (10a) and (10b) hold. Then, using (3b), (9) and Fact III.4 yields that either $[P]_{i\bullet}$ or $[-P]_{i\bullet}$ is a feasible solution to (10c) for all $w \in \mathcal{W}$ and for all $i \in \mathcal{L}$, which completes the proof. ■

The LP that corresponds to checking feasibility of (10) has $ln+1$ variables and $lF(2(l-1)+2n+W)$ inequalities.

Example 1 [9]: Consider a switched linear system under arbitrary switching with $A_1 = \begin{pmatrix} 0.3 & 0.7 \\ -2.3 & -2.3 \end{pmatrix}$, $A_2 = \begin{pmatrix} -1.8 & 1.0 \\ -0.8 & 0.1 \end{pmatrix}$. The conic partition was generated for $l = 13$, $F = 2$. The feasibility of the resulting LP with 5 variables and 781 constraints was tested using 3 different solvers (i.e., *linprog*, *SeDuMi* and *CDD*). The constraint $c \geq 1$ was imposed and the feasible solution $c = 2.2373$ was obtained for $\rho = 0.001$. In [9] this example was proposed as a test of conservativeness in terms of the complexity of the resulting polyhedral sublevel set. The simplest polyhedral set obtained in [9] has 28 vertices. Applying the approach proposed in this section, a polyhedral set with 26 vertices was obtained. The polytope $\mathcal{P}_1$ is plotted in yellow in Figure 1 and Figure 2, where trajectories obtained for all the vertices of $\mathcal{P}_1$ as initial conditions and the "extreme" dynamics $\dot{x} = A_1 x$ and $\dot{x} = A_2 x$, respectively, are also plotted.

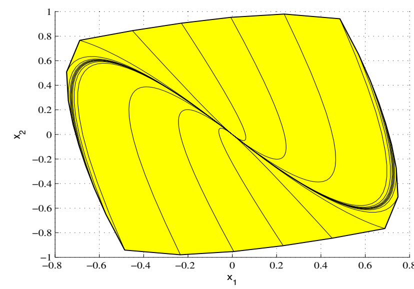

Fig. 1. Simulation results (A_1 dynamics) - Example 1.

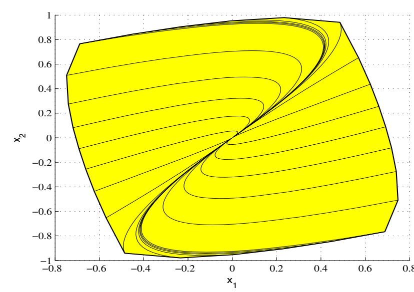

Fig. 2. Simulation results (A_2 dynamics) - Example 1.

B. Quadratic nonlinear systems

Consider systems of the form

$$\dot{x}(t) = \Phi(x(t)), \quad t \in \mathbb{R}_+, \tag{11}$$

where $\Phi : \mathbb{R}^n \to \mathbb{R}^n$, $\Phi(x) := \bar{A}(x)x$ and

$$\bar{A}(x) := A + \begin{pmatrix} B_1^\top x & \dots & B_n^\top x \end{pmatrix}^\top,$$

$A, B_i \in \mathbb{R}^{n \times n}$ for all $i \in \mathcal{N} = \mathbb{Z}_{[1,n]}$. Bilinear differential equations have numerous relevant real-life applications, as they arise naturally in models of power electronic circuits and systems biology reactions. The solution for bilinear dynamics presented in [12], which corresponds to the classical approach of [3], employs an iterative algorithm that requires solving a min-max non-convex optimization problem subject to a bilinear constraint. Next, the approach of Theorem III.6 is applied to bilinear dynamics, which yields a single LP.

Theorem IV.2 *Let $l \in \mathbb{Z}_{\geq n}$, $P \in \mathbb{R}^{l \times n}$, $F \in \mathbb{R}_{\geq n}$ and $\mathcal{F} := \mathbb{Z}_{[1,F]}$. Suppose that there exists a set of points*

$\{\{x_i^e\}_{e\in\mathcal{F}}\}_{i\in\mathcal{L}}$ *with* $x_i^e \in \mathbb{R}^n$, $x_i^e \neq 0$ *for all* $(i,e) \in \mathcal{L}\times\mathcal{F}$ *that induces a proper* l*-conic partition of* $\mathbb{R}^n$*, i.e.,* $\{\mathcal{C}_i\}_{i\in\mathcal{L}}$ *with* $\mathcal{C}_i := \mathrm{Co}(\{r(x_i^1),\ldots,r(x_i^F)\})$ *for all* $i \in \mathcal{L}$*. Furthermore, let* $\{\mathcal{C}_i^{\perp}\}_{i\in\mathcal{L}}$ *be the set of matching dual cones with corresponding matrices* $\{E_i\}_{i\in\mathcal{L}}$ *and suppose that there exist* $c \in \mathbb{R}_{>0}$ *and* $\rho \in \mathbb{R}_{>0}$ *such that the following inequalities hold* for all $i \in \mathcal{L}$*:*

$$E_i([P]_{i\bullet} \pm c[I_n]_{j\bullet})^\top \geq 0, \quad \forall j \in \mathcal{N}, \tag{12a}$$

$$E_i([P]_{i\bullet} \pm [P]_{j\bullet})^\top \geq 0, \quad \forall j \in \mathcal{L}\setminus\{i\}, \tag{12b}$$

$$\begin{aligned}(-\rho[P]_{i\bullet} - [P]_{i\bullet}\bar{A}(0))x_i^{e_2} \geq 0, &\quad \forall e_2 \in \mathcal{F},\\ \big(-\rho[P]_{i\bullet} - [P]_{i\bullet}\bar{A}(x_i^{e_1})\big)x_i^{e_2} \geq 0, &\quad \forall (e_1,e_2) \in \mathcal{F}\times\mathcal{F}.\end{aligned} \tag{12c}$$

Let $\mathcal{V} := \mathrm{Co}\left(\{\{x_i^e\}_{e\in\mathcal{F}}\}_{i\in\mathcal{L}}\right)$, $\mathcal{P}_\lambda := \{x \in \mathbb{R}^n \mid \|Px\| \leq \lambda\}$ *and let* $\lambda^* := \sup\{\lambda \in \mathbb{R}_{>0} \mid \mathcal{P}_\lambda \subseteq \mathcal{V}\}$.

Then, the function $V(x) = ||Px||$ *is a Lyapunov function in* $\mathcal{P}_{\lambda^*}$ *for system* (11).

Proof: Observing that $x \in \mathcal{C}_i \cap \mathcal{V}$ implies $x \in \mathrm{Co}(\{0,\{x_i^e\}_{e\in\mathcal{F}}\})$ and as $\bar{A}(x)$ is an affine function of x, it follows that for any fixed $e_2 \in \mathcal{F}$, (12c) implies that

$$(-\rho[P]_{i\bullet} - [P]_{i\bullet}\bar{A}(x))x_i^{e_2} \geq 0, \quad \forall x \in \mathcal{C}_i \cap \mathcal{V}. \tag{13}$$

Then, as by (12c) we also have that (13) holds for all $e_2 \in \mathcal{F}$, yields that

$$(-\rho[P]_{i\bullet} - [P]_{i\bullet}\bar{A}(x))x \geq 0, \quad \forall x \in \mathcal{C}_i \cap \mathcal{V}.$$

From the above inequality, using (4) and applying the same reasoning as in the proof of Theorem III.6, it can be shown that (3b) holds for all $x \in \mathcal{C}_i \cap \mathcal{V}$. Observing that $\mathcal{P}_{\lambda^*} \subseteq \mathcal{V}$ is a PI set for system (11) and $\{\mathcal{C}_i\}_{i\in\mathcal{L}}$ is a proper l-conic partition of $\mathbb{R}^n$ yields the desired result. ■

The conditions of Theorem IV.2 yield a local infinity norm Lyapunov function for (11) and lead to a LP with $ln+1$ variables and $lF(2l+2n+F-1)$ inequalities. Note that the set $\mathcal{P}_{\lambda^*}$ can be enlarged by enlarging the set $\mathcal{V}$, as long as the corresponding LP remains feasible.

Example 2 [12]: Consider the nonlinear quadratic system (11) with $A = \begin{pmatrix} -50 & -16 \\ 13 & -9 \end{pmatrix}$, $B_1 = \begin{pmatrix} 0 & 6.9 \\ 6.9 & 0 \end{pmatrix}$ and $B_2 = \begin{pmatrix} 0 & 2.75 \\ 2.75 & 0 \end{pmatrix}$. In [12], a polyhedral Lyapunov function was computed for this system using nonlinear programming, which resulted in a polyhedral invariant set with 20 vertices. Additionally, therein it was established that the box $\mathcal{B} := \mathbb{R}_{[-1.2,1.2]} \times \mathbb{R}_{[-2.8,2.8]}$ is a subset of the domain of attraction. Using the approach of Theorem IV.2, a conic partition was generated for $l = 17$ and $F = 2$. The feasibility of the resulting LP with 5 variables and 1327 constraints was tested using 3 different solvers (i.e., *linprog*, *SeDuMi* and *CDD*). The constraint $c \geq 1$ was imposed and the feasible solution $c = 1$ was obtained for $\rho = 0.3$. In Figure 3, the cyan polytope denotes the set $\mathcal{V}$ obtained as the convex hull of the points used to generate the conic partition. The polytopic domain of attraction $\mathcal{P}_3$, which has only 12 vertices, is plotted in Figure 3 in yellow, together with the system trajectories obtained for its vertices as initial conditions. In the same figure, the box $\mathcal{B}$ is plotted with a dotted line.

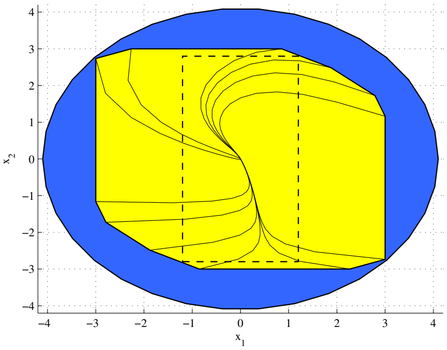

Fig. 3. Simulation results - Example 2.

The polyhedral domain of attraction $\mathcal{P}_3$ corresponding to Theorem IV.2 also establishes that $\mathcal{B}$ is a subset of the region of attraction.

V. Non-symmetric polyhedral Lyapunov functions

Non-symmetric polyhedral Lyapunov functions are particularly of relevance for real-life applications. Even in the case of asymptotically stable linear dynamics, where existence of symmetric polyhedral LFs, such as infinity norm ones, is a necessary condition, often hard constraints on the states are specified by non-symmetric polyhedra. In such a situation, symmetric domains of attraction offer an overly conservative solution. This motivates the interest for the construction of non-symmetric domains of attraction.

One of the earliest results in this direction can be found in [13]. Therein, it was shown how symmetric norm Lyapunov functions (i.e., defined by a 1-, 2- or ∞-norm) can be modified to yield non-symmetric domains of attraction for linear dynamics. More recent results on non-symmetric norm LFs can be found in [14], where a generalization of several previous contributions was attained. For a complete overview, the interested reader is referred to the monograph [1] and the references in [14].

In what follows it is indicated how the approach proposed in this paper can be extended to non-symmetric polyhedral LFs. To this end, the following correspondent of a proper l-conic partition of $\mathbb{R}^n$ is defined.

Definition V.1 Let $l \in \mathbb{Z}_{\geq n+1}$ and let $\mathcal{L} := \mathbb{Z}_{[1,l]}$. A finite set of cones $\{\mathcal{C}_i\}_{i\in\mathcal{L}}$ is called *a non-symmetric proper* l*-conic partition of* $\mathbb{R}^n$ if $\cup_{i\in\mathcal{L}}\mathcal{C}_i = \mathbb{R}^n$, $\mathcal{C}_i$ is a proper cone for all $i \in \mathcal{L}$ and $\mathrm{int}(\mathcal{C}_i) \cap \mathrm{int}(\mathcal{C}_j) = \emptyset$ for all $(i,j) \in \mathcal{L}\times\mathcal{L}$ with $i \neq j$.

The corresponding main results for polytopic differential inclusions and nonlinear quadratic systems are stated next.

Theorem V.2 *Let* $l \in \mathbb{Z}_{\geq n+1}$, $\mathcal{L} := \mathbb{Z}_{[1,l]}$, $\mathcal{N} := \mathbb{Z}_{[1,n]}$ *and* $P \in \mathbb{R}^{l\times n}$*. The following statements are equivalent.*

(i) The function $V(x) = \max_{j\in\mathcal{L}}\{[P]_{j\bullet}x\}$ *is a global Lyapunov function for system* (5).

(ii) There exist $c \in \mathbb{R}_{>0}$, $\rho \in \mathbb{R}_{>0}$ and a non-symmetric proper l-conic partition of $\mathbb{R}^n$, i.e., $\{\mathcal{C}_i\}_{i \in \mathcal{L}}$, with the set of matching dual cones $\{\mathcal{C}_i^{\perp}\}_{i \in \mathcal{L}}$ and corresponding matrices $\{E_i\}_{i \in \mathcal{L}}$, such that the following inequalities hold for all $i \in \mathcal{L}$:

$$E_i([P]_{i\bullet} \pm c[I_n]_{j\bullet})^\top \geq 0, \quad \forall j \in \mathcal{N}, \tag{14a}$$
$$E_i([P]_{i\bullet} - [P]_{j\bullet})^\top \geq 0, \quad \forall j \in \mathcal{L} \setminus \{i\}, \tag{14b}$$
$$E_i(-\rho[P]_{i\bullet} - [P]_{i\bullet} A_w)^\top \geq 0, \quad \forall w \in \mathcal{W}. \tag{14c}$$

The proof is omitted for brevity. The sufficiency part obviously follows using the same reasoning as for the symmetric case. The necessity part is obtained by observing that an asymptotically stable polytopic differential inclusion always admits a symmetric polyhedral LF, i.e., $\tilde{V}(x) := \|\tilde{P}x\|$, which in turn yields a feasible solution to the inequalities (14) of the form $P := \begin{pmatrix}\tilde{P} & -\tilde{P}\end{pmatrix}^\top$.

Theorem V.3 *Let $l \in \mathbb{Z}_{\geq n+1}$, $P \in \mathbb{R}^{l \times n}$, $F \in \mathbb{R}_{\geq n}$ and $\mathcal{F} := \mathbb{Z}_{[1,F]}$. Suppose that there exists a set of points $\{\{x_i^e\}_{e \in \mathcal{F}}\}_{i \in \mathcal{L}}$ with $x_i^e \in \mathbb{R}^n$, $x_i^e \neq 0$ for all $(i,e) \in \mathcal{L} \times \mathcal{F}$ that induces a non-symmetric proper l-conic partition of $\mathbb{R}^n$, i.e., $\{\mathcal{C}_i\}_{i \in \mathcal{L}}$ with $\mathcal{C}_i := \mathrm{Co}(\{r(x_i^1), \ldots, r(x_i^F)\})$ for all $i \in \mathcal{L}$. Furthermore, let $\{\mathcal{C}_i^{\perp}\}_{i \in \mathcal{L}}$ be the set of matching dual cones with corresponding matrices $\{E_i\}_{i \in \mathcal{L}}$ and suppose that there exist $c \in \mathbb{R}_{>0}$ and $\rho \in \mathbb{R}_{>0}$ such that the following inequalities hold* for all $i \in \mathcal{L}$:

$$E_i([P]_{i\bullet} \pm c[I_n]_{j\bullet})^\top \geq 0, \quad \forall j \in \mathcal{N}, \tag{15a}$$
$$E_i([P]_{i\bullet} - [P]_{j\bullet})^\top \geq 0, \quad \forall j \in \mathcal{L} \setminus \{i\}, \tag{15b}$$
$$\begin{aligned} (-\rho[P]_{i\bullet} - [P]_{i\bullet}\bar{A}(0))x_i^{e_2} \geq 0, \quad \forall e_2 \in \mathcal{F}, \\ \big(-\rho[P]_{i\bullet} - [P]_{i\bullet}\bar{A}(x_i^{e_1})\big)x_i^{e_2} \geq 0, \quad \forall (e_1, e_2) \in \mathcal{F} \times \mathcal{F}. \end{aligned} \tag{15c}$$

Let $\mathcal{V} := \mathrm{Co}\,(\{\{x_i^e\}_{e \in \mathcal{F}}\}_{i \in \mathcal{L}})$,

$$\mathcal{P}_\lambda := \{x \in \mathbb{R}^n \mid \max_{j \in \mathcal{L}}\{[P]_{j\bullet}x\} \leq \lambda\}$$

and let $\lambda^* := \sup\{\lambda \in \mathbb{R}_{>0} \mid \mathcal{P}_\lambda \subseteq \mathcal{V}\}$.

Then, the function $V(x) = \max_{j \in \mathcal{L}}\{[P]_{j\bullet}x\}$ is a Lyapunov function in $\mathcal{P}_{\lambda^*}$ for system (11).

The proof is omitted for brevity and it follows similarly to the symmetric case. In fact, the only difference in the non-symmetric case comes for the non-symmetric proper conic partition, which in turn only affects conditions in (14b) and (15b), respectively. What remains of interest for further research, is the maximization of the domain of attraction under non-symmetric polytopic constraints.

VI. Conclusions

This paper considered the synthesis of infinity norm Lyapunov functions for continuous-time dynamical systems. A proper conic partition of the state-space was employed to construct a finite set of linear inequalities in the elements of the Lyapunov weight matrix. For dynamics described by linear and polytopic differential inclusions, it was proven that the feasibility of the derived set of linear inequalities is necessary and sufficient for the existence of an infinity norm Lyapunov function. Furthermore, it was shown that the developed solution naturally applies to relevant classes of continuous-time nonlinear systems. An extension to non-symmetric polyhedral Lyapunov functions was also presented. Ongoing research deals with the synthesis of polyhedral control Lyapunov functions in conjunction with piecewise affine control laws.

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