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# An optimal solution to an $\mathcal{H}_-/\mathcal{H}_\infty$ fault detection problem

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***Abstract*— In this contribution, we derive an optimal solution to an $\mathcal{H}_-/\mathcal{H}_\infty$ fault detection (FD) problem for linear time-invariant systems. Firstly, a novel $\mathcal{H}_\infty$-based performance index is formulated to minimize the relative sensitivity of the residual to disturbances with respect to the sensitivity to faults. Then a class of optimal filters for the $\mathcal{H}_-/\mathcal{H}_\infty$ FD problem is obtained by solving a Linear Matrix Inequality (LMI) optimization. Further specifications such as fault isolation can be achieved by using the remaining degrees of freedom in the optimal FD filter design. Moreover, weighting filters are considered to improve the results. Finally, two examples demonstrate the effectiveness of the proposed scheme.**

## I. INTRODUCTION

As many control systems and engineering processes become more and more complex and integrated, there is a growing need for on-line supervision and fault detection to increase their reliability, safety and fault tolerance capabilities and many different approaches have been proposed to solve the FD problem [4], [15], [16], [3], [9], [18], [17], [7].

Recently, mixed-norm FD problems have attracted a great deal of attention and various approaches and schemes have been proposed in many contributions. In [14], an $\mathcal{H}_-/\mathcal{H}_\infty$ FD problem is investigated and a suboptimal solution is given, which is due to that a bilinear matrix inequality formulation is employed so that some variables are needed to be pre-defined to solve their optimization problem. Wang *et al* [12] considers the $\mathcal{H}_\infty/\mathcal{H}_\infty$ FD problem and give a suboptimal solution due to the fact that a common Lyapunov function is used to solve their matrix inequalities, which results in a conservative FD observer design. While in [10], both $\mathcal{H}_\infty/\mathcal{H}_\infty$ and $\mathcal{H}_-/\mathcal{H}_\infty$ FD problems are considered and a suboptimal solution is given since the FD problem is formulated as a quasi-linear matrix inequality formulation that necessitates an iterative method to approximate the optimal solution.

This work considers an $\mathcal{H}_-/\mathcal{H}_\infty$ FD problem for linear time-invariant systems subject to faults and disturbances. First, we propose a new performance index that captures the requirements of fault detection and disturbance rejection. The disturbance rejection performance is measured, using the $\mathcal{H}_\infty$ norm, by the size of the disturbance to residual dynamics. Using a static observer, we give a class of (not necessarily stable) solutions to a novel FD problem in the form of a simple LMI with two degrees of freedom. Then, the freedoms in this class of solutions are used to derive the optimal filter to the $\mathcal{H}_-/\mathcal{H}_\infty$ FD problem. Under certain conditions, we show that fault isolation can also be achieved. We then improve the results with the use of weighting filters.

Mainly, this work has made progress in the following two aspects: (1) we give the optimal design of the $\mathcal{H}_-/\mathcal{H}_\infty$ FD filter; (2) the cost function in this work combines both fault detection and disturbance attenuation requirements for which only the 'ratio' between these two objectives is optimized and needed to be fixed. So that a class of optimal filters with two degrees of freedom is given and can be used to achieve further specifications such as fault isolation.

The structure of the work is as follows. After defining the notation, we formulate the $\mathcal{H}_-/\mathcal{H}_\infty$ problem as well as a novel FD problems using a static observer structure in Section II. Section III gives a class of solutions to the novel FD problem. In section IV, we present further specifications that can be achieved by the filters and we give the optimal filter design for the $\mathcal{H}_-/\mathcal{H}_\infty$ FD problem. Section V gives a technique to improve the results through the use of weighting filters. Finally, numerical examples are presented in Section VI to validate our approaches, and Section VII summarizes our results.

The notation we use is fairly standard. The set of real $n\times m$ matrices is denoted by $\mathcal{R}^{n\times m}$. For $A \in \mathcal{R}^{n\times m}$ we use the notation $A^T$ to denote transpose. For a matrix $A \in \mathcal{C}^{n\times n}$, $\bar{\lambda}(A)$ denotes the largest and $\underline{\lambda}(A)$ the smallest eigenvalue of $A$, respectively. For $A \in \mathcal{C}^{n\times m}$, $\bar{\sigma}(A)$ denotes the largest, and $\underline{\sigma}(A)$ the smallest, singular values of $A$, respectively. For $A = A^T \in \mathcal{R}^{n\times n}$, $A > 0$ ($A < 0$) denotes that $A$ is positive (negative) definite, that is, all the eigenvalues of $A$ are greater (less) than zero. The $n \times n$ identity matrix is denoted as $I_n$ and the $n \times m$ null matrix is denoted as $0_{n,m}$ with the subscripts occasionally dropped if they can be inferred from context.

A transfer matrix $G(s) = D + C(sI - A)^{-1}B$ will be denoted as $G(s) \stackrel{s}{=} (A, B, C, D)$ or $G(s) \stackrel{s}{=} \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ and we define $G^\sim(s) := G^T(-s)$ to be the para-hermitian complex conjugate of $G(s)$. Transfer matrix dependence on the variable $s$ will be normally suppressed. For a (not necessarily stable) transfer matrix $G$ we define

$$\|G\|_\infty = \sup_{\omega\in\mathcal{R}} \bar{\sigma}\left(G(j\omega)\right), \qquad \|G\|_- = \inf_{\omega\in\mathcal{R}} \underline{\sigma}\left(G(j\omega)\right).$$

## II. FAULT DETECTION PROBLEM FORMULATION

Consider a linear time-invariant dynamic system subject to disturbances, modeling errors and process, sensor and

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actuator faults modeled as

$$\begin{aligned} \dot{x}(t) &= Ax(t)+Bu(t)+B_f f(t)+B_d d(t), \\ y(t) &= Cx(t)+Du(t)+D_f f(t)+D_d d(t), \end{aligned} \tag{1}$$

where $x(t) \in \mathcal{R}^n$, $u(t) \in \mathcal{R}^{n_u}$ and $y(t) \in \mathcal{R}^{n_y}$ are the process state, input and output vectors, respectively, and where $f(t) \in \mathcal{R}^{n_f}$ and $d(t) \in \mathcal{R}^{n_d}$ are the fault and disturbance vectors, respectively. Here, $B_f \in \mathcal{R}^{n \times n_f}$ and $D_f \in \mathcal{R}^{n_y \times n_f}$ are the component and instrument fault distribution matrices, respectively, while $B_d \in \mathcal{R}^{n \times n_d}$ and $D_d \in \mathcal{R}^{n_y \times n_d}$ are the corresponding disturbance distribution matrices [2]. Consider a residual generator using a static observer of the form

$$\begin{aligned} \dot{\hat{x}}(t) &= A\hat{x}(t)+Bu(t)-L\left(y(t)-C\hat{x}(t)-Du(t)\right), \\ r(t) &= H(y(t)-C\hat{x}(t)-Du(t)), \end{aligned} \tag{2}$$

where $\hat{x}(t) \in \mathcal{R}^n$ is the observer state and $r(t) \in \mathcal{R}^{n_r}$ is the residual signal. Here $L \in \mathcal{R}^{n \times n_y}$ and $H \in \mathcal{R}^{n_r \times n_y}$ are the observer and residual gain matrices, respectively, and are to be determined. Define the state estimation error signal as $e(t) = x(t) - \hat{x}(t)$. It follows that the residual dynamics are given by

$$\begin{aligned} \dot{e}(t) &= (A+LC)e(t) + (B_f+LD_f)f(t) + (B_d+LD_d)d(t), \\ r(t) &= HCe(t) + HD_f f(t) + HD_d d(t). \end{aligned}$$

By taking Laplace transforms, $r(s) = T_{rf}(s)f(s) + T_{rd}(s)d(s)$, where

$$\left[T_{rf}\ T_{rd}\right] \stackrel{s}{=} \left[\begin{array}{c|cc} A+LC & B_f+LD_f & B_d+LD_d \\ \hline HC & HD_f & HD_d \end{array}\right] = F\left[G_f\ G_d\right] \tag{3}$$

are the transfer matrices from faults and disturbances to the residual, respectively, and where

$$F \stackrel{s}{=} \left[\begin{array}{c|c} A+LC & L \\ \hline HC & H \end{array}\right], \left[\, G_f \quad G_d \,\right] \stackrel{s}{=} \left[\begin{array}{c|cc} A & B_f & B_d \\ \hline C & D_f & D_d \end{array}\right] \tag{4}$$

are the FD filter and the fault and disturbance dynamics, respectively. With these preliminaries we formulate the $\mathcal{H}_-/\mathcal{H}_\infty$ FD problem as follows.

*Problem 2.1:* With all variables as defined above, assume that:

- A1. The pair $(A, C)$ is detectable (see Remark 2.2.)
- A2. $G_f$ has full column rank over the extended imaginary axis (see Remarks 2.3 and 2.4, and Section V.)
- A3. $n_y \geq n_f$ (see Remark 2.5.)
- A4. $n_r = n_f$ (see Remark 2.6.)

Find

$$\gamma_0 := \min\ \left\{\frac{\|T_{rd}\|_\infty}{\|T_{rf}\|_-} : A+LC \text{ is stable}\right\}.$$

*Remark 2.1:* An interpretation of Problem 2.1 is that $\bar{\sigma}(T_{rd}(j\omega))$ is required to be smaller than or equal to $\gamma_0\underline{\sigma}(T_{rf}(j\omega))$ for all $\omega \in \mathcal{R}$. This captures the requirement of ensuring insensitivity to the disturbance and sensitivity to faults. □

Since

$$\left\|T_{rf}^{-1}T_{rd}\right\|_\infty \leq \left\|T_{rf}^{-1}\right\|_\infty \|T_{rd}\|_\infty = \|T_{rd}\|_\infty / \|T_{rf}\|_-,$$

then,

$$\gamma_0 \geq \min\ \left\{\left\|T_{rf}^{-1}T_{rd}\right\|_\infty : A+LC \text{ is stable}\right\} =: \gamma_1. \tag{5}$$

It follows that $\gamma_1$ is a lower bound on $\gamma_0$, which motivates the following problem:

*Problem 2.2:* Let all variables and assumptions be as given in Problem 2.1. For a given $\gamma > 0$ find $L$ and $H$, if they exist, such that $A + LC$ is stable and

$$\left\|T_{rf}^{-1}T_{rd}\right\|_\infty < \gamma. \tag{6}$$

*Remark 2.2:* Assumption A1 is needed to guarantee the existence of at least one $L$ such that $A + LC$ is stable. □

*Remark 2.3:* The cost function in (6) can be expressed as $T_{rf}^{-1}T_{rd}T_{rd}^{\sim}T_{rf}^{-\sim} < \gamma^2$ or, equivalently, as

$$T_{rd}T_{rd}^{\sim} < \gamma^2 T_{rf}T_{rf}^{\sim}, \tag{7}$$

If $G_f$ loses rank over the extended imaginary axis, then so does $T_{rf} = FG_f$ and (7) cannot be satisfied for any $\gamma > 0$. In particular, the assumption implies that $D_f$ has full column rank which is necessary for $T_{rf}^{-1}$ to have a proper state-space realization. This assumption can be (somewhat) relaxed if we change the cost function in (7) to $T_{rd}T_{rd}^{\sim} - \gamma^2 T_{rf}T_{rf}^{\sim} \leq 0$ and if $G_d$ loses rank over the imaginary axis whenever $G_f$ does. Dealing with this situation will necessitate an intricate analysis of spectral factorizations with imaginary axis zeros [1], which is outside the scope of this work. In Section V we relax this assumption by modifying the cost function in (6) and introducing weighting functions. □

*Remark 2.4:* An alternative approach in the case that $G_f$ loses rank over the external imaginary axis is to design the FD scheme over a specific finite frequency range inside which $G_f$ has full rank [11], [13], [8], although the solution involves non-convex optimization. □

*Remark 2.5:* Suppose that $n_y < n_f$, then assumptions A1 and A2, together with Theorem 13.32 in [19] imply that $G_f = \hat{G}_f G_i$ where $\hat{G}_f \in \mathcal{RL}_\infty^{n_y \times n_y}$ and $G_i \in \mathcal{RL}_\infty^{n_y \times n_f}$ such that $G_i G_i^{\sim} = I_{n_y}$. Furthermore, $\hat{G}_f(s)$ can be chosen to have the same $A$ and $C$ matrices as $G_f$. Using (4) and (7) we have

$$\begin{aligned} T_{rd}T_{rd}^{\sim} < \gamma^2 T_{rf}T_{rf}^{\sim} \quad &\Leftrightarrow \quad FG_dG_d^{\sim}F^{\sim} < \gamma^2 FG_fG_f^{\sim}F^{\sim} \\ &\Leftrightarrow \quad FG_dG_d^{\sim}F^{\sim} < \gamma^2 F\hat{G}_f\hat{G}_f^{\sim}F^{\sim}. \end{aligned}$$

Thus we may replace $G_f(s)$ by $\hat{G}_f(s)$ and so there is no loss of generality in assuming that $n_f \leq n_y$. □

*Remark 2.6:* We opt for $n_r = n_f$ for the following reasons:

1) There is no need for $n_r > n_f$ since our interest is in increasing the sensitivity of the residual to faults by increasing the singular values of $T_{rf}$ relative to those of $T_{rd}$, and there are at most $n_f$ singular values of $T_{rf}$ since $n_y \geq n_f$.
2) Since the set of all optimal filters when $n_r < n_f$ is a subset of those when $n_r = n_f$, we get maximal sensitivity to faults when $n_r = n_f$.
3) This will prove useful when we consider fault isolation in Section IV below. □

In the following sections, the optimal solution of Problem 2.1 is derived in two stages. First we construct a class of optimal solutions to Problem 2.2 using LMI techniques. Then, in Section IV, the freedoms in this class of solutions are employed to derive the optimal $\mathcal{H}_-/\mathcal{H}_\infty$ FD problem and we show that, by an appropriate choice of the freedoms, further specifications such as fault isolation can also be achieved.

## III. A class of optimal FD filters for Problem 2.2

In this section, we characterize a set of (not necessarily stable) filters of the form given in (4) such that (6) is satisfied. Our approach is to use the bounded real lemma to derive necessary and sufficient conditions for (6) in the form of a (nonlinear) matrix inequality. We then introduce a change of variables to derive an equivalent LMI.

Since $D_f \in \mathcal{R}^{n_y \times n_f}$ has full column rank, there exist $D_f^\dagger \in \mathcal{R}^{n_f \times n_y}$ and $D_f^\perp \in \mathcal{R}^{(n_y-n_f)\times n_y}$ such that

$$\begin{bmatrix} D_f^\dagger \\ D_f^\perp \end{bmatrix} D_f = \begin{bmatrix} I_{n_f} \\ 0 \end{bmatrix}, \qquad \text{rank}\left(\begin{bmatrix} D_f^\dagger \\ D_f^\perp \end{bmatrix}\right) = n_y.$$

Let

$$H = U_1 D_f^\dagger + U_2 D_f^\perp, \qquad L = (Z_1 - B_f)D_f^\dagger + Z_2 D_f^\perp \quad (8)$$

where $U_1 \in \mathcal{R}^{n_f \times n_f}$, $U_2 \in \mathcal{R}^{n_f \times (n_y - n_f)}$, $Z_1 \in \mathcal{R}^{n \times n_f}$ and $Z_2 \in \mathcal{R}^{n \times (n_y - n_f)}$ are free parameters, with $U_1$ nonsingular to ensure that $HD_f$ is invertible. Then

$$HD_f = U_1,\ B_f + LD_f = Z_1,\ (HD_f)^{-1}H = D_f^\dagger + U_0 D_f^\perp \quad (9)$$

where $U_0 = U_1^{-1}U_2$. By taking some matrix manipulations, it shows that

$$T_{rf}^{-1} \stackrel{s}{=} \left[\begin{array}{c|c} A+LC-(B_f+LD_f)(HD_f)^{-1}HC & -(B_f+LD_f)(HD_f)^{-1} \\ \hline (HD_f)^{-1}HC & (HD_f)^{-1} \end{array}\right].$$

Then

$$T_{rf}^{-1}T_{rd} \stackrel{s}{=} \left[\begin{array}{c|c} A_c & B_c \\ \hline C_c & D_c \end{array}\right]$$

where

$$\begin{aligned} A_c &= A+LC-(B_f+LD_f)(HD_f)^{-1}HC, \\ B_c &= (B_d+LD_d)-(B_f+LD_f)(HD_f)^{-1}(HD_d), \\ C_c &= (HD_f)^{-1}HC, \\ D_c &= (HD_f)^{-1}(HD_d). \end{aligned}$$

By using the rearrangements of the variables as given in (9), we obtain

$$T_{rf}^{-1}T_{rd} \stackrel{s}{=} \left[\begin{array}{c|c} A_1 + Z_0 C_2 & B_{d,1} + Z_0 D_{d,2} \\ \hline C_1 + U_0 C_2 & D_{d,1} + U_0 D_{d,2} \end{array}\right], \quad (10)$$

where we have defined

$$\left[\begin{array}{c|c} A_1 & B_{d,1} \\ \hline C_1 & D_{d,1} \\ C_2 & D_{d,2} \end{array}\right] = \left[\begin{array}{c|c} A - B_f D_f^\dagger C & B_d - B_f D_f^\dagger D_d \\ \hline D_f^\dagger C & D_f^\dagger D_d \\ D_f^\perp C & D_f^\perp D_d \end{array}\right] \quad (11)$$

and $Z_0 = Z_2 - Z_1 U_0$. The next result gives the solution of Problem 2.2.

*Theorem 3.1:* Let all variables and assumptions be as given in Problem 2.2. For some $\gamma > 0$, there exist $L$ and $H$ such that $A+LC$ is stable and (6) is satisfied if and only if there exist $P = P^T$, $Q$ and $U_0$ such that the following LMI

$$\begin{bmatrix} A_1^T P + C_2^T Q^T + PA_1 + QC_2 & \star & \star \\ B_{d,1}^T P + D_{d,2}^T Q^T & -\gamma I & \star \\ C_1 + U_0 C_2 & D_{d,1} + U_0 D_{d,2} & -\gamma I \end{bmatrix} < 0, \quad (12)$$

is satisfied, where $\star$ represents terms readily inferred from symmetry.

**Proof.** From a generalized bounded real lemma in [5], there exist $L$ and $H$ such that (6) is satisfied if and only if there exists $P = P^T$ (not necessarily positive definite) such that

$$\begin{bmatrix} A_c^T P + PA_c & PB_c & C_c^T \\ B_c^T P & -\gamma I & D_c^T \\ C_c & D_c & -\gamma I \end{bmatrix} < 0. \quad (13)$$

Substituting (10) in (13) and then the nonlinear matrix inequality is linearized by taking the change of variables $Q = PZ_0$. It follows that there exist $L$ and $H$ such that (6) is satisfied if and only if there exist $P = P^T$, $Q$ and $U_0$ such that (12) is satisfied. Furthermore, if (12) has feasible solutions $P$, $Q = PZ_0$ and $U_0$ and we define

$$H_0 = D_f^\dagger + U_0 D_f^\perp, \qquad L_0 = -B_f D_f^\dagger + Z_0 D_f^\perp, \quad (14)$$

so that $L$ and $H$ in (8) can be written as

$$L = L_0 + Z_1 H_0, \qquad H = U_1 H_0, \quad (15)$$

then we get a class of (not necessarily stable) FD filters $F$ of the form (4) parameterized by free variables $U_1$ and $Z_1$, subject to $U_1$ nonsingular. It can be seen that $A + LC = (A + L_0 C) + Z_1(H_0 C)$. The rest of the proof, which shows that it is always possible to stabilize $A+LC$ by choosing the freedom $Z_1$ in the filter design, is omitted due to the length limitation and can be obtained from the authors. □

*Remark 3.1:* If $P$ is singular, then, because of the strict inequality in (12), we can perturb $P$ such that the perturbed $P$ is nonsingular and inequality (12) is still satisfied. This shows that we can always recover $Z$ from $P$ and $Q$. □

## IV. Design of an optimal $\mathcal{H}_-/\mathcal{H}_\infty$ FD filter

In the previous section, we derived a set of fault detection filters and residual dynamics parameterized by free $Z_1$ and $U_1$, with $U_1$ nonsingular that solve Problem 2.2. This section shows that, through a suitable choice of $Z_1$ and $U_1$, the optimal $\mathcal{H}_-/\mathcal{H}_\infty$ FD filter is derived and stabilized and further specifications such as fault isolation can also be achieved in some circumstances.

Firstly, note that since $B_f + L_0 D_f = 0$ and $H_0 D_f = I_{n_f}$ from (14), then

$$T_{rf} \stackrel{s}{=} \left[\begin{array}{c|c} A + L_0 C + Z_1 H_0 C & Z_1 \\ \hline U_1 H_0 C & U_1 \end{array}\right] \quad (16)$$

from (3) and (15). As known that, to ensure fault isolability for all faults, it is required that the transfer matrix $T_{rf}$ is diagonal. So that in the case of $T_{rf} = I_{n_f}$ (ignoring disturbances), $r = f$ and each individual fault only affects its corresponding residual signal.

Therefore, if $A+L_0C$ is stable, we easily get $T_{rf} = I_{n_f}$ by choosing $Z_1 = 0$ and $U_1 = I_{n_f}$ such that the filter achieves optimal FDI.

If $A + L_0C$ is not stable, then it is needed to choose $Z_1$ such that the fault detection filter $F$ is stable. The rest of this section considers the stabilization of the $F$ by choosing suitable $Z_1$ and $U_1$ such that optimal FD is achieved.

Note that simple calculation using (10) and (14) shows that

$$A + L_0C = A_c. \tag{17}$$

It follows from the (1,1) block of (13) and the assumed nonsingularity of $P$ that $A + L_0C$ has no eigenvalues on the imaginary axis. It follows that we can effect a similarity transformation on the data, if necessary, such that

$$\begin{bmatrix} A + L_0C \\ H_0C \end{bmatrix} = \left[\begin{array}{cc} A_s & 0 \\ 0 & A_a \\ \hline C_s & C_a \end{array}\right]$$

where $A_s$ and $-A_a$ are stable. Then (16) implies that

$$T_{rf} \stackrel{s}{=} \left[\begin{array}{cc|c} A_s & 0 & 0 \\ Z_aC_s & A_a+Z_aC_a & Z_a \\ \hline U_1C_s & U_1C_a & U_1 \end{array}\right] \tag{18}$$

where we have set $Z_1 = [0 \;\; Z_a^T\,]^T$. Let $U_1$ be any orthogonal matrix and

$$Z_a = -P_a^{-1}C_a^T, \tag{19}$$

where $P_a = P_a^T > 0$ is the unique solution of the Lyapunov equation

$$A_a^TP_a + P_aA_a - C_aC_a^T = 0. \tag{20}$$

Note that $P_a > 0$ follows from the facts that $-A_a$ is stable and $(A_a, C_a)$ is observable, which in turn follows from the fact that $(A + L_0C, H_0C)$ is detectable. Then (18) implies that

$$T_{rf} \stackrel{s}{=} \left[\begin{array}{c|c} A_a+Z_aC_a & Z_a \\ \hline U_1C_a & U_1 \end{array}\right] = \left[\begin{array}{c|c} A_a-P_a^{-1}C_a^TC_a & -P_a^{-1}C_a^T \\ \hline U_1C_a & U_1 \end{array}\right].$$

Hence, $T_{rf}$, and so that $T_{rd}$ are stable, which follows from $(A_a+Z_aC_a)^TP_a+P_a(A_a+Z_aC_a)+C_a^TC_a = 0$ and $P_a > 0$. Since $U_1^TU_1 = I$ and given (20) is satisfied , by taking matrix calculations, it shows that $T_{rf}^{\sim}T_{rf} = I$.

As illustrated above, in either case of $A + L_0C$ is stable or instable, it is always possible to construct $F$ such that $T_{rf}^{\sim}T_{rf} = I$ which gives $\|T_{rf}\|_- = I$ and

$$\frac{\|T_{rd}\|_\infty}{\|T_{rf}\|_-} = \|T_{rd}\|_\infty = \left\|T_{rf}^{-1}T_{rd}\right\|_\infty < \gamma. \tag{21}$$

Therefore, the above result shows that a suitable choice of $Z_1$ and $U_1$ ensures $T_{rf}^{\sim}T_{rf} = I$ and then $\gamma_0 = \gamma_1$ in (5). Hence, it proves that our scheme gives the optimal solution of Problem 2.1.

*Remark 4.1:* Note that due to the specific choice of $Z_1$ in (19) to stabilize $F$, $T_{rf}$ is no longer diagonal, which implies that it is not possible to achieve fault isolation. However, we find that it is always possible to choose the free variable $U_1$ suitably such that $F$ achieves fault isolation at DC ($\omega = 0$). Set $U_1 = \left(I_{n_f} - C_a(A_a + Z_aC_a)^{-1}Z_a\right)^T$, then $T_{rf}(0) = U_1 - U_1C_a(A_a + Z_aC_a)^{-1}Z_a = I_{n_f}$, which ensures optimal FDI at DC. □

*Remark 4.2:* Note that if $A + L_0C$ stable, we get an optimal FDI filter for the $\mathcal{H}_-/\mathcal{H}_\infty$ problem. Now suppose in the LMI in (13) we add the constraint that $P = P^T > 0$ instead of just $P = P^T$. This will, in general, result in a larger optimal $\gamma$ but will ensure that $A + L_0C$ is stable from (17). Hence, it follows from the preceding argument that we can improve the fault isolation capability of our filter at the expense of its fault detection ability. □

## V. Extensions of the fault detection problem

In the previous sections, we presented a fault detection scheme that is optimal with respect to the specifications of Problem 2.1 and that also has some fault isolation properties. In this section, we briefly outline an extension to the proposed scheme which involves the introduction of frequency weighting to improve the fault detection properties and remove assumption $A2$ in Problem 2.1.

### *A. Further improvement in FD performance*

In the fault detection scheme developed in the previous section, the performance of the FD filter depends on the value of $\gamma$. If $\gamma < 1$, then the smallest singular value of $T_{rf}$ is larger than the largest singular value of $T_{rd}$ over all frequencies and the performance of the FD scheme is expected to be satisfactory. On the other hand, if $\gamma >> 1$, then the fault detection may not be effective or inadequate.

The problem is in our requirement that

$$\bar{\sigma}(T_{rf}(j\omega)^{-1}T_{rd}(j\omega)) < \gamma \tag{22}$$

for all frequencies $\omega \in \mathcal{R}$. In practice, for the purpose of fault detection, we only need (22) over a limited frequency range, or even at a single frequency. Let $W_{\omega_0}$ be a stable and proper minimum-phase single-input single-output bandpass filter with a peak gain of one at frequency $\omega_0 \in \mathcal{R}$. Suppose we replace the cost function in (6) by $\left\|(FG_fW_{\omega_0}^{-1})^{-1}(FG_dW_{\omega_0})\right\|_\infty < \gamma_{\omega_0}$. This can be recast as Problem 2.2 by replacing $\begin{bmatrix} G_f & G_d \end{bmatrix}$ by $\begin{bmatrix} G_fW_{\omega_0}^{-1} & G_dW_{\omega_0} \end{bmatrix}$, which because of our assumptions on $W_{\omega_0}$ inherits the assumptions of Problem 2.2.

Suppose further that we enforce that $FG_fW_{\omega_0}^{-1}$ is inner (see Section IV). Then $\sigma_i(T_{rf}(j\omega)) = |W_{\omega_0}(j\omega)|$ and $\bar{\sigma}(T_{rd}(j\omega)) < \gamma_{\omega_0}|W_{\omega_0}^{-1}(j\omega)|$, where $\sigma_i(\cdot)$ denotes the $i$th singular value. It follows that $\bar{\sigma}(T_{rd}(j\omega_0)) < \gamma_{\omega_0}\underline{\sigma}(T_{rf}(j\omega_0))$. Thus we can calculate $\gamma_{\omega_0}$ for several values of $\omega_0$ in a suitable frequency range $[\underline{\omega}, \bar{\omega}]$ to minimize $\gamma_{\omega_0}$. Since $\gamma_{\omega_0} \le \gamma$ for all $\omega_0$, we may thus be able to improve the fault detection properties at a single frequency $\omega_0$. The fault detection filter $F$ may now be followed by a second filter $W_{\omega_0}$ to shape the residual signal $r$.

## B. Removal of assumption A2 in Problem 2.1

A major assumption we made is that $G_f$ has no zeros over the extended imaginary axis, and in particular that $D_f$ has full column rank. This excludes important situations such as no sensor faults ($D_f = 0$). The problem is introduced by our choice of the cost function in (7): in the case that $G_f(j\omega_1)$ loses rank at $\omega_1 \in \mathcal{R}$, then so does $T_{rf}(j\omega_1)$, so that (7) cannot in general be satisfied at $j\omega_1$. In the following, a discussion is given for relaxing Assumption A2 and ameliorating the problem of loss rank on $G_f$.

Suppose that $G_f$ has zeros on the extended imaginary axis. Then $G_f$ can be written as $G_f = \hat{G}_f G_z$ where $\hat{G}_f$ has no zeros over the extended imaginary axis and $G_z$ has all its zeros on the extended imaginary axis [6]. Suppose now that we replace the cost function in (6) by $\left\|\hat{T}_{rf}^{-1}T_{rd}\right\|_\infty < \gamma$ where $\hat{T}_{rf} = F\hat{G}_f$ and we enforce that $\hat{T}_{rf}$ is inner as shown in Section IV. Then $\sigma_i(T_{rf}(j\omega)) = \sigma_i(\hat{T}_{rf}(j\omega)G_z(j\omega)) = \sigma_i(G_z(j\omega))$, since $\hat{T}_{rf}$ is inner, and $\bar{\sigma}(T_{rd}(j\omega)) < \gamma$. Note that the singular values of $T_{rf}(j\omega)$ are set equal to those of $G_z(j\omega)$ instead of being set equal to 1 in the original scheme, therefore $\gamma$ is the fault detection level relative to $\sigma_i(G_z(j\omega))$ rather than 1 as given in (21).

# VI. Numerical examples

To illustrate the effectiveness of the proposed FD filter scheme, we consider two examples. The first one, from the literature, compares our FD filter with other design techniques. The second example is randomly generated, and highlights the specifications given in section IV and section V.

## A. Example from the literature

Consider example 2 given in [12] and also considered in [10]. Both approaches employed an $\mathcal{H}_\infty/\mathcal{H}_\infty$ FD formulation. The state-space model of the LTI system is described as follows.

$$A = \begin{bmatrix} -10 & 0 & 5 & 0 \\ 0 & -5 & 0 & 2.5 \\ 0 & 0 & -2.5 & 0 \\ 0 & 5 & 0 & -3.75 \end{bmatrix}, B_d = \begin{bmatrix} 0.8 & 0.04 \\ -2.4 & 0.08 \\ 1.6 & 0.08 \\ 0.8 & 0.08 \end{bmatrix}, B_f = \begin{bmatrix} 4 \\ 4 \\ 8 \\ -8 \end{bmatrix},$$

$$C = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}, \quad D_d = \begin{bmatrix} 0.2 & 0.04 \\ 0.4 & 0.06 \end{bmatrix}, \quad D_f = \begin{bmatrix} 2 \\ -1 \end{bmatrix}.$$

Our algorithm gives $\gamma_0 = 0.2392$ and a stable $A + L_0 C$. We set $Z_1 = 0$ and $U_1 = 2.6I$ and obtained

$$L = \begin{bmatrix} -2.0034 & -1.0735 \\ 0.7778 & 5.4963 \\ -4.8810 & -1.6664 \\ 2.1436 & -2.8805 \end{bmatrix}, \quad H = \begin{bmatrix} 1.0385 & -0.5231 \end{bmatrix}.$$

This setting is optimal for the $\mathcal{H}_-/\mathcal{H}_\infty$ FDI following Remark 4.2. First we consider the scenario given in [12]: the disturbance is taken as $d(t) = [1.2\sin(2t)e^{-0.1t} \quad 1.5\cos(2t)e^{-0.1t}]^T$ and the fault $f_1$ is defined as $f_1(t) = 0.5,\ 5s \le t \le 10s$ and $f_1(t) = 0$ elsewhere. Figure 1 shows the residuals and highlights the fact that our residual based on $\mathcal{H}_-/\mathcal{H}_\infty$ is more robust to disturbances than the others. Note that the residuals in [12] and [10] are of dimension 2 since in their scheme the dimension of the residual is the same as the number of outputs.

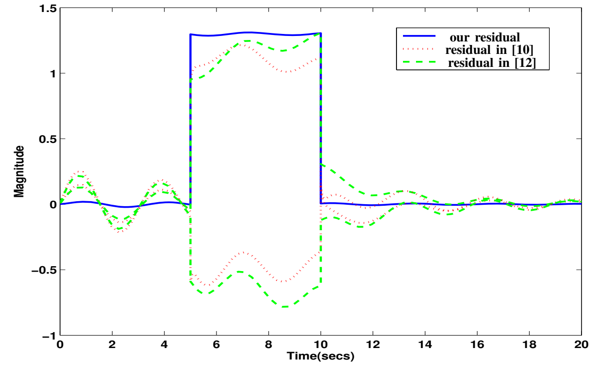


Fig. 1. residual time responses

Next, we consider the same system but this time subject to two white noise disturbances of mean zero and covariance 7 connected at $t = 0$. A fault is simulated by a unit positive jump connected at $t = 22s$. Figure 2 shows the residuals. This example justifies the use of $\mathcal{H}_-/\mathcal{H}_\infty$ schemes over $\mathcal{H}_\infty/\mathcal{H}_\infty$.

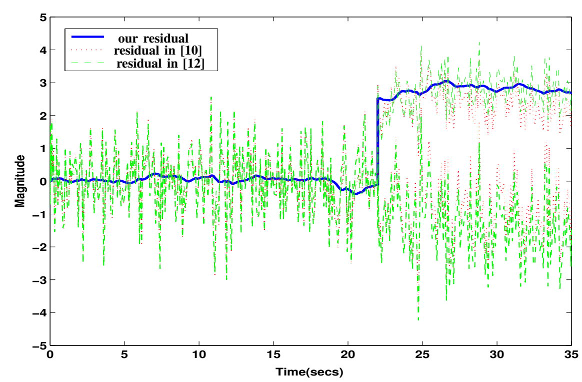


Fig. 2. residual time responses

## B. Randomly generated example

We consider the following randomly generated data:

$$A = \begin{bmatrix} -30.4438 & 4.8715 & -0.3643 & -23.6674 & -27.0508 \\ 9.8449 & -7.1563 & 6.7607 & 5.6055 & 1.1667 \\ -0.4460 & -2.0473 & -5.2344 & 0.0314 & 0.5579 \\ 23.0620 & 3.3600 & 5.1258 & 19.5067 & 27.8028 \\ -2.9175 & -8.1727 & -6.1518 & -4.5557 & -10.6214 \end{bmatrix},$$

$$C = \begin{bmatrix} -0.9800 & -2.1080 & -0.6891 & -0.4471 & 0.3988 \\ -0.3647 & -0.8006 & 0.4282 & 0.2585 & 1.4254 \\ 0.9081 & 0.5181 & -1.1179 & 0.2686 & 0.9972 \end{bmatrix},$$

$$B_f = \begin{bmatrix} -0.8999 & 1.7625 \\ 0.1478 & 0.1409 \\ -1.3598 & 1.1176 \\ -1.3020 & 0.7315 \\ 0.8723 & -0.4119 \end{bmatrix}, \quad D_f = \begin{bmatrix} -0.8032 & -0.1423 \\ -0.9889 & 1.4751 \\ -1.0297 & -0.4817 \end{bmatrix},$$

$$B_d = \begin{bmatrix} 0.0880 & 1.2801 \\ -0.9907 & 0.2294 \\ 1.3235 & 0.2285 \\ 1.1952 & 0.5443 \\ -0.9682 & 0.9220 \end{bmatrix}, \quad D_d = \begin{bmatrix} 0.0905 & -0.3783 \\ 0.8423 & -1.8179 \\ 0.4173 & 1.1086 \end{bmatrix}.$$

Our algorithm gives $\gamma = 0.8170$ and an unstable $A + L_0C$. It follows that fault isolation is not possible; however, from section IV, we can get the optimal solution and ensure fault isolation at DC by setting

$$Z_1 = \begin{bmatrix} 0.9353 & 0.2914 \\ -0.3267 & -0.1976 \\ -0.1134 & -0.0265 \\ -2.3823 & -0.9739 \\ 1.1268 & 0.6113 \end{bmatrix}, \quad U_1 = \begin{bmatrix} 0.9970 & -0.0778 \\ 0.0778 & 0.9970 \end{bmatrix}.$$

We consider that the system is subject to a disturbance, which is a white noise with mean zero and standard deviation 1. Two faults are simulated by a unit negative and positive jump respectively, connected from the $12^{th}$ and $35^{th}$ second. Figure 3 shows the residuals. It is clear to see that fault isolation is reached at DC and the impact of disturbances on the residuals has been attenuated. Next, we use the results

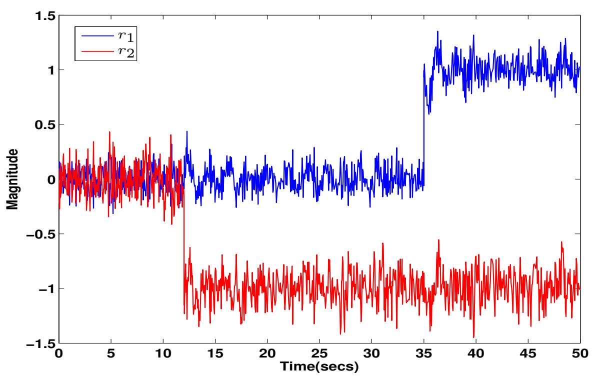


Fig. 3. residual time responses

of section V to implement a weighting filter scheme. A bandpass filter

$$W_{\omega_0} = \frac{0.516s^2 + 26.83s + 51.6}{s^2 + 26.83s + 100},$$

centered around a frequency $\omega_0 = 10$ rad/s is used. The optimal value of $\gamma$ is now $\gamma = 0.5898$. The modified residual responses are also provided in Figure 4. The result validates the employment of a weighting filter. Note that with the design incorporating the weights, it is easier to determine a residual threshold for fault detection.

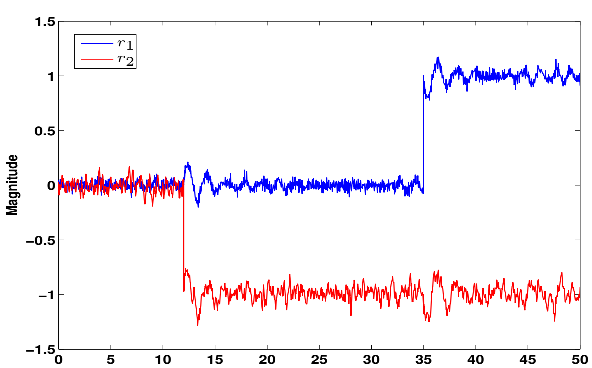


Fig. 4. residual time responses with weights

## VII. Conclusion

We have considered an $\mathcal{H}_-/\mathcal{H}_\infty$ fault detection problem for linear time-invariant systems subject to faults and disturbances using a static observer framework. Firstly, we proposed a novel $\mathcal{H}_\infty$-based performance index to minimize the sensitivity of the residual to disturbances with respect to faults. Then, by using a parameterization of the solutions of this problem, we constructed an optimal solution to the $\mathcal{H}_-/\mathcal{H}_\infty$ fault detection problem which satisfies some further fault isolation properties. Frequency weighting filters are employed to further improve the FD performance. Two examples demonstrated the effectiveness of our approach.

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