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A New Approach for Planar Tracking in a Nongaussian Setting

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Abstract— **This paper describes a new efficient approach to the conventional nonlinear tracking problem in a nongaussian setting that consists in the transformation of the nonlinear output measurement function in a linear form by the definition of a virtual measurement process. Such a procedure leads to the use of an efficient filter capable to take into account the nongaussanity of the transformed measurement noise process. This key feature is also exploited to consider and suitably manage a nongaussian and more realistic motion behaviour of the target object.**

Compared with the traditional approaches (e.g., extended Kalman filter (EKF) and unscented Kalman filter (UKF)) used in passive localization, the proposed method has potential advantages in robustness, convergence speed, and tracking accuracy.

I. Introduction

The target tracking is a widely studied problem of nonlinear filtering. The general purpose is the real-time estimation of the kinematic state of a moving object. The main difficulty of this aim arises from the strong nonlinear nature of the available measurements. Indeed, the linearization introduced by traditional solutions based on the extended Kalman filter (EKF) algorithm may return large errors in the state estimate, leading to sub-optimal performance and sometimes filter divergence [20].

This problem arises in a wide class of similar applications (e.g. SLAM [10], [5], [13]). As a consequence, a great deal of attention has been devoted to the development of solutions capable to reduce the linearization error. Prior works suggest suitable modifications of the EKF algorithm [15], [7] or iterative repetitions of the filtering steps [2]. Further works introduce more efficient solutions such as the unscented Kalman filter (UKF) [8], [9], the iterated unscented Kalman filter (IUKF) [20], and the particle filter [1].

In particular, the UKF succeeds in overcome the EKF with comparable computational complexity [11], [18]. The key idea of this technique is to approximate the probability distribution instead of the nonlinear system equations. In the application of target tracking, however, the UKF also shows weakness in robustness and tracking accuracy because of the large initial error and weak observability of the system. In the IUKF a repeated correction procedure is added to the standard UKF leading to an improved estimation accuracy. Obviously, this is paid in terms of computational complexity which grows linearly on the state dimension. For simple applications the additive computational load given by this extra iterative procedure does not represent a problem. On the contrary, for real application, it may be considered a significant limitation. New solutions able to outperform standard EKF and UKF with severe nonlinearities in the measurement process recently emerged, e.g. [16], [17].

Nonlinearity is not the only difficulty which arises in the scenario of target tracking. What makes really hard to solve this problem is the unknown behaviour of the moving object. Indeed, any dynamic model can only give weak information about the target movements based on assumptions on the maneuvering capability and external disturbance (e.g. atmospheric turbulences). However, especially when measurements are strongly corrupted by noise end/or the measure rate is low, an accurate stochastic dynamic model of target motion may have a crucial role in the estimation process. On the other hand, a model must be sufficiently simple to permit ready implementations.

Such a model is usually based on the fact that the moving objects under consideration generally follow straight line constant velocity trajectories. If the targets were not able to deviate from these trajectories than the tracking problem could be solved quickly and simply using standard filtering algorithms. However, the maneuver capability of the considered target constitutes the single feature that makes these algorithms generally unsuitable for accurate tracking [14]. This maneuver capability is usually modelled by random acceleration terms within the dynamical equations. The most correct representation of the target behaviour arises with a nongaussian statistical characterization of these random terms [12]. Existing methods generally use a Gaussian approximation of the acceleration terms tacking into account the only first and second order moments of the statistical distributions. If the target behaviour is really nongaussian, a second order approximation may be not sufficient for accurate tracking.

In this paper a new approach to the planar tracking is suggested. This new method attempts to deal with both the mentioned difficulties of the problem. The main idea is to use the range and angle measurements as time-varying parameters of the output matrix and of a virtual output vector. This simple manipulation allows the proposed algorithm to avoid any linearization of the measurements. On the other hand, it makes the measurement noise nongaussian. In [6] a similar technique was introduced showing accurate performances on a Gaussian setting. The nongaussianity of the transformed measurement process was tackled with a polynomial approach which is able to take into account statistical moments up to a certain order [4]. This feature of

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the proposed algorithm naturally holds for the state noise. This means that such a new approach is also capable to manage the nongaussian behaviour of the target in a more accurate way with respect to the existing methods. In this work a modified quadratic form of the basic algorithm in [6] is developed. The resulting technique is able to outperform both the EKF and the UKF in the considered nongaussian setting, as shown by numerical results.

The paper is organized as follows. In section II the tracking problem is formalized. Section III introduces the new approach. Results are provided in section IV. Finally, conclusions are summarized in Section V.

II. Problem Formulation

In this paper, a kinematic model for planar motion is considered where the state is given by position, velocity and acceleration in the 2-D plane, namely

$$x = [x_1, x_2, \dot{x}_1, \dot{x}_2, \ddot{x}_1, \ddot{x}_2]^T.$$

The state equation in Cartesian Coordinate System (CCS) coordinates (x_1, x_2) can be represented by the vector-matrix equation in the form

$$x(k+1) = Ax(k) + Fa_k, \tag{1}$$

with

$$A = \begin{bmatrix} I_2 & \Delta I_2 & \frac{\Delta^2}{2} I_2 \\ 0_2 & I_2 & \Delta I_2 \\ 0_2 & 0_2 & I_2 \end{bmatrix}, \; F = \begin{bmatrix} 0_2 \\ 0_2 \\ I_2 \end{bmatrix}, \tag{2}$$

where I_n and 0_n denote the $n \times n$ identity matrix and the null block matrix, respectively, and $a_k = [a_{1,k}, a_{2,k}]^T$ is the random acceleration noise along the x_1 and x_2 axis. It is assumed that the sequence $\{a_k\}$ is white and that the accidental acceleration perturbations $a_{1,k}$ and $a_{2,k}$ at each measurement time step are completely uncorrelated. Moreover, the object is supposed to experience a disturbance of $\pm A_m$ P_1 of time, zero acceleration P_2 of time, and for the rest of time a uniform distributed acceleration disturbance, as shown in Fig. 1. The statistical moments of this nongaussian distribution are given by the following equation:

$$\psi_{a_\ell}^{(i)} := \mathbb{E}\left[a_{\ell,k}^i\right] = \begin{cases} 0 & i \text{ odd;} \\ A_m^i \left(\frac{1-(P_2+2P_1)}{i+1} + 2P_2\right) & i \text{ even;} \end{cases} \tag{3}$$

for any order $i = 1, 2, \ldots$ and $\ell = 1, 2$.

As usual, the measurements process, at each time instant k, is given by the noisy values of the radius $\rho_m(k)$ and the angle position $\theta_m(k)$ of the target, respectively:

$$\rho_m(k) = \rho(k) + n_{\rho,k} = \sqrt{x_1^2(k) + x_2^2(k)} + n_{\rho,k} \tag{4}$$

$$\theta_m(k) = \theta(k) + n_{\theta,k} = \tan^{-1}\left(\frac{x_2(k)}{x_1(k)}\right) + n_{\theta,k} \tag{5}$$

where $n_{\rho,k}$ and $n_{\theta,k}$ denote the measurement errors, assumed to be independent zero-mean white sequence with σ_ρ^2 and σ_θ^2 variances, respectively.

Since the measurement process is nonlinear, the state estimation requires a nonlinear algorithm, that is, in general,

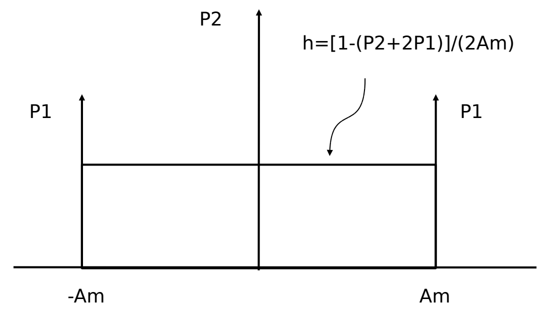

Fig. 1. Probability density function of range acceleration

an infinite dimensional problem. Therefore only suboptimal algorithms can be used for engineering applications, for example EKF and UKF. Here, a new approach to planar tracking is adopted which is based on a suitable transformation of the measurements in order to obtain a linear output model. This will be performed at the expense of loosing the Gaussianity of the measurement noise and the time-invariant property of the output matrix [6]. Finally a quadratic filter for the linear nongaussian model could be applied to improve the performance of Kalman filtering. The quadratic form is here preferred to the linear one since in [6] it is proved that performances increase with the filter order.

III. The New Filter

A. The Virtual Measurement Map

Taking account that

$$\begin{cases} x_1(k)cos(\theta(k)) = \rho(k)\ cos^2(\theta(k)) \\ x_2(k)sin(\theta(k)) = \rho(k)\ sin^2(\theta(k)) \end{cases} \tag{6}$$

and the first definition in (4)-(5), it readily follows that

$$\begin{aligned} &\rho_m(k) - n_{\rho,k} = \rho(k) = \\ &= cos(\theta_m(k) - n_{\theta,k})x_1(k) + sin(\theta_m(k) - n_{\theta,k})x_2(k) \\ &= C_1(\theta_m(k), n_{\theta,k})x(k), \end{aligned} \tag{7}$$

where

$$C_1(\theta, n) = \left[\begin{array}{cccccc} cos(\theta - n) & sin(\theta - n) & 0 & 0 & 0 & 0 \end{array}\right].$$

Moreover, from (4)-(5), it follows:

$$\theta(k) = \theta_m(k) - n_{\theta,k}, \qquad \frac{sin(\theta(k))}{cos(\theta(k))} = \frac{x_2(k)}{x_1(k)},$$

from which:

$$\begin{aligned} 0 &= -sin(\theta_m(k) - n_{\theta,k})x_1(k) + cos(\theta_m(k) - n_{\theta,k})x_2(k) \\ &= C_2(\theta_m(k), n_{\theta,k})x(k), \end{aligned} \tag{8}$$

where

$$C_2(\theta, n) = \left[\begin{array}{cccccc} -sin(\theta - n) & cos(\theta - n) & 0 & 0 & 0 & 0 \end{array}\right].$$

Equations (7) and (8) realize in a linear form the nonlinear output measurements (4) in a nongaussian setting. Now, let the virtual output be the sequence

$$y_v(k) = \begin{bmatrix} \rho_m(k) \\ 0 \end{bmatrix},$$

so that the following output measurement map holds true:

$$y_v(k) = C(\theta_m(k), n_{\theta,k})x(k) + Gn_{\rho,k}, \tag{9}$$

where

$$C(\theta_m(k), n_{\theta,k}) = \begin{bmatrix} C_1(\theta_m(k), n_{\theta,k}) \\ C_2(\theta_m(k), n_{\theta,k}) \end{bmatrix}, G = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \tag{10}$$

Remark 1: Note that the definition of a virtual output leads to rewrite the nonlinear measurement process in the linear form (9) where the output matrix becomes to be stochastic. For such kind of systems, sub-optimal filtering algorithms are available in literature (e.g. [19]).

For the sequel, it will be useful to recognize that the matrix in (10) can be factorized as

$$C(\theta, n) = R(\theta - n)S, \tag{11}$$

where

$$R(\alpha) = \begin{bmatrix} cos(\alpha) & sin(\alpha) \\ -sin(\alpha) & cos(\alpha) \end{bmatrix}, \ S = \begin{bmatrix} I_2 & 0_2 & 0_2 \end{bmatrix}. \tag{12}$$

Moreover, note that the rotation matrix $R(\alpha)$ satisfies the following properties for any $\alpha, \beta \in \mathbb{R}$:

$$R(\alpha + \beta) = R(\alpha)R(\beta), \tag{13}$$
$$R^{-1}(\alpha) = R^T(\alpha) = R(-\alpha). \tag{14}$$

The described manipulation makes the considered setting nongaussian for both the state and the measurement noises. Because of this nongaussianity, the performances of the filter could be not full satisfying for specifically applications. Significant improvements could be derived by using specific algorithms which take into account the shape of the distributions as suggested in [6]. Next section will be devoted to the definition of a filter which takes into account higher statistical moments of the noise sequences.

B. Quadratic Extension

In order to suitably solve the introduced nonlinear nongaussian problem, a polynomial filtering approach [4] can be used. In particular, such a nonlinear estimate can be obtained through a filtering process computed on a system whose output vector carries the Kronecker powers of the original output vector up to a certain order. In this work this order is set equal to two. The resulting algorithm will be thus quadratic and able to take into account the statistical moments of the noise sequences up to the fourth order.

At first, the extended state vector is defined as

$$\mathcal{X}(k) := \begin{bmatrix} x(k) \\ x^{[2]}(k) \end{bmatrix} \in \mathbb{R}^N,$$

where $N = 6+6^2 = 42$ is the extended state dimension. Furthermore, the statistical moments of the state noise sequence are given by

$$\psi_a^{(i)} := \mathbb{E}\left[a_k^{[i]}\right].$$

Note that the i-th moment of the vector a_k carries elements of the form $\psi_{a_1}^{(p)} \cdot \psi_{a_2}^{(q)}$, $p + q = i$, i.e. the i-th moments of the nongaussian state noise. Finally, let $\mathring{a}_k^{(i)}$ be the random zero-mean vector

$$\mathring{a}_k^{(i)} := a_k^{[i]} - \psi_a^{(i)}.$$

Taking into account these last definitions, properties in Lemmas 1 and 2, and (1), the second-order Kronecker power of the state vector $x(k)$ (i.e. the second block-entry of the extended state) results to satisfy the following equation:

$$\begin{aligned} x^{[2]}(k+1) =& A^{[2]}x^{[2]}(k) + F^{[2]}a_k^{[2]} \\ &+ M_6((Ax(k)) \otimes (Fa_k)) \\ =& A^{[2]}x^{[2]}(k) + F^{[2]}\psi_a^{(2)} + F^{[2]}\mathring{a}_k^{(2)} \\ &+ M_6((Ax(k)) \otimes (Fa_k)). \end{aligned} \tag{15}$$

From (1) and (15) the extended state equation is obtained as follows:

$$\mathcal{X}(k+1) = \mathcal{A}\mathcal{X}(k) + \mathcal{U} + \mathcal{V}_k, \tag{16}$$

where

$$\mathcal{A} = \begin{bmatrix} A & 0_{6\times 6^2} \\ 0_{6^2\times 6} & A^{[2]} \end{bmatrix}$$

is the extended dynamic matrix ($0_{n\times m}$ denoting the $n \times m$ null matrix),

$$\mathcal{U} = \begin{bmatrix} 0_{6\times 1} \\ F^{[2]}\psi_a^{(2)} \end{bmatrix}$$

is a deterministic term, and

$$\mathcal{V}_k = \begin{bmatrix} Fa_k \\ F^{[2]}\mathring{a}_k^{(2)} + M_6((Ax(k)) \otimes (Fa_k)) \end{bmatrix}$$

is a is a zero-mean multiplicative state noise vector. It can be proved that the random sequence $\{\mathcal{V}(k)\}$ is a white sequence. Moreover, the corresponding covariance matrix can be computed exploiting property (27) in Lemma 2. The resulting expression follows:

$$Q(k) = \mathbb{E}\left[\mathcal{V}_k\mathcal{V}_k^T\right] = \begin{bmatrix} Q_{11}(k) & Q_{12}(k) \\ Q_{12}^T(k) & Q_{22}(k) \end{bmatrix}$$

with

$$\begin{aligned} Q_{11}(k) &= Fst_2^{-1}(\psi_a^{(2)})F^T \\ Q_{12}(k) &= \left(\left(A^T\mathbb{E}[x^T(k)]\right) \otimes \left(Fst_2^{-1}\left(\psi_a^{(2)}\right)F^T\right)\right)M_6^T \\ Q_{22}(k) &= F^{[2]}\left(st_4^{-1}(\psi_a^{(4)}) - \psi_a^{(2)}\psi_a^{(2)T}\right)F^{T[2]} \\ &+M_6\left(\left(A\Psi_x(k)A^T\right) \otimes \left(Fst_2^{-1}\left(\psi_a^{(2)}\right)F^T\right)\right)M_6^T \end{aligned} \tag{17}$$

where $\Psi_x(k) = \mathbb{E}[x(k)x^T(k)]$. Note that this covariance matrix depends on the statistical moments of the nongaussian state noise up to the fourth order.

The definition of the extended output vector does not differ from that in [6]. However, to be more clear, the particular second order case is described in the sequel.

Preliminary definition are given for $i = 1, 2$:

$$\psi_{n_\rho}^{(i)} := \mathbb{E}\left[n_{\rho,k}^i\right], \tag{18}$$

$$\overline{R}_{\sigma_\theta}^{(i)} := \mathbb{E}\left[R^{[i]}\left(-n_{\theta,k}\right)\right], \tag{19}$$

$$\mathring{n}_{\rho,k}^i := n_{\rho,k}^{(i)} - \psi_{n_\rho}^{(i)}, \tag{20}$$

$$\mathring{R}^{(i)}\left(-n_{\theta,k}\right) := R^{(i)}\left(-n_{\theta,k}\right) - \overline{R}_{\sigma_\theta}^{(i)}. \tag{21}$$

Note that the mean value of the random quantities introduced in (18) and (19) can be easily computed basing on the knowledge of the statistical moments of the measurement noise sequences, which are supposed to be Gaussian. Moreover, note that the quantities in (20) and (21) are composed by zero-mean random variables.

Exploiting (13)-(14), (9) can be rewritten as

$$R^T(\theta_m(k))y_v(k) = R(-n_{\theta,k})Sx(k) + R^T(\theta_m(k))Gn_{\rho,k}. \tag{22}$$

The extended output vector will be composed by the left-side hand of (22) and its quadratic Kronecker power. Therefore, (22) will be manipulated for linear and quadratic components, respectively, taking into account (18)-(21) and Lemmas 1 and 2.

Linear components:

$$\begin{aligned} R^T(\theta_m(k))y_v(k) =& \overline{R}_{\sigma_\theta}^{(1)}Sx(k) + \mathring{R}^{(1)}\left(-n_{\theta,k}\right)Sx(k) \\ &+ R^T(\theta_m(k))Gn_{\rho,k} \\ =& \mathcal{C}_1\mathcal{X}(k) + \mathcal{W}_{1,k}, \end{aligned} \tag{23}$$

with

$$\mathcal{C}_1 = \left[\begin{array}{cc} \overline{R}_{\sigma_\theta}^{(1)}S & 0_{2\times 6^2} \end{array}\right],$$
$$\mathcal{W}_{1,k} = \mathring{R}^{(1)}\left(-n_{\theta,k}\right)Sx(k) + R^T(\theta_m(k))Gn_{\rho,k}.$$

Quadratic components:

$$\begin{aligned} & R^{[2]T}(\theta_m(k))y_v^{[2]}(k) = \\ &\quad = R^{[2]}(-n_{\theta,k})S^{[2]}x^{[2]}(k) + R^{[2]T}(\theta_m(k))G^{[2]}n_{\rho,k}^2 \\ &\quad + M_2\left(\left(R(-n_{\theta,k})Sx(k)\right) \otimes \left(R^T(\theta_m(k))Gn_{\rho,k}\right)\right) \\ &\quad = \overline{R}_{\sigma_\theta}^{(2)}S^{[2]}x^{[2]}(k) + \mathring{R}^{(2)}(-n_{\theta,k})S^{[2]}x^{[2]}(k) \\ &\quad + R^{[2]T}(\theta_m(k))G^{[2]}\psi_{n_\rho}^{(2)} + R^{[2]T}(\theta_m(k))G^{[2]}\mathring{n}_{\rho,k}^2 \\ &\quad + M_2\left(\left(\mathring{R}^{(1)}(-n_{\theta,k})Sx(k)\right) \otimes \left(R^T(\theta_m(k))Gn_{\rho,k}\right)\right) \\ &\quad + M_2\left(\left(\overline{R}_{\sigma_\theta}^{(1)}Sx(k)\right) \otimes \left(R^T(\theta_m(k))Gn_{\rho,k}\right)\right). \end{aligned} \tag{24}$$

Equation (24) can be rewritten as

$$R^{[2]T}(\theta_m(k))\left(y_v^{[2]}(k) - G^{[2]}\psi_{n_\rho}^{(2)}\right) = \mathcal{C}_2\mathcal{X}(k) + \mathcal{W}_{2,k}, \tag{25}$$

with

$$\mathcal{C}_2 = \left[\begin{array}{cc} 0_{2^2\times 6} & \overline{R}_{\sigma_\theta}^{(2)}S^{[2]} \end{array}\right],$$
$$\begin{aligned} \mathcal{W}_{2,k} =& \mathring{R}^{(2)}(-n_{\theta,k})S^{[2]}x^{[2]}(k) \\ &+ R^{[2]T}(\theta_m(k))G^{[2]}\mathring{n}_{\rho,k}^2 \\ &+ M_2\left(\left(\mathring{R}^{(1)}(-n_{\theta,k})Sx(k)\right) \otimes \left(R^T(\theta_m(k))Gn_{\rho,k}\right)\right) \\ &+ M_2\left(\left(\overline{R}_{\sigma_\theta}^{(1)}Sx(k)\right) \otimes \left(R^T(\theta_m(k))Gn_{\rho,k}\right)\right). \end{aligned}$$

Finally, by defining the extended virtual output vector as

$$\mathcal{Y}_v(k) = \left[\begin{array}{c} R^T(\theta_m(k))y_v(k) \\ R^{[2]T}(\theta_m(k))\left(y_v^{[2]}(k) - G^{[2]}\psi_{n_\rho}^{(2)}\right) \end{array}\right] \in \mathbb{R}^q,$$

$q = 2 + 2^2 = 6$, from (23) and (25), the extended output results to have the form

$$\mathcal{Y}_v(k) = \mathcal{C}\mathcal{X}(k) + \mathcal{W}_k, \tag{26}$$

where

$$\mathcal{C} = \left[\begin{array}{c} \mathcal{C}_1 \\ \mathcal{C}_2 \end{array}\right] \text{ and } \mathcal{W}_k = \left[\begin{array}{c} \mathcal{W}_{1,k} \\ \mathcal{W}_{2,k} \end{array}\right]$$

are the extended output matrix and the output zero mean noise sequence. This last can be proved to be a nongaussian white random sequence.

The system modelled by (16) and (26) satisfies the requirements for applying the standard linear Kalman algorithm that, in this case, constitutes the optimal linear filter with respect to the virtual output process. It is remarkable that Kalman filtering for (16) and (26) is only a linear optimal technique because of the nongaussianity of state and output noises. On the other hand, as shown in this section, the new filter takes into account the statistical moments of the noise sequences up to the fourth order, promising to better represent the nongaussian behaviour of the target and of the transformed output noise.

C. Reduced state-space filter

A considerable reduction of the filter state-space dimension can be obtained by eliminating the redundancy contained in the vector $\mathcal{X}(k)$. In fact, the second block entry of $\mathcal{X}(k)$ is composed by monomials in the form $x_l(k)x_p(k)$, $1 \le l \le p \le 6$. These terms do not change their values with a permutation of the indices l and p so that the same value can be repeated many times. This can be avoided by using a suitable definition of Kronecker power, instead of the classical one, which eliminates all redundancies, as suggested in [3]. This helps in reducing both memory space and computation time.

For the particular case of this paper, it is sufficient to define two suitable matrices $\tilde{T}$ and T such that:

$$x_{[2]}(k) = \tilde{T}x^{[2]}(k), \quad x^{[2]}(k) = Tx_{[2]}(k),$$
$$x^{[2]}(k) = T\tilde{T}x^{[2]}(k), \quad x_{[2]}(k) = \tilde{T}Tx_{[2]}(k),$$

where $x_{[2]}(k)$ is the reduced quadratic Kronecker power of $x(k)$ which contains the same quantities of $x^{[2]}(k)$ but without redundancies. Since the number of possible monomials is here 21, $x_{[2]}(k) \in \mathbb{R}^{21}$, $\tilde{T} \in \mathbb{R}^{21\times 36}$ and $T \in \mathbb{R}^{36\times 21}$. For details the reader is referred to [3].

Now, let

$$\tilde{T}_e = \begin{bmatrix} I_6 & 0 \\ 0 & \tilde{T} \end{bmatrix} \text{ and } T_e = \begin{bmatrix} I_6 & 0 \\ 0 & T \end{bmatrix}.$$

The reduced state can be so the vector

$$\mathcal{X}_r(k) = \begin{bmatrix} x(k) \\ x_{[2]}(k) \end{bmatrix} = \tilde{T}_e \mathcal{X}(k)$$

whose dimension is 27. The reduced filter can be finally obtained by applying the standard Kalman algorithm to the reduced system

$$\mathcal{X}_r(k+1) = \mathcal{A}_r \mathcal{X}_r(k) + \mathcal{U}_r + \mathcal{V}_{r,k},$$
$$\mathcal{Y}_v(k) = \mathcal{C}_r \mathcal{X}_r(k) + \mathcal{W}_k,$$

where

$$\mathcal{A}_r = \tilde{T}_e \mathcal{A} T_e, \quad \mathcal{U}_r = \tilde{T}_e \mathcal{U},$$
$$\mathcal{V}_{r,k} = \tilde{T}_e \mathcal{V}_k, \quad \mathcal{C}_r = \mathcal{C} T_e.$$

IV. Simulation Results

In this section it is presented the performance of the new filter VOKF (Virtual Output Kalman Filter) for simulated typical tracking settings. Moreover a comparison with the classical EKF and UKF solutions is shown.

In the simulation, the angle standard deviation (σ_θ) is considered within the set $[0.87, 1.31, 1.75, 2.18, 2.62]e-2$ rad and the radius standard deviation (σ_ρ) is 0.35 m. The initial state for the filters is calculated from the first measurements. The random acceleration noise is assumed identically distributed for two axis, with $A_m = ge^{-4}$ ($g = 9.8m/s^2$), $P_2 = 30\%$ and $P_1 = 10\%$.

The evaluation metric of interest is the standard relative position error ($RPE_i(k)$) which is defined for each sample measurement noise realization i at time k as

$$RPE_i(k) = \frac{\sqrt{\left(\hat{e}_1^{(i)}(k)\right)^2 + \left(\hat{e}_2^{(i)}(k)\right)^2}}{\sqrt{\left(x_1^{(i)}(k)\right)^2 + \left(x_2^{(i)}(k)\right)^2}} \cdot 100\%,$$

with $\hat{e}_l^{(i)}(k) = x_l^{(i)}(k) - \hat{x}_l^{(i)}(k),\ l = 1,2,\ \hat{x}_l^{(i)}(k)$ being the estimated state vector.

The average value of $RPE_i(k)$ with respect the subset of samples corresponding to stable behaviour of the filters, over a total number of 100 samples will be denoted by $RPE(k)$. The average value of $RPE(k)$ with respect time will be indicated with RPE. A run is considered to be convergent only if the $RPE_i(k) < 15\%$ at the end of any simulation.

These parameters were computed for 100 different trajectories (i.e. 100 different realizations of the nongaussian state noise), obtaining strictly similar results.

Below the results of two simulations with different kinematic range are presented considering the movement of a slow (e.g. planar robot) and a fast (e.g. airplane) object.

TABLE I
NUMERICAL RPE RESULTS

	σ_θ				
	0.87e-2	1.31e-2	1.75e-2	2.18e-2	2.62e-2
VOKF	0.3695	0.5195	0.6939	0.8342	0.9141
UKF	0.3788	0.5894	0.8244	1.0405	1.2012
EKF	0.3789	0.5932	0.8413	1.0869	1.3041

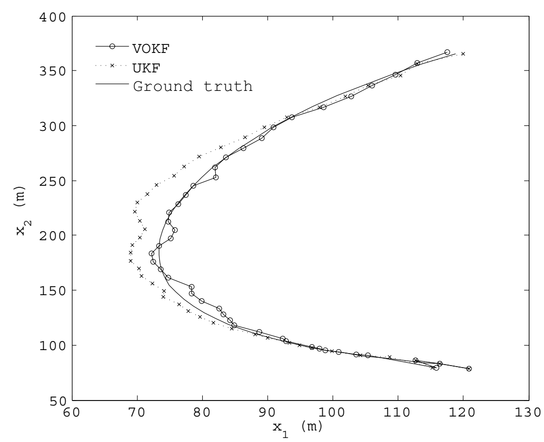

Fig. 2. Estimation result for a typical trajectory in CCS [$x_1(0) = 120$, $x_2(0) = 80$, $\sigma_\theta = 2.63e-2$ rad, $\sigma_\rho = 0.35m$]

A. Slow object

It is assumed that the observer is located at the axis origin, and the target moves at a nearly constant velocity in the 2-D plane, with initial state:

$$x(0) = \begin{bmatrix} 120m & 80m & -1.5\frac{m}{s} & 1\frac{m}{s} & 0\frac{m}{s^2} & 0\frac{m}{s^2} \end{bmatrix}^T$$

and the measurement process is available with a sampling time of 2 s.

In Table I the RPE for the given set of different σ_θ values for VOKF, UKF and EKF algorithms is presented. It appears clear that the VOKF has superior performance for higher values of angle noise variance, while EKF shows worst behaviour for high noise when the linearization generates significant approximation errors.

In Fig. 2 the estimation result for a typical trajectory in CCS is graphically represented.

B. Fast object

In this simulation, the target moves with initial state

$$x(0) = \begin{bmatrix} 12km & 8km & -120\frac{m}{s} & 15\frac{m}{s} & 0\frac{m}{s^2} & 0\frac{m}{s^2} \end{bmatrix}^T.$$

The EKF results to be non stable because for 95% of simulations is unable to converge, meanwhile both VOKF and UKF are stable. This is due to a larger initial estimation error.

In Table II the RPE for the given set of different σ_θ values for VOKF and UKF algorithms is presented.

In Fig 3 a typical behaviour of $RPE(k)$ is represented showing that the VOKF is superior to the UKF both in convergence time and tracking error.

TABLE II

NUMERICAL RPE RESULTS

	σ_θ				
	0.87e-2	1.31e-2	1.75e-2	2.18e-2	2.62e-2
VOKF	0.2371	0.3500	0.4881	0.6456	0.8047
UKF	0.2408	0.4299	0.6359	1.0551	1.0329

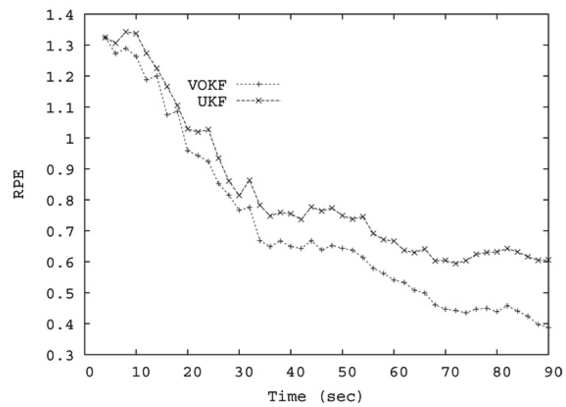

Fig. 3. Comparison of tracking results for VOKF and UKF algorithms with measurement precision [$\sigma_\theta = 1.75e-2$ rad, $\sigma_\rho = 0.35m$]

V. CONCLUSIONS

The idea here presented of the definition of a virtual output process capable to transform the nonlinear measurement output function with respect the system state in a linear time varying one, even losing the gaussianity property of the noise, seems to be very promising and effective.

Future works will be devoted to develop higher order filter to better manage nongaussanities.

APPENDIX

In the following Lemmas a few useful properties are summarized.

Lemma 1: For any matrices A, B, C, and D, it is

$$\begin{aligned}&(A+B)\otimes(C+D)=\\ &\quad=(A\otimes C)+(A\otimes D)+(B\otimes C)+(B\otimes D),\\ &(A\otimes B)\cdot(C\otimes D)=(A\cdot C)\otimes(B\cdot D),\\ &(A\otimes B)^T=A^T\otimes B^T,\end{aligned}$$

where $\cdot$ and $st(\cdot)$ denote the standard matrix product and the stack operation on a given matrix, respectively.

Lemma 2: Let $x, y \in \mathbb{R}^n$, it is

$$st(xx^T)=x^{[2]} \Leftrightarrow xx^T=st_n^{-1}(x^{[2]}), \qquad (27)$$

$$(x+y)^2=x^{[2]}+y^{[2]}+M_n(x\otimes y), \qquad (28)$$

where $st_n^{-1}(\cdot)$ denotes the inverse stack operation on a given vector and M_n is a suitably dimensioned commutation matrix.

For a detailed proof of Lemma 1 and 2, see [4].

REFERENCES

[1] M. S. Arulampalam, S. Maskell, N. Gordon, and T. Clapp. A tutorial on particle filters for online nonlinear/non-gaussian bayesian tracking. *IEEE Trans. Signal Processing*, 50(2), February 2002.

[2] Y. Bar-Shalom, X. R. Li, and T. Kirubarajan. *Estimation with Application to Tracking and Navigation*. Theory, Algorithm, and Software. Wiley, New York, 2001.

[3] F. Carravetta, A. Germani, and M. Raimondi. Polynomial filtering for linear discrete tiem non-gaussian systems. *SIAM J. Control and Optimization*, 34(5):1666–1690, September 1996.

[4] F. Carravetta, A. Germani, and M. Raimondi. Polynomial filtering of discrete-time stochastic linear systems with multiplicative state noise. *IEEE Transaction on Automatic Control*, 42(8):1106–1126, August 1997.

[5] J. A. Castellanos, J. Neira, and J. D. Tardos. Limits to consistency of the EKF-based SLAM. In *Intelligent Autonomous Vehicle (IAV 2004)*, Lisbon, Portugal, July 2004.

[6] F. Conte, V. Cusimano, and A. Germani. Optimal polynomial filtering for planar tracking via virtual measurement process. In *2011 International Federation of Automatic Control world congress (IFAC WC2011) (to be presented)*, 2011.

[7] F. Guo, Z. Sun, and K. Huangfu. A modified covariance extended Kalman filtering algorithm in passive location. In *Proceedings of IEEE International Conference on Robotics, Intelligent Systems and Signal Processing*, pages 307–311, October 2003.

[8] S. J. Julier and J. K. Uhlmann. A new extension of the Kalman filter to nonlinear systems. In *Proceedings of AeroSense: The 11th International Symposium on Aerospace/Defence Sensing, Simulation and Controls*, Orlando, FL, 1997.

[9] S. J. Julier and J. K. Uhlmann. A new method for the nonlinear transformation of means and covariances in filters and estimators. *IEEE Transaction on Automatic Control*, 45(3):477–482, March 2000.

[10] S. J. Julier and J. K. Uhlmann. A counter example to the theory of simultaneous localization and map building. In *International Conference on Robotics and Automation (ICRA01)*, Seoul, Korea, 2001.

[11] S. J. Julier and J. K. Uhlmann. Unscented filtering and nonlinear estimation. In *Proceedings of the IEEE*, volume 92, March 2004.

[12] Eyung W. Kang. *Radar system analysis, design, and simulation*. Artech house, Boston, 2008.

[13] A. Martinelli, V. Nguyen, N. Tomatis, and R. Siegwart. A relative map approach to SLAM based on shift and rotation invariants. *Robotics and Autonomous Systems*, 55:50–61, 2006.

[14] R. A. Singer. Estimating optimal tracking filter performance for manned maneuvering targets. *IEEE Transaction on Aerospace and Electronic Systems*, AES-6(4), 1970.

[15] T. L. Song and J. Speyer. A stochastic analysis of a modified gain extended Kalman filter with applications to estimation with bearings only measurement. *IEEE Transaction on Automatic Control*, AC-30(10):940–949, 1985.

[16] X. Tian and Y. Bar-Shalom. Coordinate conversion and tracking for very long range radars. *IEEE Transaction on Aerospace and Electronic Systems*, AES-45(3):1073–1088, July 2009.

[17] X. Tian, Y. Bar-Shalom, G. Chen, E. Blasch, and K. Pham. A novel filtering approach for the general contact lens problem with range rate measurements. In *13th International Conference on Information Fusion*, Edimburgh, Scotland, July 2010.

[18] R. van der Merwe, E. A. Wan, and S. J. Julier. Sigma-point Kalman filters for nonlinear estimation and sensor-fusion: Applications to integrated navigation. In *Proceedings of AIAA Guidance Navigation and Controls Conference*, Providence, RI, 2004.

[19] R. Won-Sang, W. Ick-Ho, and P. Jin Bae. Less conservative robust kalman filtering using noise corrupted measurement matrix for discrete linear time-varying system. In *IEEE Intern. Conf. on Industrial Technology (ICT)*, February 2009.

[20] R. Zhan and J. Wan. Iterated unscented Kalman filter for passive target tracking. *IEEE Transaction on Aerospace and Electronic Systems*, 43(3):1155–1162, July 2007.