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SVD-Based Computation of Zeros of Polynomial Matrices

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***Abstract*— We present an algorithm for determining the zeros of polynomial matrices of arbitrary order, normal rank, and dimension. Specifically, we use the singular value decomposition to reduce the problem to an eigenvalue problem.**

I. Introduction

Polynomials, as the basis for ordinary differential and difference equations, pervade almost every aspect of engineering [1–3]. Regardless of whether one is working in the continuous-time domain, where one considers polynomials in the differentiation operator, in the discrete-time domain, where one considers polynomials in the backshift operator, or in any other domain, the zeros of a polynomial typify the dynamics and overall stability of the problem at hand [3–5]. Hence the ability to compute the zeros of a polynomial reliably is of prime importance for practical problems.

When handling scalar polynomials, as is the case for SISO systems, the problem of determining zeros robustly is well understood, and various algorithms are available for computing the zeros of a scalar polynomial [6–8]. However, when dealing with polynomial matrices, the problem is not as clear. Although a theoretical basis for the zeros of a polynomial matrix is provided via the Smith and Hermite forms, or the determinant if the the polynomial matrix is square, their computation is typically carried out symbolically [9–11], and hence is not amenable to many practical applications. Furthermore, although an extensive treatment of linearizations of polynomial matrices can be found [11–14], much of the attention has been devoted to computing the generalized eigenvalues of polynomial matrix linearizations [11, 15, 16]. However, a simple example (Example 2.1) we provide in the present paper shows that the generalized eigenvalues are not necessarily the same as the zeros, even though it appears that this fact is known [11, 14]. In fact, an entire literature has sprung up regarding these "infinite zeros" which are responsible for the difference between the generalized eigenvalues of polynomial matrix linearizations and the zeros of polynomial matrices [17, 18].

Here we present a direct, numerical algorithm for computing the zeros of a polynomial matrix which relies solely on the most basic properties of polynomial matrices and does not encite the need to discuss these "infinite zeros" or other unnecessary facts such as row/column reducedness. The contents of the paper are as follows. First, we present the necessary preliminaries concerning polynomial matrices, allowing us to build the rest of the paper from the most basic polynomial matrix facts. Then, after introducing the problem statement, we present our numerical algorithm for computing the zeros of a polynomial matrix. Finally, we present several numerical examples, and conclusions.

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II. Definitions

In this section, we introduce polynomial matrices, normal rank, and zeros. Although many of these definitions can be found in the literature [11], we repeat them here for completeness.

Definition *2.1:* Let $C_0, C_1, \ldots, C_n \in \mathbb{R}^{p\times m}$ and

$$C(\boldsymbol{\lambda}) \triangleq C_n\boldsymbol{\lambda}^n + \cdots + C_1\boldsymbol{\lambda} + C_0. \tag{1}$$

Then $C \in \mathbb{R}^{p\times m}[\boldsymbol{\lambda}]$.

Definition *2.2:* Let $C \in \mathbb{R}^{p\times m}[\boldsymbol{\lambda}]$ and let n be the smallest nonnegative integer such that $C(\boldsymbol{\lambda})$ is of the form (1). Then the *order of* $C(\boldsymbol{\lambda})$ is n if $C(\boldsymbol{\lambda})$ is nonzero, and $-\infty$ if $C(\boldsymbol{\lambda})$ is zero.

Remark *2.1:* In the literature, (1) is sometimes referred to as a matrix pencil, with the common case being the linear (or first-order) matrix pencil $C(\boldsymbol{\lambda}) = A - \boldsymbol{\lambda}B$.

Definition *2.3:* Let $C \in \mathbb{R}^{p\times m}[\boldsymbol{\lambda}]$. Then the *normal rank of* $C(\boldsymbol{\lambda})$ is $\max_{z\in\mathbb{C}} \operatorname{rank}\big[C(z)\big]$. Specifically, we write

$$\operatorname{nrank}\big[C(\boldsymbol{\lambda})\big] \triangleq \max_{z\in\mathbb{C}} \operatorname{rank}\big[C(z)\big].$$

Furthermore, $C(\boldsymbol{\lambda})$ has *full normal rank* if $m = p$ and $\operatorname{nrank}\big[C(\boldsymbol{\lambda})\big] = p$.

Definition *2.4:* Let $C \in \mathbb{R}^{p\times m}[\boldsymbol{\lambda}]$. Then $z \in \mathbb{C}$ is a *zero of* $C(\boldsymbol{\lambda})$ if

$$\operatorname{rank}\left[C(z)\right] < \operatorname{nrank}\left[C(\boldsymbol{\lambda})\right].$$

Definition 2.4 implies that the zero polynomial matrix and all other constant matrices have no zeros. Furthermore, the problem of determining the zeros of a linear matrix pencil with full normal rank is equivalent to the generalized eigenvalue problem. However, when a first-order polynomial matrix is rectangular or does not have full normal rank, then the generalized eigenvalue problem does not, in general, return the zeros. This is true for generalized eigenvalue solvers that uses the QZ decomposition [7, 8], as we demonstrate in the following example.

Example *2.1:* Let

$$C(\boldsymbol{\lambda}) \triangleq \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\boldsymbol{\lambda} - \begin{bmatrix} 0 & \alpha \\ 0 & 0 \end{bmatrix}.$$

Then $\operatorname{nrank}\left[C(\boldsymbol{\lambda})\right] = 1$ and the only zero of $C(\boldsymbol{\lambda})$ is α. However, the QZ-algorithm leaves $C(\boldsymbol{\lambda})$ unchanged since $C(\boldsymbol{\lambda})$ is already upper-triangular. Hence generalized eigenvalue solvers that employ the QZ decomposition return, as generalized eigenvalues, the ratios $0/0$ and $0/0$ [7]. □

III. Problem Formulation

Given a polynomial matrix $P \in \mathbb{R}^{p\times m}[\boldsymbol{\lambda}]$, determine all of the zeros of $P(\boldsymbol{\lambda})$.

IV. Zeros of a Polynomial Matrix

In this section we present a method for computing the zeros of a polynomial matrix. Although some of these results may again be found in the literature [11], we present them again for completeness with our own proofs, so as to guide the reader in the development of the algorithm.

***Lemma** 4.1:* Let $A \in \mathbb{C}^{p\times m}$, $B \in \mathbb{C}^{n\times m}$, and

$$C \triangleq \begin{bmatrix} A \\ B \end{bmatrix}.$$

Also, let $V \in \mathbb{C}^{m\times \ell}$ be a basis for the nullspace of B. Then

$$\text{nullity}\,(C) = \text{nullity}\,(AV)\,. \tag{2}$$

Proof: Let $T \in \mathbb{C}^{\ell\times k}$ be a basis for the nullspace of AV. Then letting $U \triangleq VT$, it follows that $CU = 0_{(p+n)\times k}$.

Next, suppose that U is not a complete basis for the nullspace of C, that is, suppose there exists an $x \in \mathcal{N}\,(C)$ such that $x \notin \mathcal{R}\,(U)$, where $\mathcal{N}\,(\cdot)$ and $\mathcal{R}\,(\cdot)$ denote the nullspace and rangespace, respectively. Then since $\mathcal{N}\,(C) \subseteq \mathcal{N}\,(B)$, it follows that there exists a $y \in \mathbb{C}^{\ell\times 1}$ such that $x = Vy$ and $y \notin \mathcal{R}\,(T)$. However, since $y \notin \mathcal{R}\,(T)$, it follows that $y \notin \mathcal{N}\,(AV)$, that is, $AVy = Ax \neq 0$, which contradicts the assumption that $x \in \mathcal{N}\,(C)$. Hence U is a complete basis for the nullspace of C.

Finally, since V is a basis, V has full column rank. Hence U has full column rank. Furthermore, since the dimension of the nullspace of C and AV both equal k, we have (2). □

***Fact** 4.1:* Let $A \in \mathbb{R}^{p\times m}[\boldsymbol{\lambda}]$, $B \in \mathbb{R}^{n\times m}$, and

$$C(\boldsymbol{\lambda}) \triangleq \begin{bmatrix} A(\boldsymbol{\lambda}) \\ B \end{bmatrix}.$$

Also, let $V \in \mathbb{C}^{m\times \ell}$ be a basis for the nullspace of B. Then $z \in \mathbb{C}$ is a zero of $C(\boldsymbol{\lambda})$ if and only if z is a zero of $A(\boldsymbol{\lambda})V$.

Proof: From Lemma 4.1, for every $x \in \mathbb{C}$, we have that

$$\text{nullity}\big(C(x)\big) = \text{nullity}\big(C(x)V\big) = \text{nullity}\big(A(x)V\big),$$

and hence

$$\text{rank}\big(C(x)\big) = \text{rank}\big(A(x)V\big) + m - \ell.$$

Therefore, $z \in \mathbb{C}$ is a zero of $C(\boldsymbol{\lambda})$ if and only if z is a zero of $A(\boldsymbol{\lambda})V$. □

***Fact** 4.2:* Let $A \in \mathbb{R}^{m\times p}[\boldsymbol{\lambda}]$, $B \in \mathbb{R}^{m\times n}$, and

$$C(\boldsymbol{\lambda}) \triangleq \begin{bmatrix} A(\boldsymbol{\lambda}) & B \end{bmatrix}.$$

Also, let $V \in \mathbb{C}^{\ell\times m}$ be a basis for the left nullspace of B. Then $z \in \mathbb{C}$ is a zero of $C(\boldsymbol{\lambda})$ if and only if z is a zero of $VA(\boldsymbol{\lambda})$.

Proof: Let $A_1(\boldsymbol{\lambda}) \triangleq A^T(\boldsymbol{\lambda})$, $B_1 \triangleq B^T$, $V_1 \triangleq V^T$, and $C_1(\boldsymbol{\lambda}) \triangleq C^T(\boldsymbol{\lambda})$. Then from Fact 2.4, $z \in \mathbb{C}$ is a zero of $C_1(\boldsymbol{\lambda})$ if and only z is a zero of $A_1(\boldsymbol{\lambda})V_1$. Furthermore, since for every $x \in \mathbb{C}$, $\text{rank}\,(C(x)) = \text{rank}\,(C_1(x))$ and $\text{rank}\,(A_1(x)V_1) = \text{rank}\,(VA(x))$, it follows that z is a zero of $C(\boldsymbol{\lambda})$ if and only if z is a zero of $C_1(\boldsymbol{\lambda})$, and z is a zero of $A_1(\boldsymbol{\lambda})V_1$ if and only if z is a zero of $VA(\boldsymbol{\lambda})$. Hence $z \in \mathbb{C}$ is a zero of $C(\boldsymbol{\lambda})$ if and only if z is a zero of $VA(\boldsymbol{\lambda})$. □

Next, we show that the zeros of a polynomial matrix of arbitrary order are equivalent to the zeros of an easily constructed first-order polynomial matrix. Furthermore, since the problem of determining the zeros of a first-order polynomial matrix can be viewed as a special case of the generalized eigenvalue problem, the problem of determining the zeros of a polynomial matrix of arbitrary order can be viewed as a special case of the generalized eigenvalue problem.

***Fact** 4.3:* Let $C \in \mathbb{R}^{p\times m}[\boldsymbol{\lambda}]$ be given by (1), and let

$$E \triangleq \begin{bmatrix} C_n & 0_{p\times m} & \cdots & \cdots & 0_{p\times m} \\ 0_m & I_m & 0_m & \cdots & 0_m \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & & \ddots & \ddots & 0_m \\ 0_m & \cdots & \cdots & 0_m & I_m \end{bmatrix},$$

$$F \triangleq \begin{bmatrix} -C_{n-1} & \cdots & \cdots & \cdots & -C_0 \\ I_m & 0_m & \cdots & \cdots & 0_m \\ 0_m & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0_m \\ 0_m & \cdots & 0_m & I_m & 0_m \end{bmatrix},$$

where 0_m denotes the $m\times m$ zero matrix, and $0_{p\times m}$ denotes the $p \times m$ zero matrix. Also, let

$$A(\boldsymbol{\lambda}) \triangleq E\boldsymbol{\lambda} - F.$$

Then $z \in \mathbb{C}$ is a zero of $C(\boldsymbol{\lambda})$ if and only if z is a zero of $A(\boldsymbol{\lambda})$.

Proof: Let $G \in \mathbb{R}^{p\times nm}[\boldsymbol{\lambda}]$ denote the first p rows of $A(\boldsymbol{\lambda})$, and let $H \in \mathbb{R}^{m(n-1)\times nm}[\boldsymbol{\lambda}]$ denote the final $m(n-1)$ rows of $A(\boldsymbol{\lambda})$, that is,

$$A(\boldsymbol{\lambda}) = \begin{bmatrix} G(\boldsymbol{\lambda}) \\ H(\boldsymbol{\lambda}) \end{bmatrix},$$

where

$$G(\boldsymbol{\lambda}) \triangleq \begin{bmatrix} (C_n\boldsymbol{\lambda} + C_{n-1}) & C_{n-2} & \cdots & C_0 \end{bmatrix},$$
$$H(\boldsymbol{\lambda}) \triangleq \begin{bmatrix} 0_{(n-1)m\times m} & I_{(n-1)m} \end{bmatrix}\boldsymbol{\lambda} - \begin{bmatrix} I_{(n-1)m} & 0_{(n-1)m\times m} \end{bmatrix}.$$

Then for every $x \in \mathbb{C}$,

$$U(x) \triangleq \begin{bmatrix} x^{n-1}I_m & \cdots & xI_m & I_m \end{bmatrix}^T,$$

is a basis for the nullspace of $H(x)$. Furthermore,

$$G(x)U(x) = C(x).$$

Hence from Lemma 4.1, for every $x \in \mathbb{C}$, we have that

$$\text{nullity}\big(A(x)\big) = \text{nullity}\big(G(x)U(x)\big) = \text{nullity}\big(C(x)\big).$$

Therefore $z \in \mathbb{C}$ is a zero of $C(\boldsymbol{\lambda})$ if and only if z is a zero of $A(\boldsymbol{\lambda})$. □

***Remark** 4.1:* Fact 4.3 shows that the zeros of a polynomial matrix are equivalent to the zeros of an easily constructed first-order polynomial matrix. However, even though the problem has been reduced to a first-order matrix pencil, a generalized eigenvalue solver does not necessarily return the zeros of the original polynomial matrix, as demonstrated in Example 2.1. The following Proposition, however, provides a method for computing the zeros of first-order matrix pencil by first reducing the problem to a standard eigenvalue problem.

***Proposition** 4.1:* Let $C \in \mathbb{R}^{p\times m}[\boldsymbol{\lambda}]$ be given by (1), and let $A(\boldsymbol{\lambda}) \triangleq E_0\boldsymbol{\lambda} - F_0$ be given by Fact 4.3. Furthermore,

i) Let $i \triangleq 0$, $\ell_0 \triangleq (p + [n-1]m)$, and $k_0 \triangleq mn$.
ii) Compute the singular value decomposition of E_i, that is, compute the unitary $U_i \in \mathbb{R}^{\ell_i\times\ell_i}$, unitary $V_i \in \mathbb{R}^{k_i\times k_i}$, and quasi-diagonal $S_i \in \mathbb{R}^{\ell_i\times k_i}$ such that $E_i = U_iS_iV_i$.
iii) Let $r_i \triangleq \operatorname{rank}[E_i]$.
iv) If $r_i = 0$, then go to Step vii). Otherwise, continue.
v) If $r_i = \ell_i$, then go to Step vi). Otherwise,
   a) Let $F_i' \in \mathbb{R}^{\ell_i - r_i\times k_i}$ denote the last $\ell_i - r_i$ rows of the product $U_i^TF_i$.
   b) Compute a basis $W_i \in \mathbb{R}^{k_i\times j_i}$ for the nullspace of F_i' using the singular value decomposition.
   c) Increment i, and let $\ell_i \triangleq r_i$ and $k_i \triangleq j_i$.
   d) Let E_i and F_i denote the first r_i rows of the products $U_{i-1}^TE_{i-1}W_{i-1}$ and $U_{i-1}^TF_{i-1}W_{i-1}$, respectively.
   e) Return to Step ii).
vi) If $r_i = k_i$, then go to Step vii). Otherwise,
   a) Let $F_i' \in \mathbb{R}^{\ell_i\times k_i - r_i}$ denote the last $k_i - r_i$ columns of the product $F_iV_i^T$.
   b) Compute a basis $T_i \in \mathbb{R}^{j_i\times\ell_i}$ for the left nullspace of F_i' using the singular value decomposition.
   c) Increment i, and let $\ell_i \triangleq j_i$ and $k_i \triangleq r_i$.
   d) Let E_i and F_i denote the first r_i columns of the products $T_{i-1}E_{i-1}V_{i-1}^T$ and $T_{i-1}F_{i-1}V_{i-1}^T$, respectively.
   e) Return to Step ii).
vii) If E_i is zero, then $C(\boldsymbol{\lambda})$ has no zeros. Otherwise, $z \in \mathbb{C}$ is a zero of $C(\boldsymbol{\lambda})$ if and only if z is an eigenvalue of $E_i^{-1}F_i$.

Proof: First, from Fact 4.3, $z \in \mathbb{C}$ is a zero of $C(\boldsymbol{\lambda})$ if and only if z is a zero of $A(\boldsymbol{\lambda}) = E_0\boldsymbol{\lambda} - F_0$.

Next, suppose that $r_0 < \ell_0$. Then

$$U_0^T(E_0\boldsymbol{\lambda} - F_0) = \begin{bmatrix} E_0''\boldsymbol{\lambda} - F_0'' \\ F_0' \end{bmatrix}.$$

Furthermore, since U_0 is unitary, U_0 has full rank, and it follows that $z \in \mathbb{C}$ is a zero of $E_0\boldsymbol{\lambda} - F_0$ if and only if z is a zero of $U_0^T(E_0\boldsymbol{\lambda} - F_0)$. Additionally, since W_0 denotes a basis for the nullspace of F_0', from Fact 4.1, we have that $z \in \mathbb{C}$ is a zero of $U_0^T(E_0\boldsymbol{\lambda} - F_0)$ if and only if z is a zero of $(E_0''\boldsymbol{\lambda} - F_0'')W_0 = E_1\boldsymbol{\lambda} - F_1$. Hence, $z \in \mathbb{C}$ is a zero of $C(\boldsymbol{\lambda})$ if and only if z is a zero of $E_1\boldsymbol{\lambda} - F_1$.

Similarly, suppose that $r_0 = \ell_0 < k_0$. Then

$$(E_0\boldsymbol{\lambda} - F_0)V_0^T = \begin{bmatrix} E_0''\boldsymbol{\lambda} - F_0'' & F_0' \end{bmatrix},$$

and from Fact 4.2, we have that $z \in \mathbb{C}$ is a zero of $C(\boldsymbol{\lambda})$ if and only if z is a zero of $T_0(E_0''\boldsymbol{\lambda} - F_0'') = E_1\boldsymbol{\lambda} - F_1$. Hence, by induction, for every $j \in [0, i-1]$, $z \in \mathbb{C}$ is a zero of $E_j\boldsymbol{\lambda} - F_j$ if and only if z is a zero of $E_{j+1}\boldsymbol{\lambda} - F_{j+1}$. Therefore, it follows that $z \in \mathbb{C}$ is a zero of $C(\boldsymbol{\lambda})$ if and only if z is a zero of $E_i\boldsymbol{\lambda} - F_i$.

Finally, if E_i is zero, then there are no points in $\mathbb{C}$ at which the pencil $E_i\boldsymbol{\lambda} - F_i = F_i$ drops rank. Hence $C(\boldsymbol{\lambda})$ has no zeros. However, if E_i is not zero, then it is square and nonsingular. Hence $z \in \mathbb{C}$ is a zero of $C(\boldsymbol{\lambda})$ if and only if z is an eigenvalue of $E_i^{-1}F_i$. □

***Remark** 4.2:* Proposition 4.1 reduces the problem of determining the zeros of an arbitrary polynomial matrix to the square, regular eigenvalue problem $E_i^{-1}F_ix = \lambda x$.

***Remark** 4.3:* If the final E_i in Proposition 4.1 has full normal rank, but is ill-conditioned, then more accurate estimates of the zeros of $C(\boldsymbol{\lambda})$ may be obtained by computing the generalized eigenvalues of (F_i, E_i), as opposed to computing the eigenvalues of $E_i^{-1}F_i$.

V. Numerical Examples

Here we demonstrate the algorithm presented in Proposition 4.1 with two examples. We begin by returning to Example 2.1.

***Example** 5.1:* Let

$$C(\boldsymbol{\lambda}) \triangleq \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\boldsymbol{\lambda} - \begin{bmatrix} 0 & \alpha \\ 0 & 0 \end{bmatrix}.$$

Then, $\operatorname{nrank}[C(\boldsymbol{\lambda})] = 1$ and the only zero of $C(\boldsymbol{\lambda})$ is α. Furthermore, the QZ-algorithm yields, as generalized eigenvalues, the ratios $0/0$ and $0/0$.

Next, consider Proposition 4.1. Then

$$E_0 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad F_0 = \begin{bmatrix} 0 & \alpha \\ 0 & 0 \end{bmatrix}.$$

Furthermore, computing the singular value decomposition of E_0, we find that

$$U_0 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad S_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad V_0 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Therefore, from Step v), we have that

$$F_0' = \begin{bmatrix} 0 & 0 \end{bmatrix}, \quad W_0 = I_2,$$

and hence

$$E_1 = \begin{bmatrix} 0 & 1 \end{bmatrix}, \quad F_1 = \begin{bmatrix} 0 & \alpha \end{bmatrix}.$$

Finally, returning to Step ii) and computing the singular value decomposition of E_1, we find that

$$U_1 = 1, \quad S_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}, \quad V_1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$

Therefore, from Step vi), we have that

$$F_1' = 0, \quad T_1 = 1,$$

and hence

$$E_2 = 1, \quad F_2 = \alpha.$$

Thus α is the only eigenvalue of $E_2^{-1}F_2 = \alpha$ and the only zero of $C(\boldsymbol{\lambda})$. □

Next, we demonstrate how Proposition 4.1 is used to computed the zeros of a higher order polynomial matrix.

Although, Proposition 4.1 can be applied to problems of arbitrary dimension, normal rank, and order, we consider a problem with full normal rank and low enough dimensions so that we can compute the determinant symbolically, and compare the zeros computed using both methods. Furthermore, note that when a matrix does not have full normal rank, then one can not compute the zeros by symbolically computing the determinant.

***Example** 5.2:* Let

$$C(\boldsymbol{\lambda}) \triangleq \begin{bmatrix} 6\boldsymbol{\lambda}^3+4\boldsymbol{\lambda}^2+\boldsymbol{\lambda}+6 & 8\boldsymbol{\lambda}^3+7\boldsymbol{\lambda}^2+3\boldsymbol{\lambda}+5 \\ 3\boldsymbol{\lambda}^3+4\boldsymbol{\lambda}^2+4\boldsymbol{\lambda}+8 & 4\boldsymbol{\lambda}^3+7\boldsymbol{\lambda}^2+5\boldsymbol{\lambda}+1 \end{bmatrix}.$$

Then the order of $C(\boldsymbol{\lambda})$ is 3 and the matrix coefficients of $C(\boldsymbol{\lambda})$ are given by

$$C_3 = \begin{bmatrix} 6 & 8 \\ 3 & 4 \end{bmatrix}, \quad C_2 = \begin{bmatrix} 4 & 7 \\ 4 & 7 \end{bmatrix},$$
$$C_1 = \begin{bmatrix} 1 & 3 \\ 4 & 5 \end{bmatrix}, \quad C_0 = \begin{bmatrix} 6 & 5 \\ 8 & 1 \end{bmatrix}.$$

Furthermore, symbolically computing the determinant of $C(\boldsymbol{\lambda})$, we find that

$$\det\left[C(\boldsymbol{\lambda})\right] = 5\boldsymbol{\lambda}^5 - 7\boldsymbol{\lambda}^4 - 62\boldsymbol{\lambda}^3 - 37\boldsymbol{\lambda}^2 - 13\boldsymbol{\lambda} - 34,$$

and hence the zeros of $C(\boldsymbol{\lambda})$ are

$$z\left(C(\boldsymbol{\lambda})\right) = \left\{ \begin{array}{l} \ \ 4.537 \\ -2.433 \\ -1.103 \\ \ \ 0.1996 \pm \jmath 0.7202 \end{array} \right\}. \qquad (3)$$

For this low order example, we can now use these values as a baseline against which to check the algorithm we have proposed in Proposition 4.1. Specifically, from Proposition 4.1, we have that

$$E_0 = \begin{bmatrix} C_3 & 0_{2\times2} & 0_{2\times2} \\ 0_{2\times2} & I_2 & 0_{2\times2} \\ 0_{2\times2} & 0_{2\times2} & I_2 \end{bmatrix},$$
$$F_0 = \begin{bmatrix} -C_2 & -C_1 & -C_0 \\ I_2 & 0_{2\times2} & 0_{2\times2} \\ 0_{2\times2} & I_2 & 0_{2\times2} \end{bmatrix}.$$

Furthermore, computing the singular value decomposition of E_0, we find that $r_0 = 5 < l_0 = 6$. Therefore, after performing Step v), we have that

$$E_1 = \begin{bmatrix} -4.800 & 4.144 & 4.144 & 5.92 & -1.776 \\ 0.06214 & 0.06214 & 0.06214 & 0.08877 & 0.9734 \\ -0.1450 & 0.8550 & -0.1450 & -0.2071 & 0.06214 \\ -0.1450 & -0.1450 & 0.8550 & -0.2071 & 0.06214 \\ -0.2071 & -0.2071 & -0.2071 & 0.7041 & 0.08877 \end{bmatrix},$$
$$F_1 = \begin{bmatrix} 3.103 & -3.605 & -1.369 & -0.03930 & 7.614 \\ -0.1450 & -0.1450 & 0.8550 & -0.2071 & 0.06214 \\ -0.4244 & -0.4244 & -0.4244 & -0.6063 & 0.1819 \\ 0.8550 & -0.1450 & -0.1450 & -0.2071 & 0.06214 \\ -0.1450 & 0.8550 & -0.1450 & -0.2071 & 0.06214 \end{bmatrix}.$$

Finally, since the singular values of E_1 are

$$\sigma(E_1) = \left\{9.79, \quad 1, \quad 1, \quad 1, \quad 0.0693\right\},$$

we conclude that $r_1 = 5 = l_1 = k_1$, that is, E_1 is square and nonsingular. Hence

$$E_1^{-1}F = \begin{bmatrix} 0.02797 & -6.427 & 6.204 & 6.216 & 2.481 \\ -0.3071 & -2.562 & 1.469 & 1.338 & 1.225 \\ 0.9723 & -2.282 & 1.749 & 1.737 & 1.105 \\ 0.02261 & -2.198 & 2.560 & 2.570 & 1.552 \\ -0.1953 & 0.7710 & 0.04348 & -1.040 & -0.3848 \end{bmatrix},$$

and we have that the eigenvalues of $E_1^{-1}F_1$ are

$$z\left(E_1^{-1}F_1\right) = \left\{ \begin{array}{l} \ \ 4.537 \\ -2.433 \\ -1.103 \\ \ \ 0.1996 \pm \jmath 0.7202 \end{array} \right\}. \qquad (4)$$

Therefore comparing (3) and (4), we find that the zeros of $C(\boldsymbol{\lambda})$ and the eigenvalues of $E_1^{-1}F_1$ are equal, that is, the algorithm presented in Proposition 4.1 has indeed returned the correct zeros of $C(\boldsymbol{\lambda})$. Furthermore, in this case, the QZ-algorithm applied directly to (F_0, E_0) yields the generalized eigenvalues (4) with an additional eigenvalue at infinity. However, in general, there is no guarantee that the generalized eigenvalues of the polynomial matrix linearization (F_0, E_0) will be a subset of the zeros of polynomial matrix as Example 2.1 demonstrates. □

VI. Conclusions

We have presented an algorithm for determining the zeros of polynomial matrices of arbitrary order, normal rank, or dimension. Specifically, we used the singular value decomposition to reduce the problem to an eigenvalue problem.

References

[1] C. T. Chen, *Linear System Theory and Design*, 3rd ed. Oxford University Press, 1999.
[2] K. Ogata, *Modern Control Engineering*, 4th ed. Upper Saddle River, NJ: Prentice-Hall, 2002.
[3] G. E. P. Box, G. M. Jenkins, and G. C. Reinsel, *Time Series Analysis: Forecasting and Control*, 4th ed. Wiley, 2008.
[4] M. L. Abell and J. P. Braselton, *Modern Differential Equations*, 2nd ed. Orlando: Harcourt College, 2001.
[5] J. Tokarzewski, *Finite Zeros in Discrete Time Control Systems*. Springer, 2006.
[6] J. M. McNamee, *Numerical Methods for Roots of Polynomials*. Cambridge, MA: Elsevier Science, 2007, vol. 14.
[7] G. H. Golub and C. F. V. Loan, *Matrix Computations*, 3rd ed. Baltimore: The Johns Hopkins University Press, 1996.
[8] G. W. Stewart, *Matrix Algorithms Volume II: Eigensystems*. Philadelphia: SIAM, 2001.
[9] A. Vardulakis, *Linear Multivariable Control: Algebraic Analysis and Synthesis Methods*. Wiley, 1991.
[10] P. J. Antsaklis and A. N. Michel, *Linear Systems*. Birkhauser, 2006.
[11] P. L. I. Gohberg and L. Rodman, *Matrix Polynomials*. Philadelphia: SIAM, 2009.
[12] R. M. C. A. Amiraslani and P. Lancaster, "Linearization of matrix polynomials expressed in polynomial bases," *IMA J. Num. Anal.*, vol. 29, pp. 141–157, 2009.
[13] N. M. N. J. Higham, D. S. Mackey and F. Tisseur, "Symmetric linearizations for matrix polynomials," *SIAM J. Matrix Anal. Appl.*, vol. 29, pp. 143–159, 2006.
[14] F. D. Teran and F. M. Dopico, "Sharp lower bounds for the dimension of linearizations of matrix polynomials," *Electron. J. Linear Algebra*, vol. 17, pp. 518–531, 2008.
[15] P. Lancaster, "Linearization of regular matrix polynomials," *Electron. J. Linear Algebra*, vol. 17, pp. 21–27, 2008.

[16] N. J. Higham and F. Tisseur, "Bounds for eigenvalues of matrix polynomials," *Linear Algebra and its Applications*, vol. 358, pp. 5–22, Jan. 2003.

[17] D. Henrion and J. C. Zuniga, "Detecting Infinite Zeros in Polynomial Matrices," *IEEE Transactions on Circuits and Systems II: Express Briefs*, vol. 52, no. 11, pp. 744 – 745, 2005.

[18] D. Henrion, J. Ruiz Leon, M. Sebek, "Extraction of Infinite Zeros of Polynomial Matrices," in *IEEE Conference on Decision and Control*, vol. 5, 2000, pp. 4221 – 4226.