Region of Attraction Comparison for Gradient Projection Anti-windup Compensated Systems

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Abstract— The *gradient projection anti-windup* (GPAW) scheme was recently proposed for saturated multi-input-multi-output (MIMO) *nonlinear* systems driven by MIMO nonlinear controllers, a topic recognized as an open problem in a recent survey paper. Thus far, stability results for GPAW compensated systems are restricted to the simple case of a saturated first order linear time invariant (LTI) plant driven by a first order LTI controller. Here, we present a region of attraction (ROA) comparison result for general GPAW compensated regulatory systems. The ROA comparison result is demonstrated on a simple planar nonlinear system, which also highlights the limitations of existing state-of-the-art anti-windup results.

I. INTRODUCTION

The *gradient projection anti-windup* (GPAW) scheme was proposed in [1] for saturated multi-input-multi-output (MIMO) *nonlinear* systems driven by MIMO nonlinear controllers, a topic recognized as an open problem in the recent survey paper [2]. The GPAW scheme is a generalization of the well-known *conditional integration* method [3, p. 38], [4] to the case of MIMO nonlinear controllers [5, p. 52]. The construction of the GPAW compensated controller is detailed in [5, Chapter 2, pp. 33 – 70] along with some of its fundamental properties. It was shown in [6], [5, Theorem 2.5.3, p. 50] that the GPAW scheme achieves *controller state-output consistency*, a property unique to the GPAW scheme while being an implicit objective for the majority of anti-windup schemes [5, Remark 2.21, pp. 51 – 52].

The *general* anti-windup problem for nonlinear systems is presented in [5, Section 1.3, pp. 23 – 26]. Briefly, it states that given some nominal controller designed for the *unconstrained* system, nominal performance is recovered when no controls saturate, and that stability and performance of the anti-windup compensated system must be no worse than those of the saturated *uncompensated* system.[1] One crucial difference with the problem considered in current anti-windup literature is that results *relative* to the uncompensated system are sought. More specifically, results which show that the stability and performance of the uncompensated system can only be maintained/improved with anti-windup compensation. While such requirements appear fundamental, it has not been specified in present anti-windup literature to the best of our knowledge. These requirements are further

motivated in [5, Section 3.7, pp. 94 – 96], where the superiority of such *relative* results are made apparent.

In this new framework, strong stability results have been obtained when restricted to the case of a saturated first order linear time invariant (LTI) plant driven by a first order LTI controller. In particular, the region of attraction (ROA) of the uncompensated system can only be maintained/enlarged by GPAW compensation [5, Proposition 3.5.13, p. 90], [7], [8]. In this paper, we present an ROA comparison result for MIMO nonlinear systems/controllers, which states conditions to ensure that the ROA (or ROA estimate) of the uncompensated system is never reduced by GPAW compensation. We construct a simple nonlinear planar system to demonstrate the application of this ROA comparison result. This same system is shown to elude current state-of-the-art anti-windup methods for LTI systems [2], [9], [10]; feedback linearizable nonlinear systems[2] [11]; and Euler-Lagrange systems [12].

The rest of the paper is organized as follows. A general ROA comparison result is presented in Section II, followed by the main result in Section III. The example presented in Section IV is significant as it highlights the limitations of current state-of-the-art anti-windup results. The Appendix which follows the Conclusions summarizes the construction of the GPAW compensated controller.

We will adopt the following conventions. For any integer $i > 0$, we define $\mathcal{I}_i := \{1, 2, \ldots, i\}$. Inequalities involving vectors are to be interpreted element-wise. The symbol $\wedge$ denotes logical conjunction.

II. A GENERAL ROA COMPARISON RESULT

Here, we present a general result that allows the size of the ROA of an autonomous system to be inferred from that of a related autonomous system. First, we recall some basic definitions. Consider the n-th order autonomous system

$$\dot{x} = f(x), \qquad (1)$$

where for some domain $D \subset \mathbb{R}^n$, we assume the function $f \colon D \to \mathbb{R}^n$ satisfies some regularity conditions to ensure existence and uniqueness of solutions for all *forward* times. When f is continuous, classical results based on Lipschitz continuity of f, e.g. [13, Section 3.1, pp. 88 – 95], are available to ensure existence and uniqueness of solutions for all (forward and reverse) times. If f is *discontinuous*,[3]

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[1]For regulatory systems, this requires the region of attraction (ROA) of the uncompensated system to be *contained within* the ROA of the anti-windup compensated system.

[2]The majority of anti-windup schemes for nonlinear systems are constructed for feedback linearizable systems. See [5, Section 1.4.2, p. 27] for a more comprehensive list of references.

[3]The vector field of GPAW compensated systems will be discontinuous in general (see the switching conditions in (19) and [5, p. 76]).

regularity conditions are available to ensure existence and uniqueness of solutions for all *forward* times [14, §2.7, pp. 75 – 86, and §2.10, pp. 106 – 117]. Let $\phi(t, x_0)$ be the unique solution of system (1) starting from x_0 at time $t = 0$. A set M is a *positively invariant set* for system (1) if the solution starting within M stays in M for all *forward* times, i.e. $x_0 \in M \Rightarrow (\phi(t, x_0) \in M, \forall t \geq 0)$ [13, p. 127].

We assume that $x_{eq} \in D$ is an asymptotically stable equilibrium point for system (1). The ROA of the equilibrium x_{eq} for system (1) is defined by [13, p. 314]

$$R_A(x_{eq}) := \{\bar{x} \in \mathbb{R}^n \mid \phi(t, \bar{x}) \to x_{eq} \text{ as } t \to \infty\}.$$

In other words, the ROA is the set of all points such that the solution starting from any point within it converges to the equilibrium. Clearly, $R_A(x_{eq})$ must contain the equilibrium x_{eq}. A set $\Omega \subset R_A(x_{eq})$ is said to be an *estimate* of $R_A(x_{eq})$ if every solution starting in Ω approaches x_{eq} as $t \to \infty$ [13, p. 316], i.e. $x_0 \in \Omega \Rightarrow \lim_{t \to \infty} \phi(t, x_0) = x_{eq}$. Clearly, a set is an estimate of $R_A(x_{eq})$ if and only if it is a subset of $R_A(x_{eq})$ (including the *subset* $R_A(x_{eq})$). Numerous methods are available to estimate ROAs, e.g. [15], [16]. Our purpose is *not* to estimate ROAs, but, *given* an ROA estimate for a system, to find conditions for which the same ROA estimate is valid for a related system. Recall that a continuous function $\alpha \colon [0, a) \to [0, \infty)$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$ [13, Definition 4.2, p. 144].

An ROA estimate $\Omega \subset R_A(x_{eq})$ is said to be associated with a Lyapunov function $V \colon \Omega \to \mathbb{R}$ for system (1) if

$$V(x_{eq}) = 0, \qquad V(\bar{x}) > 0, \qquad \forall \bar{x} \in \Omega \setminus \{x_{eq}\},$$
$$\dot{V}(\bar{x}) = \frac{\partial V(\bar{x})}{\partial x} f(\bar{x}) \leq -\alpha(\|\bar{x} - x_{eq}\|), \qquad \forall \bar{x} \in \Omega, \tag{2}$$

for some class $\mathcal{K}$ function $\alpha \colon [0, a) \to [0, \infty)$, where $a := \sup_{\bar{x} \in \Omega} \|\bar{x} - x_{eq}\|$. In this case, we indicate the association by denoting such an estimate by Ω_V. Note that numerous ROA estimation methods yield estimates that are associated with Lyapunov functions.

The next result gives sufficient conditions for an ROA estimate of a system associated with some Lyapunov function to be a valid ROA estimate for a related system, i.e. the estimate is contained within the ROA of this related system.

Lemma 1 ([5, Lemma 4.3.1, p. 122]): Consider two n-th order autonomous systems

$$\dot{x} = f_1(x), \tag{3}$$
$$\dot{x} = f_2(x). \tag{4}$$

For some $D \subset \mathbb{R}^n$, assume $f_1 \colon D \to \mathbb{R}^n$ and $f_2 \colon D \to \mathbb{R}^n$ are such that solutions $\phi_1(t, x_0)$ and $\phi_2(t, x_0)$ to systems (3) and (4) respectively exist and are unique for all $x_0 \in D$ and all $t \geq 0$. Assume $x_{eq} \in D$ is an asymptotically stable equilibrium point for both systems, and let $R_{A1}(x_{eq})$, $R_{A2}(x_{eq})$ be the ROAs of systems (3) and (4) respectively. Let $\Omega_V \subset R_{A1}(x_{eq})$ be an estimate of $R_{A1}(x_{eq})$ associated with a Lyapunov function $V \colon \Omega_V \to \mathbb{R}$ satisfying (2) (with f_1, Ω_V in place of f, Ω). If Ω_2 is a subset of Ω_V (possibly $\Omega_2 = \Omega_V$) and also a positively invariant set for system (4), and in addition,

$$\frac{\partial V(\bar{x})}{\partial x} f_2(\bar{x}) \leq \frac{\partial V(\bar{x})}{\partial x} f_1(\bar{x}), \qquad \forall \bar{x} \in \Omega_2, \tag{5}$$

then Ω_2 is an estimate of $R_{A2}(x_{eq})$, i.e. $\Omega_2 \subset R_{A2}(x_{eq})$.

Proof: The proof has some similarities with the proof of [17, Lemma 1]. We need to show that $\lim_{t \to \infty} \phi_2(t, x_0) = x_{eq}$ for all $x_0 \in \Omega_2$. Since x_{eq} is an asymptotically stable equilibrium for system (4), $R_{A2}(x_{eq})$ is non-empty and necessarily contains x_{eq}, i.e. $x_{eq} \in R_{A2}(x_{eq}) \neq \emptyset$. Since $R_{A2}(x_{eq})$ is open [13, Lemma 8.1, p. 314], it contains some sufficiently small neighborhood of x_{eq}. Hence there exists a sufficiently small $\epsilon > 0$ such that $B_\epsilon := \{\bar{x} \in \mathbb{R}^n \mid V(\bar{x}) \leq \epsilon\} \subset R_{A2}(x_{eq})$. Fix any such $\epsilon > 0$.

From $\Omega_2 \subset \Omega_V$, (5), and (2), we have for all $\bar{x} \in \Omega_2$,

$$\frac{\partial V(\bar{x})}{\partial x} f_2(\bar{x}) \leq \frac{\partial V(\bar{x})}{\partial x} f_1(\bar{x}) \leq -\alpha(\|\bar{x} - x_{eq}\|).$$

Let ∂B_ϵ be the boundary of B_ϵ, i.e. $\partial B_\epsilon = \{\bar{x} \in \mathbb{R}^n \mid V(\bar{x}) = \epsilon\}$, $\beta := \min_{\bar{x} \in \partial B_\epsilon} \|\bar{x} - x_{eq}\|$, and $\delta := \alpha(\beta)$. Clearly, we have $\beta > 0$ and $\delta > 0$ due to $\epsilon > 0$ and (2). Then for all $\bar{x} \in \Omega_2 \setminus B_\epsilon$, we have $\|\bar{x} - x_{eq}\| \geq \beta$, $\alpha(\|\bar{x} - x_{eq}\|) \geq \alpha(\beta) = \delta$, and

$$\frac{\partial V(\bar{x})}{\partial x} f_2(\bar{x}) \leq -\alpha(\|\bar{x} - x_{eq}\|) \leq -\delta. \tag{6}$$

Let $x_0 \in \Omega_2$. Since Ω_2 is a positively invariant set for system (4), we have $\phi_2(t, x_0) \in \Omega_2$ for all $t \geq 0$. The Lyapunov function V evaluated along $\phi_2(t, x_0)$ is given by

$$V(\phi_2(t, x_0)) = V(x_0) + \int_0^t \frac{\partial V(\phi_2(\tau, x_0))}{\partial x} f_2(\phi_2(\tau, x_0)) \, d\tau,$$

for all $t \geq 0$. Clearly, $\phi_2(t, x_0)$ can remain in $\Omega_2 \setminus B_\epsilon$ only for a finite amount of time. Otherwise, $\phi_2(t, x_0) \in \Omega_2 \setminus B_\epsilon$ for all $t \geq 0$ and (6) imply

$$V(\phi_2(t, x_0)) \leq V(x_0) - \delta \int_0^t d\tau = V(x_0) - \delta t, \qquad \forall t \geq 0,$$

which shows that $V(\phi_2(t, x_0)) < 0$ for sufficiently large t, a contradiction to V being positive definite. Hence $\phi_2(t, x_0)$ must enter B_ϵ at some finite time, and must approach x_{eq} since $B_\epsilon \subset R_{A2}(x_{eq})$, i.e. $\lim_{t \to \infty} \phi_2(t, x_0) = x_{eq}$. ∎

Remark 1: Note that the ROA estimate Ω_V in Lemma 1 need not be compact. Moreover, the conclusion $\Omega_2 \subset R_{A2}(x_{eq})$ shows that the (possibly unknown) ROA $R_{A2}(x_{eq})$ is at least as large as Ω_2. □

Lemma 1 will be specialized in the next section to GPAW compensated systems.

III. ROA COMPARISON FOR GPAW COMPENSATED SYSTEMS

For any well-posed anti-windup problem, the input-constrained nominal system must be stable in some sense at least locally, e.g. locally asymptotically stable with respect to some equilibrium. When it is indeed locally asymptotically stable with respect to some equilibrium, the fundamental question is whether an anti-windup scheme can

maintain/enlarge the ROA while achieving performance enhancements in the presence of control saturation.

Here, we present the main result which provides conditions under which the ROA of the *regulatory* GPAW compensated system is "lower bounded" by some ROA estimate of the uncompensated system. First, we describe the closed-loop systems. The open-loop plant is described by

$$\dot{x} = f(x, \text{sat}(u)), \qquad x(0) = x_0,$$
$$y = g(x, \text{sat}(u)), \tag{7}$$

where $x, x_0 \in \mathbb{R}^n$ are its state and initial state, $y \in \mathbb{R}^p$ is the measurement, $u = [u_1, u_2, \ldots, u_m] \in \mathbb{R}^m$ is the plant input, and the saturation function $\text{sat} \colon \mathbb{R}^m \to \mathbb{R}^m$ is defined by

$$\text{sat}(u) = [\rho_1(u_1), \rho_2(u_2), \ldots, \rho_m(u_m)]^{\text{T}},$$
$$\rho_i(u_i) = \max\{\min\{u_i, u_{\text{max},i}\}, u_{\text{min},i}\}, \qquad \forall i \in \mathcal{I}_m,$$

for some $u_{\text{max},i}, u_{\text{min},i} \in \mathbb{R}$ satisfying $u_{\text{min},i} < u_{\text{max},i}$ for all $i \in \mathcal{I}_m$. The nominal controller is

$$\dot{x}_c = f_c(x_c, y, r), \qquad x_c(0) = x_{c0},$$
$$u_c = g_c(x_c), \tag{8}$$

where $x_c, x_{c0} \in \mathbb{R}^q$ are its state and initial state, $u_c \in \mathbb{R}^m$ is the controller output, and $r := r(t) \in \mathbb{R}^{n_r}$ is the instantaneous value of some exogenous input.

Making appropriate substitutions in (7) and (8), the nominal uncompensated system with $u = u_c$ can be written as

$$\Sigma_n \colon \begin{cases} \dot{x} = f(x, \text{sat}(g_c(x_c))), \\ \dot{x}_c = f_c(x_c, g(x, \text{sat}(g_c(x_c))), r), \end{cases} \tag{9}$$

or $\Sigma_n \colon \dot{z}_n = f_n(z_n)$. The GPAW compensated system comprising the plant (7) and GPAW compensated controller (see (21) in Appendix) with $u = u_g$ is described by

$$\Sigma_g \colon \begin{cases} \dot{x} = f(x, \text{sat}(g_c(x_g))), \\ \dot{x}_g = R_{\mathcal{I}^*} f_c(x_g, g(x, \text{sat}(g_c(x_g))), r), \end{cases} \tag{10}$$

where $R_{\mathcal{I}^*} f_c(x_g, y, r) := R_{\mathcal{I}^*}(x_g, y, r) f_c(x_g, y, r)$. Similarly, this can be written as $\Sigma_g \colon \dot{z}_g = f_g(z_g)$. In (9), $f_n \colon \mathbb{R}^{n+q} \to \mathbb{R}^{n+q}$ is the vector field of the nominal system with state $z_n := (x, x_c)$, while in (10), $f_g \colon \mathbb{R}^{n+q} \to \mathbb{R}^{n+q}$ is the vector field of the GPAW compensated system with state $z_g := (x, x_g)$.

Controller state-output consistency is a fundamental property of GPAW compensated controllers, stated as Theorem 3 in the Appendix. It implies that the unsaturated region

$$\mathbb{R}^n \times K, \qquad K := \{\bar{x} \in \mathbb{R}^q \mid \text{sat}(g_c(\bar{x})) = g_c(\bar{x})\}, \tag{11}$$

is a positively invariant set for the GPAW compensated system Σ_g [5, Remark 2.18, p. 51]. When the controller state is initialized such that $x_g(0) \in K$, Theorem 3 shows that the GPAW compensated system (10) can be written as

$$\Sigma_{gu} \colon \begin{cases} \dot{x} = f(x, g_c(x_g)), \\ \dot{x}_g = R_{\mathcal{I}^*} f_c(x_g, g(x, g_c(x_g)), r), \end{cases} \tag{12}$$

or $\Sigma_{gu} \colon \dot{z}_g = f_{gu}(z_g)$, where the saturation function $\text{sat}(\cdot)$ has been eliminated. For this system, comparison would be made against the *unconstrained* system

$$\Sigma_u \colon \begin{cases} \dot{x} = f(x, g_c(x_c)), \\ \dot{x}_c = f_c(x_c, g(x, g_c(x_c)), r), \end{cases} \tag{13}$$

or $\Sigma_u \colon \dot{z}_u = f_u(z_u)$ with state $z_u := (x, x_c)$.

We assume that solutions of the nominal system Σ_n and unconstrained system Σ_u exist and are unique for all initial conditions and all times. This is a mild assumption as existence and uniqueness of solutions to the nominal and unconstrained systems are usually guaranteed in the design of the nominal controller even when not explicitly sought. For the GPAW compensated systems Σ_g and Σ_{gu}, we assume that their solutions exist and are unique for all *forward* times when started within the unsaturated region[4] $\mathbb{R}^n \times K$. We restrict its existence to *forward* times since in general, the vector field of the GPAW compensated system will be *discontinuous* at least on the saturation constraint boundary $\mathbb{R}^n \times \partial K$ (see the switching conditions in (19) and [5, p. 76]). For brevity, this assumption on existence and uniqueness of solutions will not be repeated in the sequel.

The preceding systems (9), (10), (12), and (13), are regulatory (in contrast to tracking systems) when the controller reference r is constant.[5] We are interested in asymptotic stability, for which it is meaningful only if the constant r induces some *isolated* equilibria. Which of the (possibly multiple) isolated equilibria is to be taken as reference should be apparent for the particular application. We assume that r induces an isolated equilibrium point $z_{eq} := (\tilde{x}, \tilde{x}_c) \in \mathbb{R}^{n+q}$ for the nominal system (9) satisfying

$$f(\tilde{x}, g_c(\tilde{x}_c)) = 0,$$
$$f_c(\tilde{x}_c, g(\tilde{x}, g_c(\tilde{x}_c)), r) = 0, \tag{14}$$
$$u_{\text{min}} < g_c(\tilde{x}_c) < u_{\text{max}},$$

where $u_{\text{max}} := [u_{\text{max},1}, \ldots, u_{\text{max},m}]^{\text{T}}$ and $u_{\text{min}} := [u_{\text{min},1}, \ldots, u_{\text{min},m}]^{\text{T}}$ are the vectors of saturation limits. Notice that the last condition of (14) implies that the equilibrium z_{eq} lies in the *interior* of the unsaturated region $\mathbb{R}^n \times K$. It is clear that any isolated equilibrium within the *interior* of the unsaturated region for system (9) must also be an isolated equilibrium for systems (10), (12), and (13), since their vector fields coincide in $\mathbb{R}^n \times (K \setminus \partial K)$, i.e.

$$f_n(\bar{z}) = f_g(\bar{z}) = f_{gu}(\bar{z}) = f_u(\bar{z}), \qquad \forall \bar{z} \in \mathbb{R}^n \times (K \setminus \partial K).$$

For this r, we assume that z_{eq} is a locally asymptotically stable equilibrium for the nominal system (9), which is *a standard assumption in the anti-windup setting*. The preceding relation and (14) implies that it is also a locally asymptotically stable equilibrium for systems (10), (12), and (13).

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[4] The restriction to initial conditions within the unsaturated region $\mathbb{R}^n \times K$ is for simplicity, since this can usually be enforced.

[5] When r is a constant, it can be taken as part of the description of the function f_c in (9), (10), (12), and (13). Then we can write f_n, f_g, f_{gu}, and f_u, in (9), (10), (12), and (13) respectively, not as a function of r. These result in autonomous system descriptions.

Let the ROAs of z_{eq} for systems (9), (10), (12), and (13) be $R_n(z_{eq})$, $R_g(z_{eq})$, $R_{gu}(z_{eq})$, and $R_u(z_{eq})$, respectively. We are interested in establishing ROA containment results like [5, Proposition 3.5.13, p. 90], [8, Proposition 4], where the size of $R_g(z_{eq})$ and $R_{gu}(z_{eq})$ is to be inferred from that of $R_n(z_{eq})$ and $R_u(z_{eq})$ respectively. However, recognizing that the controller state can usually be initialized arbitrarily, and hence it is possible to ensure $x_g(0) \in K$, we are interested only in the size of $R_g(z_{eq})$ and $R_{gu}(z_{eq})$ within the unsaturated region, i.e. $R_g(z_{eq}) \cap (\mathbb{R}^n \times K)$ and $R_{gu}(z_{eq}) \cap (\mathbb{R}^n \times K)$. From the definitions of f_g (10), f_{gu} (12), and K (11), we have

$$f_g(\bar{z}) = f_{gu}(\bar{z}), \qquad \forall \bar{z} \in \mathbb{R}^n \times K,$$

while Theorem 3 shows that $\mathbb{R}^n \times K$ is a positively invariant set for systems Σ_g and Σ_{gu}. This implies

$$R_g(z_{eq}) \cap (\mathbb{R}^n \times K) = R_{gu}(z_{eq}) \cap (\mathbb{R}^n \times K).$$

Then, with the initial controller state $x_g(0) \in K$ and the initial plant state $x(0)$, the solutions of the GPAW compensated systems Σ_g and Σ_{gu} will converge to z_{eq} if $(x(0), x_g(0)) \in R_g(z_{eq}) \cap (\mathbb{R}^n \times K)$.

The following is the main ROA comparison result that provides sufficient conditions to assert that some ROA estimate Ω_V of the nominal system Σ_n (or unconstrained system Σ_u) is also an ROA estimate of the GPAW compensated system Σ_g (or Σ_{gu}). An additional restriction (in contrast to Lemma 1) has been placed on the description of the ROA estimate Ω_V, namely that it is a sublevel set of the associated Lyapunov function,[6] i.e. $\Omega_V = \{\bar{z} \in \mathbb{R}^{n+q} \mid V(\bar{z}) \leq c\}$ for some $c > 0$. Numerous ROA estimation methods yield estimates of this form, e.g. [15], [16], so that this restriction is well justified.

Theorem 2 (ROA Containment): Consider the nominal system Σ_n (9) and GPAW compensated system Σ_g (10). Assume that $z_{eq} = (\tilde{x}, \tilde{x}_c)$ is an asymptotically stable equilibrium for systems Σ_n and Σ_g that satisfies (14), and let $R_n(z_{eq})$, $R_g(z_{eq})$ be the respective ROAs for Σ_n and Σ_g. Let $\Omega_V = \{\bar{z} \in \mathbb{R}^{n+q} \mid V(\bar{z}) \leq c\} \subset R_n(z_{eq})$ for some $c > 0$ be an estimate of $R_n(z_{eq})$ associated with a Lyapunov function $V \colon \mathbb{R}^{n+q} \to \mathbb{R}$, $z = (x, x_c) \mapsto V(x, x_c) = V(z)$, satisfying

$$
\begin{aligned}
V(z_{eq}) &= 0, \qquad V(\bar{z}) > 0, \qquad \forall \bar{z} \in \Omega_V \setminus \{z_{eq}\}, \\
\dot{V}(\bar{z}) &= \frac{\partial V(\bar{z})}{\partial z} f_n(\bar{z}) \leq -\alpha(\|\bar{z} - z_{eq}\|), \quad \forall \bar{z} \in \Omega_V,
\end{aligned}
\tag{15}
$$

for some class $\mathcal{K}$ function α, where f_n is the vector field of the nominal system Σ_n (9). If there exists[7] a $\Gamma = \Gamma^{\mathrm{T}} > 0 \in \mathbb{R}^{q \times q}$ such that for all $(\bar{x}, \bar{x}_c) \in \Omega_V \cap (\mathbb{R}^n \times \partial K)$,

$$\frac{\partial V(\bar{x}, \bar{x}_c)}{\partial x_c}(I - R_{\mathcal{I}^*}) f_c(\bar{x}_c, g(\bar{x}, g_c(\bar{x}_c)), r) \geq 0, \tag{16}$$

holds, then GPAW compensation with parameter Γ yields system Σ_g (10) whose ROA contains $\Omega_{VK} := \Omega_V \cap (\mathbb{R}^n \times K)$, i.e. $\Omega_{VK} \subset R_g(z_{eq})$.

Remark 2: Observe that f_c in (16) is *independent* of the saturation function $\mathrm{sat}(\cdot)$, and it needs to hold only on the *boundary* $\mathbb{R}^n \times \partial K$ of the unsaturated region $\mathbb{R}^n \times K$. $\square$

Proof: We will be applying Lemma 1 with (3) and (4) representing Σ_n (9) and Σ_g (10) respectively. First, we establish the analogue of (5) with respect to the subset $\Omega_{VK} := (\Omega_V \cap (\mathbb{R}^n \times K)) \subset \Omega_V$. Define $f(\bar{z}) := f(\bar{x}, \mathrm{sat}(g_c(\bar{x}_c)))$, $f_c(\bar{z}) := f_c(\bar{x}_c, g(\bar{x}, \mathrm{sat}(g_c(\bar{x}_c))), r)$, and observe that $\mathrm{sat}(g_c(\bar{x}_c)) = g_c(\bar{x}_c)$ for all $\bar{x}_c \in K$. Then using (9), (10), and (16), we have

$$
\begin{aligned}
\frac{\partial V(\bar{z})}{\partial z} f_g(\bar{z}) &= \frac{\partial V(\bar{x}, \bar{x}_c)}{\partial x} f(\bar{z}) + \frac{\partial V(\bar{x}, \bar{x}_c)}{\partial x_c} R_{\mathcal{I}^*} f_c(\bar{z}), \\
&\leq \frac{\partial V(\bar{x}, \bar{x}_c)}{\partial x} f(\bar{z}) + \frac{\partial V(\bar{x}, \bar{x}_c)}{\partial x_c} f_c(\bar{z}), \\
&= \frac{\partial V(\bar{z})}{\partial z} f_n(\bar{z}),
\end{aligned}
$$

for all $\bar{z} \in \Omega_V \cap (\mathbb{R}^n \times \partial K)$. For all $\bar{z}$ in the interior of the unsaturated region, i.e. $\bar{z} \in \mathbb{R}^n \times (K \setminus \partial K)$, no saturation constraints are active, so that $R_{\mathcal{I}^*} = I$ (see (22)) and hence $f_g(\bar{z}) = f_n(\bar{z})$. Together, these yield

$$\frac{\partial V(\bar{z})}{\partial z} f_g(\bar{z}) \leq \frac{\partial V(\bar{z})}{\partial z} f_n(\bar{z}), \qquad \forall \bar{z} \in \Omega_{VK}, \tag{17}$$

which is analogous to (5).

Next, we show that Ω_{VK} is a positively invariant set for Σ_g. Since $\Omega_V = \{\bar{z} \in \mathbb{R}^{n+q} \mid V(\bar{z}) \leq c\}$, we can express $\Omega_{VK} = \Omega_V \cap (\mathbb{R}^n \times K)$ as $\Omega_{VK} = \{\bar{z} \in \mathbb{R}^n \times K \mid V(\bar{z}) \leq c\}$. Theorem 3 shows $\mathbb{R}^n \times K$ to be a positively invariant set for Σ_g, so that for any $z_0 \in \Omega_{VK}$, the solution $\phi_g(t, z_0)$ of Σ_g satisfy $\phi_g(t, z_0) \in \mathbb{R}^n \times K$ for all $t \geq 0$. Conditions (17) and (15) show that $\frac{\partial V(\bar{z})}{\partial z} f_g(\bar{z}) \leq 0$ for all $\bar{z} \in \Omega_{VK}$. Together, these show Ω_{VK} to be a positively invariant set for Σ_g.

By making the following identifications with quantities in Lemma 1,

$$
\begin{aligned}
f_1 &\sim f_n, & R_{A1}(x_{eq}) &\sim R_n(z_{eq}), & x_{eq} &\sim z_{eq}, \\
f_2 &\sim f_g, & R_{A2}(x_{eq}) &\sim R_g(z_{eq}), & \Omega_2 &\sim \Omega_{VK},
\end{aligned}
$$

it can be verified that all its hypotheses are satisfied. Application of Lemma 1 then yields $\Omega_{VK} \subset R_g(z_{eq})$. $\blacksquare$

Remark 3: From [13, Lemma 4.3, p. 145], we see that a class $\mathcal{K}$ function α exists to satisfy the second condition of (15) when $\dot{V}$ is continuous and negative definite with respect to the equilibrium z_{eq} in Ω_V, i.e. when $\dot{V}$ is continuous and

$$\dot{V}(z_{eq}) = 0, \qquad \dot{V}(\bar{z}) < 0, \qquad \forall \bar{z} \in \Omega_V \setminus \{z_{eq}\}.$$

Clearly, $\dot{V}$ is continuous when V is continuously differentiable and f_n is continuous. $\square$

Remark 4: The main condition in Theorem 2 is (16), which as mentioned, has f_c independent of the saturation function. It can be verified that the proof is valid with minor modifications[8] when Σ_n, Σ_g, f_n, f_g, $R_n(z_{eq})$, $R_g(z_{eq})$,

---

[6]Observe that Ω_V having this form implies that it contains the equilibrium z_{eq} in its interior.

[7]Recall that $R_{\mathcal{I}^*}$ is defined with parameter Γ (see (22) in Appendix).

[8]The only modifications needed are that the condition $\mathrm{sat}(g_c(\bar{x}_c)) = g_c(\bar{x}_c)$ for all $\bar{x}_c \in K$ is *not needed*, and define $f(\bar{z}) := f(\bar{x}, g_c(\bar{x}_c))$, $f_c(\bar{z}) := f_c(\bar{x}_c, g(\bar{x}, g_c(\bar{x}_c)), r)$ without the saturation function.

are replaced by Σ_u, Σ_{gu}, f_u, f_{gu}, $R_u(z_{eq})$, $R_{gu}(z_{eq})$, respectively. $\qquad\square$

The observation of Remark 4 means that the ROA of the GPAW compensated system can be lower bounded in size by two ways: comparing against an ROA estimate of the nominal (saturated) system, or an ROA estimate of the *unconstrained* system. When comparing against an ROA estimate of the nominal system, condition (16) is a sufficient condition for GPAW compensation to yield ROA improvements over the nominal system with respect to the estimate.[9] Usually, the ROA of the unconstrained system will be at least as large as the ROA of the constrained nominal system. There may exist a Γ satisfying (16), yet no Γ such that the analogue of (16) can hold with respect to a larger ROA estimate for the unconstrained system. ROA comparison against the nominal system is a relative result, while comparisons against the ROA of the unconstrained system is in some sense an "absolute" result, prevalent in current anti-windup literature. As shown in [5, Section 3.7, pp. 94 – 96], such "absolute" results may give some confidence in the application of the anti-windup scheme, but may not reveal any advantages gained by its adoption.

Theorem 2 gives sufficient conditions for GPAW compensation to yield ROA improvements, without any indication as to the existence of Γ satisfying (16), nor its choice. Moreover, the definition of $R_{\mathcal{I}*}$ requires the solution of an optimization problem in general (see Appendix), on every point in $\Omega_V \cap (\mathbb{R}^n \times \partial K)$. In general, it would not be easy to find a Γ satisfying (16). Next, we show that Theorem 2 can be applied at least to a simple nonlinear system.

IV. An Illustrative Nonlinear Example

Here, we apply Theorem 2 on a simple second order nonlinear system. This example also highlights the limitations of three classes of current state-of-the-art anti-windup results, in the sense that they cannot be applied to this system.

The following nonlinear system with a first order saturated linear plant and a first order nonlinear nominal controller is adapted from [13, Example 8.9, pp. 318 – 320]

$$\Sigma_n\colon \begin{cases} \dot{x} = -\mathrm{sat}(u), \\ \dot{u} = x + (x^2 - 1)u, \end{cases}$$

which corresponds to the constrained nominal system (9). The objective is to regulate the system state about the origin $z_{eq} = (0,0)$. We take $u_{\max} = 1$ and $u_{\min} = -1$. Using closed-form expressions for the GPAW compensated controller (see [5, equation (A.5), p. 187]), we obtain the GPAW compensated system as

$$\Sigma_g\colon \begin{cases} \dot{x} = -\mathrm{sat}(u), \\ \dot{u} = \begin{cases} 0, & \text{if } A_{\max}, \\ 0, & \text{if } A_{\min}, \\ x + (x^2 - 1)u, & \text{otherwise}, \end{cases} \end{cases}$$

[9]If the estimate is *exact* within the unsaturated region, then we have $\Omega_{VK} = (R_n(z_{eq}) \cap (\mathbb{R}^n \times K)) \subset R_g(z_{eq})$.

where the controller switching conditions are

$$A_{\max} \Leftrightarrow u \geq u_{\max} \wedge x + (x^2 - 1)u > 0,$$
$$A_{\min} \Leftrightarrow u \leq u_{\min} \wedge x + (x^2 - 1)u < 0.$$

Observe that the preceding has similarities with the well-known *conditional integration* method [3, p. 38], [4]. This is expected because the GPAW scheme is an extension of the conditional integration method.

Since an ROA estimate for the associated unconstrained system (13) $\Sigma_u\colon \dot{z} = f_u(z)$ or

$$\Sigma_u\colon \begin{cases} \dot{x} = -u, \\ \dot{u} = x + (x^2 - 1)u, \end{cases} \qquad (18)$$

is readily available in [13, Example 8.9, pp. 318 – 320], we will use it to demonstrate the application of Theorem 2 (see Remark 4). The conclusion yields a containment result for the ROA of the alternate form of the GPAW compensated system

$$\Sigma_{gu}\colon \begin{cases} \dot{x} = -u, \\ \dot{u} = \begin{cases} 0, & \text{if } A_{\max}, \\ 0, & \text{if } A_{\min}, \\ x + (x^2 - 1)u, & \text{otherwise}. \end{cases} \end{cases} \qquad (19)$$

Remark 5: The majority of anti-windup literature is focused on saturated LTI plants driven by LTI controllers [2], [9], [10]. Since the nominal controller (described by $\dot{u} = x + (x^2 - 1)u$) is nonlinear, this class of methods do not apply. For nonlinear systems, the literature is dominated by methods for feedback linearizable plants driven by a feedback linearizing controller, e.g. [11]. While the unconstrained plant (described by $\dot{x} = -u$) is indeed feedback linearizable (in fact, it is linear), the nominal controller is *not* a feedback linearizing controller, which renders this class of methods inapplicable. As for the method of [12] for Euler-Lagrange systems, a fundamental requirement is that the *unconstrained* system must be *globally* asymptotically stable. As shown in [13, Example 8.9, pp. 318 – 320], the unconstrained system (18) is only *locally* asymptotically stable, so that the method of [12] also cannot be applied. In summary, this simple second order system eludes all these state-of-the-art anti-windup schemes. We will show however, that Theorem 2 applies, suggesting its generality. $\qquad\square$

As shown in [13, Example 8.9, pp. 318 – 320], a Lyapunov function for system Σ_u (18) is

$$V(\bar{z}) = \bar{z}^{\mathrm{T}} P \bar{z}, \qquad P = \begin{bmatrix} 1.5 & -0.5 \\ -0.5 & 1 \end{bmatrix},$$

and an ROA estimate associated with V is the sublevel set $\Omega_V = \{\bar{z} \in \mathbb{R}^2 \mid V(\bar{z}) \leq c\}$ with $c = 2.25$. This choice of $c = 2.25$ ensures $\dot{V}(\bar{z}) = \frac{\partial V(\bar{z})}{\partial z} f_u(\bar{z})$ is negative definite in Ω_V. Since V is continuously differentiable and f_u is continuous, (15) holds (see Remark 3). The ROA estimate Ω_V is illustrated in Fig. 1 together with $R_u(z_{eq})$ and $R_{gu}(z_{eq})$, the true ROAs of the unconstrained system Σ_u

and GPAW compensated system Σ_{gu} respectively.[10] While not proven, Fig. 1 suggests that GPAW compensation may enlarge the ROA of the *unconstrained* system.

To apply Theorem 2, we need to verify (16). Note that because the controller is first order, it is independent of any GPAW parameter Γ (see [5, Remark B.1, pp. 201 – 202]). From (18) and (19), we see that $f_c(\bar{x}_c, g(\bar{x}, g_c(\bar{x}_c)), r)$ in (16) translates to $\bar{x} + (\bar{x}^2 - 1)\bar{x}_c$. Defining $\gamma(\bar{x}, \bar{x}_c) := \frac{\partial V(\bar{x}, \bar{x}_c)}{\partial \bar{x}_c}(I - R_{\mathcal{I}^*})f_c(\bar{x}_c, g(\bar{x}, g_c(\bar{x}_c)), r)$, we have that for all $(\bar{x}, \bar{x}_c) \in \mathbb{R}^2$,

$$\gamma(\bar{x}, \bar{x}_c) = (2\bar{x}_c - \bar{x})(I - R_{\mathcal{I}^*})f_c(\bar{x}_c, g(\bar{x}, g_c(\bar{x}_c)), r),$$

$$= \begin{cases} (2\bar{x}_c - \bar{x})(\bar{x} + (\bar{x}^2 - 1)\bar{x}_c), & \text{if } A_{\max 1}, \\ (2\bar{x}_c - \bar{x})(\bar{x} + (\bar{x}^2 - 1)\bar{x}_c), & \text{if } A_{\min 1}, \\ 0, & \text{otherwise}, \end{cases}$$

$$\text{where} \quad A_{\max 1} \Leftrightarrow \bar{x}_c \geq u_{\max} \wedge \bar{x} + (\bar{x}^2 - 1)\bar{x}_c > 0,$$
$$A_{\min 1} \Leftrightarrow \bar{x}_c \leq u_{\min} \wedge \bar{x} + (\bar{x}^2 - 1)\bar{x}_c < 0.$$

For any $(\bar{x}, \bar{x}_c) \in \mathbb{R} \times \partial K = \mathbb{R} \times \{u_{\min}, u_{\max}\}$, we have from the preceding,

$$\gamma(\bar{x}, \bar{x}_c) = \begin{cases} (2u_{\max} - \bar{x})(\bar{x} + (\bar{x}^2 - 1)u_{\max}), & \text{if } A_{\max 2}, \\ (2u_{\min} - \bar{x})(\bar{x} + (\bar{x}^2 - 1)u_{\min}), & \text{if } A_{\min 2}, \\ 0, & \text{otherwise}, \end{cases}$$

$$\text{where} \quad A_{\max 2} \Leftrightarrow \bar{x}_c = u_{\max} \wedge \bar{x} + (\bar{x}^2 - 1)u_{\max} > 0,$$
$$A_{\min 2} \Leftrightarrow \bar{x}_c = u_{\min} \wedge \bar{x} + (\bar{x}^2 - 1)u_{\min} < 0.$$

By inspection of the preceding, we see that $\gamma(\bar{x}, \bar{x}_c) \geq 0$ for all $(\bar{x}, \bar{x}_c) = (\bar{x}, u_{\max})$ when $\bar{x} \leq 2u_{\max} = 2$, and all $(\bar{x}, \bar{x}_c) = (\bar{x}, u_{\min})$ when $\bar{x} \geq 2u_{\min} = -2$. In other words, we have

$$\gamma(\bar{x}, \bar{x}_c) \geq 0, \qquad \forall (\bar{x}, \bar{x}_c) \in X_\gamma \subset \mathbb{R} \times \partial K,$$
$$X_\gamma = ((-\infty, 2] \times \{u_{\max}\}) \cup ([-2, \infty) \times \{u_{\min}\}). \quad (20)$$

Condition (16) requires $\gamma(\bar{x}, \bar{x}_c) \geq 0$ for all $(\bar{x}, \bar{x}_c) \in \Omega_V \cap (\mathbb{R} \times \partial K)$. Using the definition of Ω_V ($\Omega_V = \{\bar{z} \in \mathbb{R}^2 \mid V(\bar{z}) \leq c\}$ for $c = 2.25$), it can be verified that

$$\Omega_V \cap (\mathbb{R} \times \partial K) = (\beta_+ \times \{u_{\max}\}) \cup (\beta_- \times \{u_{\min}\}),$$

where $\beta_+ := [\frac{1}{3} - \sqrt{\frac{17}{18}}, \frac{1}{3} + \sqrt{\frac{17}{18}}] = [-0.638, 1.305]$ and $\beta_- := [-\frac{1}{3} - \sqrt{\frac{17}{18}}, -\frac{1}{3} + \sqrt{\frac{17}{18}}] = [-1.305, 0.638]$. Since $\beta_+ \subset (-\infty, 2]$ and $\beta_- \subset [-2, \infty)$, we see from (20) that $\gamma(\bar{x}, \bar{x}_c) \geq 0$ for all $(\bar{x}, \bar{x}_c) \in (\beta_+ \times \{u_{\max}\}) \cup (\beta_- \times \{u_{\min}\}) = \Omega_V \cap (\mathbb{R} \times \partial K)$, which shows that (16) holds.

All hypotheses of Theorem 2 are satisfied, and we conclude from its application that the ROA of system Σ_{gu} contains $\Omega_V \cap (\mathbb{R} \times K)$. Two sets of solutions, one starting from $z_0 = (-1.2, -0.7) \in (\Omega_V \cap (\mathbb{R} \times K)) \subset R_u(z_{eq}) \subset R_{gu}(z_{eq})$ and another starting from $z_0 = (2.8, 0) \in R_{gu}(z_{eq}) \setminus R_u(z_{eq})$ are shown in Figs. 1 and 2.

[10]The ROAs $R_u(z_{eq})$ and $R_{gu}(z_{eq})$ are found numerically by a trial and error process together with backward-in-time simulations.
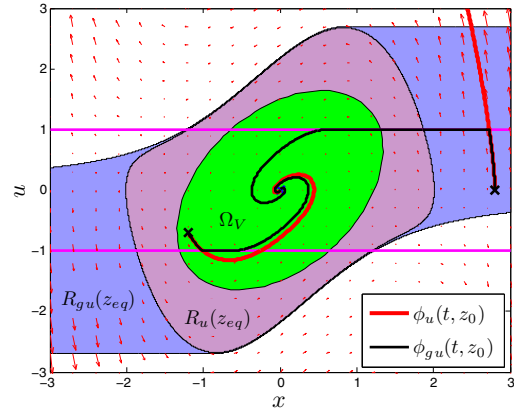

Fig. 1. ROA estimate Ω_V of a planar nonlinear system together with $R_u(z_{eq})$ and $R_{gu}(z_{eq})$, the true ROAs of the unconstrained system Σ_u and GPAW compensated system Σ_{gu} respectively.
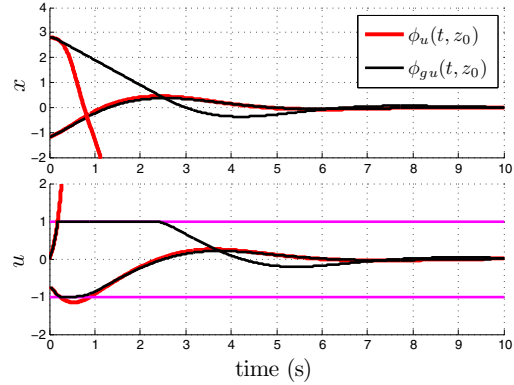

Fig. 2. Solutions of unconstrained and GPAW compensated systems.

V. CONCLUSIONS

In this paper, we extended stability results for *gradient projection anti-windup* (GPAW) compensated systems to saturated multi-input-multi-output (MIMO) nonlinear plants driven by MIMO nonlinear controllers. The fundamental question for anti-windup schemes is whether their application can improve performance without sacrificing stability. To the best of our knowledge, this question has not been explicitly addressed in the literature. For regulatory systems, this requires the region of attraction (ROA) of the uncompensated system to be *contained within* that of the anti-windup compensated system. The main result is a first step in this direction, which gives sufficient conditions for an ROA estimate of the uncompensated system to be a valid ROA estimate for the corresponding GPAW compensated system. A second order nonlinear example not only demonstrates the application of the main result, but highlights the limitations of existing state-of-the-art anti-windup results. Future work includes the search for qualitatively similar but less conservative ROA containment results.

APPENDIX
CONSTRUCTION OF GPAW COMPENSATED CONTROLLER

The construction of the GPAW compensated controller is detailed in [5, Chapter 2, pp. 33 – 70] (and also in [1], [6]– [8], [18]), which we summarize here. Due to lack of space, we direct the interested reader to these sources for details.

Consider a nominal controller (8) that has been designed to achieve some nominal stability and performance.[11] Application of GPAW compensation on the nominal controller (8) yields the GPAW compensated controller

$$\dot{x}_g = R_{\mathcal{I}^*}(x_g, y, r) f_c(x_g, y, r), \qquad x_g(0) = x_{c0},$$
$$u_g = g_c(x_g), \tag{21}$$

where $x_g \in \mathbb{R}^q$ is the state, $u_g \in \mathbb{R}^m$ is the output, (y, r, f_c, g_c) remains unaltered as in (8), and the state-and-input dependent $q \times q$ matrix $R_{\mathcal{I}^*}(x_g, y, r)$ remains to be defined.

Define the $2m$ saturation constraint functions h_i by

$$h_i(x_g) = g_{ci}(x_g) - u_{\max,i}, \qquad \forall i \in \mathcal{I}_m,$$
$$h_{i+m}(x_g) = -g_{ci}(x_g) + u_{\min,i},$$

where g_{ci} for $i \in \mathcal{I}_m$ are the elements of the controller output function $g_c = [g_{c1}, g_{c2}, \ldots, g_{cm}]^{\mathrm{T}}$ in (21). Assuming differentiability of g_c, these constraint functions have gradients

$$\nabla h_i(x_g) = -\nabla h_{i+m}(x_g) = \nabla g_{ci}(x_g), \qquad \forall i \in \mathcal{I}_m.$$

For any index set $\mathcal{I} \subset \mathcal{I}_{2m}$, define the $q \times \max\{|\mathcal{I}|, 1\}$ matrix

$$N_{\mathcal{I}}(x_g) = \begin{cases} [\nabla h_{\sigma_{\mathcal{I}}(1)}(x_g), \ldots, \nabla h_{\sigma_{\mathcal{I}}(|\mathcal{I}|)}(x_g)], & \text{if } \mathcal{I} \neq \emptyset, \\ 0, & \text{otherwise,} \end{cases}$$

where $\sigma_{\mathcal{I}} \colon \{1, 2, \ldots, |\mathcal{I}|\} \to \mathcal{I}$ is a chosen (non-unique) bijection that assigns an integer in $\mathcal{I}$ to each integer in $\{1, 2, \ldots, |\mathcal{I}|\}$. See [5, Remark 2.5, p. 39] for a more detailed description of the bijection $\sigma_{\mathcal{I}}$ and an example.

For any index set $\mathcal{I} \subset \mathcal{I}_{2m}$ such that $\mathrm{rank}(N_{\mathcal{I}}(x_g)) = |\mathcal{I}|$, define the projection matrix $R_{\mathcal{I}} \colon \mathbb{R}^q \to \mathbb{R}^{q \times q}$

$$R_{\mathcal{I}}(x_g) = I - \tilde{S}_{\mathcal{I}}(x_g),$$
$$\tilde{S}_{\mathcal{I}}(x_g) = \begin{cases} \Gamma N_{\mathcal{I}} (N_{\mathcal{I}}^{\mathrm{T}} \Gamma N_{\mathcal{I}})^{-1} N_{\mathcal{I}}^{\mathrm{T}}(x_g), & \text{if } \mathcal{I} \neq \emptyset, \\ 0, & \text{otherwise,} \end{cases} \tag{22}$$

where $\Gamma \in \mathbb{R}^{q \times q}$ is the *single* GPAW parameter, chosen to be symmetric positive definite. As shown in [5, Remark B.1, pp. 201 – 202], first order GPAW compensated controllers are fully defined independent of any parameters.

Define the index set of active saturation constraints $\mathcal{I}_{\mathrm{sat}}$, and the candidate solution set $\mathcal{J}$

$$\mathcal{I}_{\mathrm{sat}} := \mathcal{I}_{\mathrm{sat}}(x_g) = \{i \in \mathcal{I}_{2m} \mid h_i(x_g) \geq 0\},$$
$$\mathcal{J} := \mathcal{J}(x_g) = \{\mathcal{I} \subset \mathcal{I}_{\mathrm{sat}} \mid |\mathcal{I}| \leq q\}.$$

For any fixed (x_g, y, r), let $\mathcal{I}^*$ be a solution to the following *combinatorial* optimization subproblem[12]

$$\max_{\mathcal{I} \in \mathcal{J}} f_c^{\mathrm{T}}(x_g, y, r) \Gamma^{-1} R_{\mathcal{I}}(x_g) f_c(x_g, y, r),$$
$$\text{subject to} \qquad \mathrm{rank}(N_{\mathcal{I}}(x_g)) = |\mathcal{I}|, \tag{23}$$
$$N_{\mathcal{I}_{sat}}^{\mathrm{T}}(x_g) R_{\mathcal{I}}(x_g) f_c(x_g, y, r) \leq 0.$$

Then the GPAW compensated controller is defined by (21) with $R_{\mathcal{I}^*}(x_g, y, r)$ defined by (22) and $\mathcal{I}^*$. As shown in [5,

Remark 2.17, p. 50], GPAW compensation is active only when some controls saturate. *Controller state-output consistency* is a fundamental property of GPAW compensated controllers, stated below.

Theorem 3 ([5, Theorem 2.5.3, p. 50]): Consider the GPAW compensated controller defined by (21), (22), and a solution $\mathcal{I}^*$ to subproblem (23). If there exists a $T \in \mathbb{R}$ such that $\mathrm{sat}(u_g(T)) = u_g(T)$, then $\mathrm{sat}(u_g(t)) = u_g(t)$ holds for all $t \geq T$.

Proof: See [5, pp. 50 – 51]. ∎

The preceding is the basic construction of the GPAW compensated controller. See [5, pp. 111 – 115, pp. 185 – 204] for alternate equivalent realizations.

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[11]As shown in [5, Section 2.6, pp. 53 – 55], more general controllers can be approximated arbitrarily well to have the form of (8).

[12]The existence of an optimal solution $\mathcal{I}^*$ to subproblem (23) is assured by [5, Proposition 2.5.1, p. 49].