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Stabilizability of a Group of Single Integrators and Its Application to Decentralized Formation Problem

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Abstract— **This paper addresses a fundamental property for a class of multi-agent systems, i.e., stabilizability of a group of single integrators, having external control inputs, under a fixed and weighted directed network topology. A necessary and sufficient condition for the stabilizability of the multi-agent system is presented. In particular, it is shown that the multi-agent system is stabilizable if and only if the external control inputs are applied to certain agents (e.g., root node of the communication network when the network is connected). The framework proposed here puts an emphasis on its ability in decentralized control; that is, each agent uses its own and its neighbors' state information as feedback, to stabilize the multi-agent system. Based on these results, the decentralized set-point control problem with formation is also addressed.**

I. Introduction

For a decade the multi-agent systems (MASs), consisting of several identical dynamic systems and communicating some information through a local interaction among them, have received considerable attention due to their importance in biology and engineering. A major concern of the MAS is to reach an agreement among the identical systems, which is termed as consensus or synchronization problem. See, e.g., [1], [2], [4], [6], [10], [11], [16], [17], [18]. When the consensus problem is solved, the agreed value is often a weighted average (in a certain sense) of the initial conditions of the involved agents, and it is not a function of external inputs to the MAS. However, in the perspective of control engineering, there are some domains where the agreement should be controlled by certain external inputs to the system.

This notion has naturally initiated the study of controllability in [19], which is followed by [3], [7], [8], [9], [13], [14], [15]. The leader-follower controllability, discussed in the reference above, can be summarized as follows. For a group of N single integrators, represented by $\dot{x} = -Lx$ with $x := [x_1 \ \cdots \ x_N]^T \in \mathbb{R}^N$ (x_i is the state of agent i) and the Laplacian matrix L (obtained from the interaction topology of the MAS), let us suppose that the last m agents are chosen as leaders, while the first $N-m$ agents as followers. Then partitioning the overall state x as $x = [z^T \quad r^T]^T$ with $z \in \mathbb{R}^{N-m}$ and $r \in \mathbb{R}^m$ (i.e., z and r represent the states of the followers and the leaders respectively), one has the dynamics of followers as $\dot{z} = -Fz - Rr$, in which $F \in \mathbb{R}^{(N-m)\times(N-m)}$ and $R \in \mathbb{R}^{(N-m)\times m}$. The MAS is said to be controllable if the pair (F, R) is controllable; that is, the state r of the leaders is treated as an independent external control input. Once the controllability is guaranteed for a given MAS, the standard open-loop control

$$r(t) = R^T e^{F^T(t_f - t)} W^{-1}(t_f)(z_f - e^{Ft_f} z_0) \qquad (1)$$

is used for placing each state of the followers to the desired position (i.e., set-point control with formation), where $W(t) := \int_0^t e^{F\tau} RR^T e^{F^T\tau} d\tau$ is the controllability Gramian and $z_0 = z(0)$ and $z_f = z(t_f)$ are the initial and final states of the followers [13].

However, we note that controllability notion intrinsically includes the possibility of enforcement of arbitrary positioning at arbitrary time, which leads to a few drawbacks in the MAS application and raises a question about whether we are asking too much. Most of all, it has been shown in [19] that a MAS with more than two agents *under a complete graph* turns out to be uncontrollable for any choice of single leader. It is also pointed out that a MAS is always uncontrollable if the MAS is symmetric with respect to a leader (even though the network is connected and the leader is a source of the network) [13], [14], [15]. Moreover, if the control law (1) is used in practice, it is actually a *centralized* control because it requires the information (the initial and final states, i.e, z_0 and z_f) of all the followers.

These restrictions suggest us to consider a relaxed notion of 'stabilizability' for the MAS. Introducing the concept of the independent strongly connected components (iSCCs) of a graph, we show that any given MAS is stabilizable if and only if an independent external control input is applied to a node in each iSCC of the network. When the network is connected, the node happens to be a root node of the network. It is seen that stabilizability of a MAS does not depend on the weights of the network graph, but only on the structure, which is another benefit of considering stabilizability. It is also shown that the MAS can be stabilized through self-feedbacks to the suitably chosen nodes (that is, a node in each iSCC). Similarly to [16, Section 3], we provide a *decentralized* control law for the MAS that achieves the set-point control with formation.

The presentation of this paper slightly generalizes the previous results [3], [7], [8], [9], [13], [14], [15], [19]. In particular, we deal with directed graphs rather than undirected ones. And, by considering external inputs to the MAS, we could avoid the problem that the leaders do not join the formation (which can be seen from (1) where $r(t)$ is the

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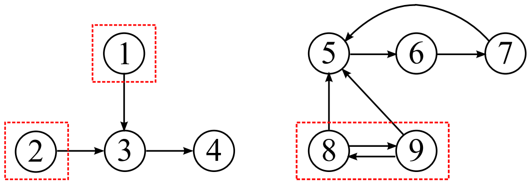

Fig. 1. A graph having nine nodes. The number in each circle denotes the node in the node set $\mathcal{N} = \{1, 2, \ldots, 9\}$. For brevity the weights α_{ih}'s are omitted. There are three iSCCs of the graph, induced by the node sets $\{1\}$, $\{2\}$, and $\{8, 9\}$, each of which are marked by the red dashed boxes.

states of leaders but used as the inputs so that they are not a part of formation).

The remainder of this paper is organized as follows. Section II presents some preliminaries of the paper, especially on the graph and matrix theory, and consensus problem. A necessary and sufficient condition for the stabilizability of the MAS is given in Section III, as well as its application to the decentralized set-point control problem with formation. After demonstrating the proposed result through an example in Section IV, the paper concludes with some concluding remarks in Section V.

II. Preliminaries

A. Basics on Graph and Matrix Theory

In order to efficiently model the local interaction among the identical systems, the graph theory is used. A (weighted directed) graph, denoted by $\mathcal{G} = (\mathcal{N}, \mathcal{E}, \mathcal{A})$, consists of a finite nonempty set of nodes $\mathcal{N} = \{1, 2, \ldots, N\}$, an edge set (excluding the self-connection) of ordered pairs of nodes $\mathcal{E} \subseteq \mathcal{N} \times \mathcal{N}$, and a weighted adjacency matrix $\mathcal{A} = [\alpha_{ih}] \in \mathbb{R}^{N\times N}$. An edge $(h, i) \in \mathcal{E}$, represented by an arrow tailed at the node h and headed toward the node i, is related with the adjacency matrix $\mathcal{A}$ by the rule that $\alpha_{ih} > 0$ if and only if $i \neq h$ and $(h, i) \in \mathcal{E}$. Otherwise $\alpha_{ih} = 0$. The set $\mathcal{N}^i$ of neighbors of the node i is defined as $\mathcal{N}^i = \{h \in \mathcal{N} : (h, i) \in \mathcal{E}\}$. A path (of length l) from the node h to the node i is a sequence of nodes of the form $\{i_0, i_1, \ldots, i_l\}$ so that $i_0 = h$, $i_l = i$, $i_k \in \mathcal{N}^{i_{k+1}}$, and i_k's are distinct. A graph $\mathcal{G}$ is complete if $(h, i) \in \mathcal{E}$ for any $h, i \in \mathcal{N}$ and $h \neq i$. A strongly connected graph $\mathcal{G}$ is the graph satisfying that for any pairs of distinct nodes h and i, there is a path from h to i in $\mathcal{G}$. A graph $\mathcal{G}$ is connected[1] if there is a node h, called a *root* or *source* of the graph, such that for any $i \in \mathcal{N} \setminus \{h\}$, a path from h to i exists. Note that any node of the strongly connected graph is a source. Note also that a strongly connected graph is connected, but not vice versa.

Given a graph $\mathcal{G} = (\mathcal{N}, \mathcal{E}, \mathcal{A})$, consider a subset $\mathcal{N}_1 \subseteq \mathcal{N}$ of nodes of $\mathcal{G}$. The subgraph $\mathcal{G}_1$ induced by the set $\mathcal{N}_1$ is the graph $\mathcal{G}_1 = (\mathcal{N}_1, \mathcal{E}_1, \mathcal{A}_1)$, in which $\mathcal{E}_1 := \{(h, i) \in \mathcal{E} : h, i \in \mathcal{N}_1\}$ and $\mathcal{A}_1$ is the submatrix obtained by deleting the ith row and column of $\mathcal{A}$ for all $i \notin \mathcal{N}_1$. An independent strongly connected component (iSCC) of $\mathcal{G} = (\mathcal{N}, \mathcal{E}, \mathcal{A})$ is an induced subgraph $\mathcal{G}_1 = (\mathcal{N}_1, \mathcal{E}_1, \mathcal{A}_1)$ such that it is maximal subject to being strongly connected and satisfies that $(h, i) \notin \mathcal{E}$ for any $h \in \mathcal{N} \setminus \mathcal{N}_1$ and $i \in \mathcal{N}_1$. Fig. 1 shows an example of iSCCs of the given graph.

The Laplacian matrix $L = [l_{ih}] \in \mathbb{R}^{N\times N}$ of a graph $\mathcal{G} = (\mathcal{N}, \mathcal{E}, \mathcal{A})$ is defined as $L := \mathcal{D} - \mathcal{A}$, where $\mathcal{D} := \mathrm{diag}(\mathcal{A}1_N)$ and $1_N \in \mathbb{R}^N$ is the column vector of all ones. Note that any Laplacian matrix contains a zero eigenvalue with the corresponding eigenvector 1_N, and all the nonzero eigenvalues lie in the open right-half complex-plane $\mathbb{C}_{>0}$ by Gershgorin disc theorem. Without loss of generality, we sort the eigenvalues of L as $0 = \lambda_1(L) \leq \mathrm{Re}(\lambda_2(L)) \leq \cdots \leq \mathrm{Re}(\lambda_N(L))$, in which $\lambda_i(L)$ denotes the ith eigenvalue of L.

Now we review reducible, irreducible, and strictly diagonally dominant matrices, which are closely related with the strongly connected graphs. A matrix $A \in \mathbb{R}^{k\times k}$ is reducible if either (a) $k = 1$ and $A = 0$; or (b) for $k \geq 2$, there exist a permutation matrix $P \in \mathbb{R}^{k\times k}$ and an integer m with $1 \leq m \leq k - 1$ such that

$$P^T A P = \begin{bmatrix} B & C \\ 0 & D \end{bmatrix}, \tag{2}$$

where $B \in \mathbb{R}^{m\times m}$ and $D \in \mathbb{R}^{(k-m)\times(k-m)}$. If a matrix A is not reducible then it is said to be irreducible. A matrix $A = [a_{ih}] \in \mathbb{R}^{k\times k}$ is strictly diagonally dominant if it satisfies that (a) $|a_{ii}| \geq \sum_{h\neq i} |a_{ih}|$ for all $i = 1, \ldots, k$; and (b) $|a_{ii}| > \sum_{h\neq i} |a_{ih}|$ for at least one i.

The following results will be used throughout the paper.

Theorem 1 ([5, Theorems 6.2.14 and 6.2.24]): Let $\mathcal{G} = (\mathcal{N}, \mathcal{E}, \mathcal{A})$. Then the adjacency matrix $\mathcal{A}$ is irreducible if and only if the Laplacian matrix L is irreducible if and only if the graph $\mathcal{G}$ is strongly connected. □

Lemma 1 ([5, Corollary 6.2.27]): Let a square matrix A be irreducible and strictly diagonally dominant. Then it is invertible. □

Theorem 2 ([21, Theorem 2.12]): Let $\mathcal{G} = (\mathcal{N}, \mathcal{E}, \mathcal{A})$ and $i \in \mathcal{N}$. Then there exists at least one iSCC $\mathcal{G}_1 = (\mathcal{N}_1, \mathcal{E}_1, \mathcal{A}_1)$ of $\mathcal{G}$ such that either $i \in \mathcal{N}_1$ or there exists a path from h to i for each $h \in \mathcal{N}_1$. □

It is worthwhile to note that by Theorem 2, any graph contains at least one iSCC. In addition, a graph possesses at most N iSCCs since $|\mathcal{N}| = N$.[2]

Theorem 3 ([21, Theorem 2.13]): Let $\mathcal{G} = (\mathcal{N}, \mathcal{E}, \mathcal{A})$ and assume that $\mathcal{G}$ has c distinct iSCCs $\mathcal{G}_j = (\mathcal{N}_j, \mathcal{E}_j, \mathcal{A}_j)$ with $j = 1, \ldots, c$. Then $\dim(\ker(L^T)) = c$ and there exist unique (modulo node permutations) vectors $r_j = [r_{1j}\ r_{2j}\ \cdots\ r_{Nj}]^T \in \mathbb{R}^N$ for $j = 1, \ldots, c$, satisfying

$$\begin{matrix} r_{ij} > 0 & \text{if } i \in \mathcal{N}_j, \\ r_{ij} = 0 & \text{if } i \notin \mathcal{N}_j, \end{matrix} \qquad j = 1, \ldots, c$$

and $r_j^T 1_N = 1$, such that $\ker(L^T) = \mathrm{span}\{r_1, \ldots, r_c\}$. □

Corollary 1 ([21, Corollary 2.14]): Let $\mathcal{G} = (\mathcal{N}, \mathcal{E}, \mathcal{A})$. Then the graph $\mathcal{G}$ is connected if and only if it contains exactly one iSCC if and only if $\lambda_2(L) \in \mathbb{C}_{>0}$. □

[1]A connected graph here differs from the one in [22], which is sometimes called weakly connected graph, in the sense that the latter uses a weak path (or a semiwalk) in defining the notion. See, e.g., [12], [22] for further details.

[2]In this paper, the operator $|\cdot|$ has two meaning. It denotes the cardinality of the set if its argument is a set, while it means the absolute value when the argument is a number.

B. Consensus Problem and Its Use in Formation Control

Consider the N identical systems described by

$$\dot{x}_i = u_i, \qquad i = 1, \dots, N, \tag{3}$$

where $x_i \in \mathbb{R}^n$ is the state and $u_i \in \mathbb{R}^n$ the interagent control. For the MAS (3), it is assumed that the interaction among the agents is modeled by a weighted directed graph $\mathcal{G} = (\mathcal{N}, \mathcal{E}, \mathcal{A})$. More precisely, each element i in the node set $\mathcal{N}$ corresponds to the identifier of the ith agent, and the edge $(h, i) \in \mathcal{E}$ and the element α_{ih} of $\mathcal{A}$ imply that the agent i receives the state information of the agent h with the interaction strength α_{ih}.

Now we define the consensus problem of the MAS (3). The consensus of the MAS is said to be reached if under a certain interagent control u_i, $\lim_{t\to\infty} \|x_h(t) - x_i(t)\| = 0$ for any $h, i \in \mathcal{N}$.

In order to achieve the consensus of the MAS, the interagent control is often given by

$$u_i = \sum_{h \in \mathcal{N}^i} \alpha_{ih}(x_h - x_i) = -\sum_{h \in \mathcal{N}} l_{ih} x_h. \tag{4}$$

Then the overall dynamics of the MAS becomes

$$\dot{x} = -(L \otimes I_n)x,$$

where $x := \mathrm{col}(x_1, \dots, x_N)$ is the stack of the vectors x_i for $i \in \mathcal{N}$, L the Laplacian matrix of $\mathcal{G}$, I_n the $n \times n$ identity matrix, and $\otimes$ denotes the Kronecker product.

Theorem 4 ([2], [17], [20]): Under the control (4), the consensus of the MAS (3) is reached if and only if the graph $\mathcal{G}$ is connected. In addition, $\lim_{t\to\infty} \|x_i(t) - X_0 r\| = 0$ for all $i \in \mathcal{N}$, in which $X_0 := [x_1(0)\ x_2(0)\ \cdots\ x_N(0)] \in \mathbb{R}^{n \times N}$ and r^T is the left eigenvector of L associated to $\lambda_1(L) = 0$ with $r^T 1_N = 1$. $\square$

Inspired by the consensus problem of the MAS, one can provide a solution [16, p. 35] to the formation problem by modifying the interagent control (4) as

$$u_i = \sum_{h \in \mathcal{N}^i} \alpha_{ih}\{(x_h - d_h) - (x_i - d_i)\},$$

in which $d_i \in \mathbb{R}^n$ is a constant displacement vector that describes the formation. In fact, defining $\tilde{x}_i := x_i - d_i$ and $\tilde{x} := \mathrm{col}(\tilde{x}_1, \dots, \tilde{x}_N)$, one obtains that $\dot{\tilde{x}} = -(L \otimes I_n)\tilde{x}$ and hence concludes that by Theorem 4, $\tilde{x}_h(t) - \tilde{x}_i(t) \to 0$ or equivalently $x_h(t) - x_i(t) \to d_h - d_i$ as $t \to \infty$, whenever $\mathcal{G}$ is connected. That is, the formation is achieved by the N agents asymptotically. Note that the actual position of the formation depends on the initial condition $x_i(0)$ of each agent and on the displacement vector d_i, since $x_i(t) \to d_i + \tilde{X}_0 r$ as $t \to \infty$ where $\tilde{X}_0 := [x_1(0) - d_1\ \cdots\ x_N(0) - d_N]$.

III. Main Result

Having a close look at the Laplacian matrix and the iSCCs of a graph $\mathcal{G}$, this section provides the stabilizability results for the MAS having external control inputs and presents their application to the decentralized set-point control problem with formation.

A. Stabilizability of a Group of Single Integrators

Consider a group of N systems given by

$$\dot{x}_i = u_i + e_i r_i, \qquad i = 1, \dots, N, \tag{5}$$

where $x_i \in \mathbb{R}^n$ is the state, $u_i \in \mathbb{R}^n$ the interagent control, and $r_i \in \mathbb{R}^n$ the external control of the agent i. The coefficient $e_i \in \mathbb{R}$ models whether the agent i is affected by the external input r_i or not. In other words, $e_i = 1$ if the external input r_i is applied to the agent i. Otherwise $e_i = 0$.

Assume that the local interaction among the agents is modeled by a weighted directed graph $\mathcal{G} = (\mathcal{N}, \mathcal{E}, \mathcal{A})$ and the interagent control u_i is given by (4). Then the overall dynamics of the MAS (5) is written as

$$\dot{x} = -(L \otimes I_n)x + (E \otimes I_n)r, \tag{6}$$

in which $x := \mathrm{col}(x_1, \dots, x_N)$, $r := \mathrm{col}(r_1, \dots, r_N)$, $E := \mathrm{diag}(e_1, \dots, e_N)$, and L is the Laplacian matrix of $\mathcal{G}$.

Now we present the following result which implies that the stabilizability of the MAS is closely related with the iSCCs of the graph.

Theorem 5: Suppose that the given graph $\mathcal{G} = (\mathcal{N}, \mathcal{E}, \mathcal{A})$ contains c distinct iSCCs $\mathcal{G}_j = (\mathcal{N}_j, \mathcal{E}_j, \mathcal{A}_j)$ with $j = 1, \dots, c$. Then the MAS (6) is stabilizable if and only if for each $j = 1, \dots, c$, there exists a node $q_j \in \mathcal{N}_j$ such that $e_{q_j} = 1$. $\square$

Theorem 5 generalizes the results of [3], [7], [8], [9], [13], [14], [15], [19] in the sense that any MAS that is controllable with suitably chosen leaders (in terms of [19]) is stabilizable if the external inputs are injected into the leaders, and that there does exist a MAS that is uncontrollable with any choice of leaders but is stabilizable.

Another important point of the theorem is about the stabilizing control r_{q_j}. In view of the conventional stabilizability, the stabilizing control is generally of the form $r_{q_j} = -K_{q_j} x$ for some $K_{q_j} \in \mathbb{R}^{n \times Nn}$, which may require the access to the state x_h for $h \notin \mathcal{N}^{q_j}$. However, as will be seen shortly, the decentralized feedback $r_{q_j} = -k_{q_j} x_{q_j}$ is enough for stabilizing the MAS (6).

Proof: Since the graph contains c distinct iSCCs, without loss of generality, we assume that L and E in (6) are of the forms[3]

$$L = \left[\begin{array}{ccc|c} L_1 & & 0 & \\ & \ddots & & 0 \\ 0 & & L_c & \\ \hline L_{01} & \cdots & L_{0c} & L_0 \end{array}\right], \quad E = \left[\begin{array}{ccc|c} E_1 & & 0 & \\ & \ddots & & 0 \\ 0 & & E_c & \\ \hline 0 & \cdots & 0 & E_0 \end{array}\right], \tag{7}$$

where for $j = 1, \dots, c$, $L_j \in \mathbb{R}^{N_j \times N_j}$ is the Laplacian matrix corresponding to the iSCC $\mathcal{G}_j$ with $N_j := |\mathcal{N}_j|$, and $L_0 \in \mathbb{R}^{N_0 \times N_0}$ with $N_0 := N - \sum_{j=1}^{c} N_j$. For $h = 0, 1, \dots, c$, the diagonal matrix $E_h \in \mathbb{R}^{N_h \times N_h}$ is appropriately defined with its diagonal entries being 0 or 1. Note that by Corollary 1, each L_j for $j = 1, \dots, c$ contains the simple zero eigenvalue

[3]One can always obtain the forms through suitable node permutations (i.e., renumbering the agents' identifiers). See the definition of iSCC.

and all the other $N_j - 1$ eigenvalues lie in $\mathbb{C}_{>0}$. Note also that $\lambda(L_0) \subset \mathbb{C}_{>0}$ since $\dim(\ker(L^T)) = c$ by Theorem 3 and $\lambda(L) = \lambda(L_1) \cup \cdots \cup \lambda(L_c) \cup \lambda(L_0)$, where $\lambda(A)$ is the set of eigenvalues of A. With this description at hand, we now prove the theorem.

(Necessity) Suppose that for some j with $1 \le j \le c$, there is no such a node q_j. We assume $j = 1$ with no loss of generality. Then $E_1 = 0$. Let $w_1^T \in \mathbb{R}^{N_1}$ be a nonzero left eigenvector associated to the zero eigenvalue of L_1, i.e., $w_1^T L_1 = 0$. With $w^T := [w_1^T\ 0] \in \mathbb{R}^N$ and any nonzero row vector $v^T \in \mathbb{R}^n$, one obtains

$$\begin{aligned}(w^T \otimes v^T) & \begin{bmatrix} sI_{Nn} + L \otimes I_n & E \otimes I_n \end{bmatrix} \\ &= \left(w^T \begin{bmatrix} sI_N + L & E \end{bmatrix}\right) \otimes v^T \\ &= \begin{bmatrix} sw_1^T + w_1^T L_1 & 0 \end{bmatrix} \otimes v^T = 0, \text{ with } s = 0,\end{aligned}$$

which contradicts the stabilizability of the MAS.

(Sufficiency) Without loss of generality, assume that the node in the theorem is that $q_j = (\sum_{h=1}^{j} N_h) - N_j + 1$. Then $E_j = \mathrm{diag}(1, *, \ldots, *)$ for all $j = 1, \ldots, c$, where the asterisks stand for the numbers of no interest.

Now suppose that the MAS (6) is not stabilizable; that is, there exist a complex number $s \in \mathbb{C}_{\ge 0}$ and a nonzero row vector $\bar{w}^T \in \mathbb{C}^{Nn}$ such that $\bar{w}^T[sI_{Nn} + L \otimes I_n\ \ E \otimes I_n] = 0$, in which $\mathbb{C}_{\ge 0}$ denotes the set of complex numbers possessing nonnegative real parts. Suppose that $\bar{w}^T$ is partitioned as $\bar{w}^T = [\bar{w}_1^T\ \cdots\ \bar{w}_c^T\ \ \bar{w}_0^T]$, where $\bar{w}_h^T \in \mathbb{C}^{N_h n}$ for $h = 0, 1, \ldots, c$. Then it follows that

$$\bar{w}_j^T(sI_{N_j n} + L_j \otimes I_n) + \bar{w}_0^T(L_{0j} \otimes I_n) = 0, \tag{8a}$$

$$\bar{w}_0^T(sI_{N_0 n} + L_0 \otimes I_n) = 0, \tag{8b}$$

$$\bar{w}_h^T(E_h \otimes I_n) = 0, \tag{8c}$$

for $j = 1, \ldots, c$ and $h = 0, 1, \ldots, c$.

We claim that $\bar{w}_0^T = 0$. If not, by (8b), it is a left eigenvector associated to the eigenvalue $s \in \mathbb{C}_{\ge 0}$ of $-(L_0 \otimes I_n)$. However, this is not possible since $-(L_0 \otimes I_n)$ is Hurwitz. The matrix $-(L_0 \otimes I_n)$ being Hurwitz is implied by the Hurwitz matrix $-L_0$. Therefore (8) becomes

$$\bar{w}_j^T(sI_{N_j n} + L_j \otimes I_n) = 0, \tag{9a}$$

$$\bar{w}_j^T(E_j \otimes I_n) = 0, \tag{9b}$$

for $j = 1, \ldots, c$ and for some $s \in \mathbb{C}_{\ge 0}$.

Moreover, we also claim that for $1 \le j \le c$, the vector $\bar{w}_j^T \in \mathbb{C}^{N_j n}$ is of the form $\bar{w}_j^T = w_j^T \otimes v_j^T$ whenever $\bar{w}_j^T \ne 0$, in which $v_j^T \in \mathbb{C}^n$ is a nonzero row vector and $w_j^T \in \mathbb{C}^{N_j}$ is a left eigenvector associated to the zero eigenvalue of L_j. For such j satisfying $\bar{w}_j^T \ne 0$, it is observed from (9a) that $\bar{w}_j^T$ is a left eigenvector corresponding to a zero eigenvalue of $-(L_j \otimes I_n)$, since $s \in \mathbb{C}_{\ge 0}$ and all the eigenvalues of $-(L_j \otimes I_n)$ lie in $\mathbb{C}_{<0} \cup \{0\}$. That is to say, $\bar{w}_j \in \ker\left(-(L_j \otimes I_n)^T\right)$. Let $\{z_1, \ldots, z_n\}$ be the standard basis in $\mathbb{R}^n$. Then the set $W_j := \{w_j \otimes z_1, \ldots, w_j \otimes z_n\}$ is linearly independent and $-(L_j \otimes I_n)^T(w_j \otimes z_h) = 0$ for $h = 1, \ldots, n$. Hence $\ker\left(-(L_j \otimes I_n)^T\right) = \mathrm{span}(W_j)$ since $\dim\left(\ker\left(-(L_j \otimes I_n)^T\right)\right) = n$.[4] Hence the claim follows.

Finally from (9b), one obtains that $(w_j^T E_j) \otimes v_j^T = 0$ whenever $\bar{w}_j^T \ne 0$. This necessarily implies $w_j^T E_j = 0$ since $v_j^T \ne 0$. On the other hand, since L_j is the Laplacian matrix corresponding to the iSCC $\mathcal{G}_j$, all the elements of w_j^T have to differ from zero by Theorem 3. But this contradicts $w_j^T E_j = 0$ because $E_j = \mathrm{diag}(1, *, \ldots, *)$. Therefore $\bar{w}_j^T = 0$ for *all* $j = 1, \ldots, c$, and $\bar{w}^T = 0$ which again contradicts the assumption that $\bar{w}^T \ne 0$. ■

Corollary 2: Suppose that the graph $\mathcal{G} = (\mathcal{N}, \mathcal{E}, \mathcal{A})$ is connected. Then the MAS (6) is stabilizable if and only if there exists a source $q \in \mathcal{N}$ of $\mathcal{G}$ such that $e_q = 1$. □

Proof: By Corollary 1, any connected graph contains exactly one iSCC. Let the iSCC be $\mathcal{G}_1 = (\mathcal{N}_1, \mathcal{E}_1, \mathcal{A}_1)$. Then, by Theorem 5, the MAS is stabilizable if and only if there is a node $q \in \mathcal{N}_1$ such that $e_q = 1$. Thus the result follows from the fact that each node in $\mathcal{N}_1$ is a source of $\mathcal{G}$ and vice versa (see Theorem 2 and the definition of iSCC). ■

Now we investigate a stabilizing external control. The proposed control r_i is a self-feedback control and hence is decentralized. In addition, in view of designing a stabilizing control, it provides considerable flexibility to the designer since the result below says that any negative self-feedback to a node in each iSCC stabilizes the MAS. The result follows from the irreducibility and strictly diagonally dominance of L induced from strongly connected graph.

Theorem 6: Let $\mathcal{G}_j = (\mathcal{N}_j, \mathcal{E}_j, \mathcal{A}_j)$, $j = 1, \ldots, c$ be the c distinct iSCCs of the graph $\mathcal{G} = (\mathcal{N}, \mathcal{E}, \mathcal{A})$, and let

$$r_i = -k_i x_i, \qquad i \in \mathcal{N}, \tag{10}$$

in which k_i's are nonnegative numbers. Then the MAS (5) with (4) and (10) is exponentially stable if for each iSCC, there is a node $q_j \in \mathcal{N}_j$ such that $e_{q_j} = 1$ and $k_{q_j} > 0$. □

Proof: Under the control of (4) and (10), the dynamics of each agent becomes $\dot{x}_i = -\sum_{j \in \mathcal{N}} l_{ij} x_j - e_i k_i x_i$ and hence the overall dynamics is written as $\dot{x} = -\{(L + EK) \otimes I_n\}x$, where $K = \mathrm{diag}(k_1, k_2, \ldots, k_N)$. Without loss of generality, it is assumed that the matrices L and E are of the forms in (7) and $K = \mathrm{diag}(K_1, \ldots, K_c, K_0)$ where K_h's are the $(N_h \times N_h)$ diagonal matrices suitably defined.

For each $j = 1, \ldots, c$, we claim that all the eigenvalues of the matrix $\bar{L}_j := L_j + E_j K_j$ lie in $\mathbb{C}_{>0}$. If $N_j = 1$ then the claim trivially follows from the definition of the Laplacian matrix (i.e., $L_j = 0$) and the assumption of the theorem. Now assume that $N_j \ge 2$. Then $\bar{L}_j$ is irreducible. If not, there exist a permutation matrix $P_j \in \mathbb{R}^{N_j \times N_j}$ such that $P_j^T \bar{L}_j P_j$ is block upper triangular as in (2). Noting that $Q^T E_j K_j Q$ is diagonal for any permutation matrix Q, one observes that $P_j^T L_j P_j = P_j^T \bar{L}_j P_j - P_j^T E_j K_j P_j$ is also block upper triangular, which is not possible because the Laplacian L_j corresponding to the iSCC $\mathcal{G}_j$ is irreducible by Theorem 1. Thus $\bar{L}_j$ is irreducible. Moreover, it is strictly diagonally

[4] One of the properties of the Kronecker product is that for any $A \in \mathbb{R}^{n \times n}$ and $B \in \mathbb{R}^{m \times m}$, the eigenvalues of $A \otimes B$ are $\lambda_i(A)\lambda_h(B)$ for $i = 1, \ldots, n$ and $h = 1, \ldots, m$. Thus the algebraic multiplicity of the zero eigenvalue of $(L_j \otimes I_n)$ is n for $j = 1, \ldots, c$.

dominant since L_j is diagonally dominant and $e_{q_j}k_{q_j} > 0$. Hence by Lemma 1, $0 \notin \lambda(\bar{L}_j)$ and by Gershgorin disc theorem, the claim follows.

Now it remains to show that $\lambda(L_0+E_0K_0) \subset \mathbb{C}_{>0}$. Define $M_0 := -[L_{01} \quad \cdots \quad L_{0c}]$, $\bar{D}_0 := \mathrm{diag}(M_0 1_{N-N_0}) + E_0K_0$, and $\bar{L}_0 := L_0 - \mathrm{diag}(M_0 1_{N-N_0})$. Note that all the elements of $\bar{D}_0$ are nonnegative and the matrix $\bar{L}_0$ corresponds to the Laplacian matrix of the subgraph induced by the set $\mathcal{N}_0$. Then, with the fact that $L_0 + E_0K_0 = \bar{L}_0 + \bar{D}_0$, one can show the invertibility of $\bar{L}_0 + \bar{D}_0$ from [21, Appendix A.3.2] with the matrices L_0 and D_0 in [21, Appendix A.3.2] replaced by $\bar{L}_0$ and $\bar{D}_0$, respectively. On the other hand, all the Gershgorin discs of $L_0+E_0K_0 = \bar{L}_0+\bar{D}_0$ are subsets of $\mathbb{C}_{>0} \cup \{0\}$, since $\bar{L}_0$ is a Laplacian matrix and the diagonal matrix $\bar{D}_0$ is nonnegative. Thus $\lambda(L_0 + E_0K_0) \subset \mathbb{C}_{>0}$.

By the claims proven so far, one concludes that $-(L + EK)$ is Hurwitz and in turn, so the matrix $-(L+EK)\otimes I_n$ is. ■

Corollary 3: Let $\mathcal{G} = (\mathcal{N}, \mathcal{E}, \mathcal{A})$ be any connected graph and let the external controls be as in (10). Then the MAS (5) with (4) and (10) is exponentially stable if there is a source node $q \in \mathcal{N}$ such that $e_q = 1$ and $k_q > 0$. □

Proof: Let $\mathcal{G}_1 = (\mathcal{N}_1, \mathcal{E}_1, \mathcal{A}_1)$ be the iSCC of the connected graph. Then the node q in the theorem belongs to $\mathcal{N}_1$. Thus the result follows from Theorem 6. ■

Remark 1: On the basis of the theorems and corollaries presented so far, it is worthwhile to mention about the number of external controls that is necessary to stabilize the MAS. In view of Theorem 5, only one external control to each iSCC is sufficient, that is, c external controls are enough in the case of the graph having c distinct iSCCs. All the other stabilizing external controls, especially including the controls acting on the nodes that do not belong to any iSCCs, are redundant. Moreover, from Theorem 6, it turns out that any negative self-feedback to a node in each iSCC stabilizes the MAS, even for arbitrarily small gains $k_{q_j} > 0$. ◊

B. Application: Decentralized Set-Point Control Problem with Formation

Consider a group of N systems (5) with the interaction topology modeled by a graph $\mathcal{G} = (\mathcal{N}, \mathcal{E}, \mathcal{A})$. Let $\mathcal{G}_j = (\mathcal{N}_j, \mathcal{E}_j, \mathcal{A}_j)$, $j = 1, \ldots, c$ be the c distinct iSCCs of the graph $\mathcal{G}$. In order to avoid notational complexity, it is assumed that for $j = 1, \ldots, c$, there is exactly one node $q_j \in \mathcal{N}_j$ with $e_{q_j} = 1$ (so that $e_i = 0$ for all $i \notin \{q_1, \ldots, q_c\}$). When there are additional external controls, one can solve the problem (defined below) by appropriately modifying the proposed scheme.

For the given N displacement vectors $d_i \in \mathbb{R}^n$, $i \in \mathcal{N}$ and a set-point $d_0 \in \mathbb{R}^n$, the *decentralized set-point control problem with formation* for the MAS (5) is said to be solved if there exist the controls u_i and r_{q_j} such that (a) each agent is allowed to use the state information of its own and its neighboring agents for feedback, as well as the desired displacements d_i's; (b) the information on the set-point d_0 is available only for the nodes q_j's; and (c) under the controls, the MAS satisfies that $\lim_{t\to\infty} \|x_i(t) - (d_i + d_0)\| = 0$ for each $i \in \mathcal{N}$. Note that the problem here differs from the formation problem given in Section II-B in the sense that the formation eventually reached by the MAS does not depend on the initial conditions $x_i(0)$'s. In addition, the formation can be put in any location by setting d_0.

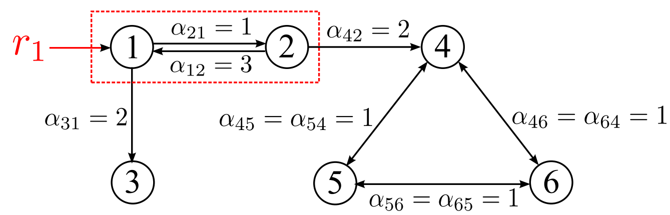

Fig. 2. An example of MAS consisting of six agents. The number in the circle denotes the agent's identifier. The red dashed box corresponds to the iSCC of the connected interaction topology. The external control, namely r_1, is assumed to be applied only to the agent 1.

Under these setups, we propose the following interagent and external controls

$$u_i = \sum_{h\in\mathcal{N}^i} \alpha_{ih}\{(x_h - d_h) - (x_i - d_i)\}, \quad i \in \mathcal{N}, \tag{11a}$$

$$r_{q_j} = -k_{q_j}(x_{q_j} - d_{q_j} - d_0), \qquad j = 1, \ldots, c, \tag{11b}$$

where k_{q_j}'s are any positive numbers.

To show that the MAS under the decentralized control (11) solves the problem, define $\tilde{x}_i := x_i - d_i - d_0$. Then the overall dynamics of the MAS is written as $\dot{\tilde{x}} = -\{(L + EK) \otimes I_n\}\tilde{x}$, where L is the Laplacian matrix of $\mathcal{G}$ and the diagonal matrices E and K are appropriately defined from e_i's and k_{q_j}'s. By Theorem 6, it follows that the overall dynamics is exponentially stable and thus, that $\lim_{t\to\infty} \|x_i(t) - (d_i + d_0)\| = 0$ for $i \in \mathcal{N}$.

IV. Example

An example of the decentralized set-point control problem with formation is addressed in this section. Consider the MAS (5) with $x_i := \mathrm{col}(x_{i1}, x_{i2}) \in \mathbb{R}^2$, consisting of six agents, and suppose that each agent of the MAS interacts with the others as depicted in Fig. 2. Suppose also that only the agent 1 has the external control r_1; namely, $e_1 = 1$ and for $i = 2, \ldots, 6$, $e_i = 0$. Note that the overall dynamics (6) derived from Fig. 2 is uncontrollable in the sense of [19] because the subgraph induced by the node set $\{4, 5, 6\}$ is unweighted and complete.

For the MAS, we consider the scenario that, starting from an arbitrary configuration, the MAS is driven by the interagent control (11a) (without the external control) so as to achieve the formation, where the displacement vectors are assumed to be given as

$$\begin{aligned}&[d_1 \quad d_2 \quad d_3 \quad d_4 \quad d_5 \quad d_6] \\ &\quad := \begin{bmatrix} 0 & -5 & 5 & -10 & 0 & 10 \\ 0 & -5\sqrt{3} & -5\sqrt{3} & -10\sqrt{3} & -10\sqrt{3} & -10\sqrt{3} \end{bmatrix}.\end{aligned}$$

After that, by additionally controlling the 1st agent through the external control (11b), we move the agent 1 to the set-point $d_0 = d_0 + d_1 = \mathrm{col}(35, 35)$ maintaining the formation.

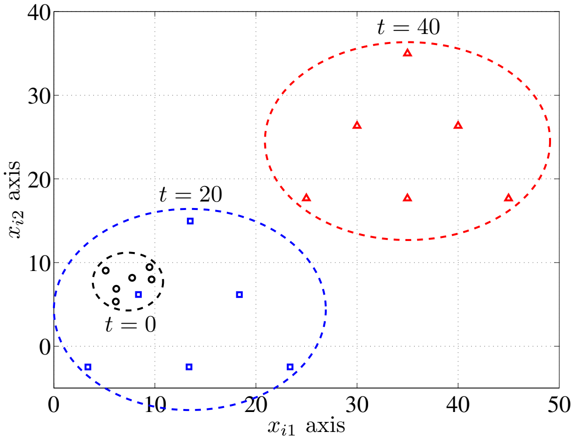

Fig. 3. Simulation result with the snapshots at the times $t = 0$, 20, and 40 seconds. The agents' identifiers are omitted for brevity and all the initial conditions of the agents are randomly chosen. The black circles, blue squares, and red triangles represent the configuration of the MAS at $t = 0$, 20, and 40 seconds, respectively.

Fig. 3 shows the simulation result. In the simulation, the external control is applied to the agent 1 in a way that $r_1(t) = 0$ for $t \in [0, 20)$ and $r_1(t) = -5(x_1(t) - d_1 - d_0)$ for $t \in [20, 40]$. From the figure, one can observe that the MAS reaches the formation at $t = 20$ seconds, although the location of the formation depends on the initial conditions and the displacement vectors of the agents as mentioned in Section II-B. On the other hand, the formation is moved to another location by successfully pulling the agent 1 to the set-point d_0 through the external control r_1, during $t \in [20, 40]$.

V. Conclusions and Future Works

In the paper, we have studied the stabilizability of a group of single integrators, having external control inputs, under the interaction topology modeled by a weighted directed graph. The key concept that characterizes the stabilizability of the multi-agent system (MAS) was shown to be the independent strongly connected components (iSCCs) of the graph. In fact, it was shown that the MAS is stabilizable if and only if in each iSCC, there is an agent that has access to an external control. This result is a generalization of the (leader-follower) controllability-based results, e.g., [3], [7], [8], [9], [13], [14], [15], [19], if the controllability of the MAS is replaced by the stabilizability. Based on the result, we proposed a decentralized external self-feedback control that stabilizes the MAS. Moreover, the decentralized set-point control problem with formation is dealt with as an application of the problem.

Finally, consideration of the extension to the MAS having general linear dynamics is under further study.

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