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Stability Margins in Traditional and Second Order Sliding Mode Control

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***Abstract*—Sliding Mode Control (SMC) and Second Order Sliding Mode (2-SMC) are robust techniques that are used in many applications including aerospace control. In order to be certified for practical applications, a control design must demonstrate not only stability in the ideal case, but stability when subjected to dynamic perturbation i.e. stability margins. In Linear Time Invariant (LTI) systems the mostly accepted stability measures are phase margin and gain margin. However, these margins cannot be strictly computed for nonlinear systems; in particular, they cannot be computed for SMC and 2-SMC controllers, and this has delayed acceptance of such techniques in practical applications. This paper is intended to open a discussion of how stability margins compatible with tradition and with modern nonlinear methods might be defined in SMC and 2-SMC systems.**

I. INTRODUCTION

Control systems must function as intended in the presence of perturbations that are unknown and uncharacterizable. Achieving stability of control is not enough. Stability margins [1] give a measure how stable a control system is. With several important caveats, a well-designed feedback control system is robust to additive disturbance. *Gain margin* [1], [2] which has a clear meaning for both linear and nonlinear controllers, describes robustness to multiplicative disturbance. It is the evaluation of sensitivity to dynamic disturbance where trouble occurs [2], [3].

The predominant measure of sensitivity to dynamic disturbance is called "*phase margin*" [1]-[4]. It is well-known that phase margin is derived from the characteristic equation of the closed-loop system. Unfortunately, the corresponding characteristic equation strictly exists only for linear systems [1]-[4] and thus *phase margin* cannot be computed, in the customary manner, for nonlinear systems [5] including systems with sliding mode control (SMC) [6], [7] and higher order sliding mode control (HOSM) [8]-[10]. The result is that innovative control techniques cannot meet established criteria, and are not considered viable alternatives.

Several justifiable algorithms exist for evaluating the stability of an *equilibrium point* in nonlinear systems subjected to dynamic/parametric disturbance. Piecewise linearization [1], [2] can be applied to so-called "soft" nonlinearities, although the effort involved may be considerable if the number of stability points is very large. Lyapunov's exponents [11], [12] and small gain theory [5] can be applied to a wide class of nonlinear systems for identifying the stability margins but often results in an unrealistically conservative approximation.

SMC [6], [7] second order sliding mode (2-SMC) [8] and HOSM [9], [10] control are obvious choices for controlling systems with bounded matched disturbances/uncertainties. The main advantages of the HOSM/2-SMC [8], [9], [13]-[16] over the classical SMC include a higher accuracy of the sliding variable stabilization and the possibility of generating continuous control laws (super-twisting or twisting, sub-optimal, prescribed control law as a filter). It is worth noting that the one of the main motivation for the development of 2-SMC control algorithms was eliminating chattering, which always is the major drawback of the classical SMC [6], [7]. A price for achieving the robustness/insensitivity to the matched bounded disturbances/uncertainties in systems with SMC/HOSM control is introduction of a *limit cycle* [5] into the control system with (theoretically) infinity frequency and zero amplitude with respect to the system's output called a sliding variable. It is well known [5] that the stable limit cycles are attractive (asymptotically stable) in some domain and this explains the robustness of sliding modes. The sliding mode itself can be treated as an equilibrium point, which stability is enforced by meeting the sliding mode existence condition [6], [7] that can be interpreted as the existence condition for an aforementioned limit cycle.

Therefore, looking for the stability margins in systems with SMC/HOSM [6], [7], [8]-[10] control we should take into account that the equilibrium point, which stability margin we intend to identify, is a limit cycle that can exist in nonlinear systems only [5].

Perhaps most promising techniques that reconcile the linear and nonlinear systems stability analysis are the locus of a perturbed relay system (LPRS) [13], [16] and describing function technique (DF) [5], [13]. These techniques give frequency domain analysis of existence and stability of limit cycles in nonlinear systems. In this paper, the DF technique is used (due to its simplicity) for identifying the stability margins in systems with SMC/2-SMC control. However, we have to acknowledge that DF technique gives only approximate limit cycle analyses and the stability margin identification [5].

The structure of the paper is as follows. Stability margins in systems with classical sliding mode control are studied in Section II. Stability margins for 2-SMC super-twisting control algorithm are discussed in Section III. Section IV demonstrates via examples. The conclusions are presented in Section V.

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II. STABILITY MARIN IN SYSTEMS WITH CLASSICAL SMC

A. Problem Formulation

Consider an uncertain linear time invariant single input-single output system

$$\dot{x} = Ax + b(u + f(x,t)),\ \sigma = Cx \quad (1)$$

where $x \in \mathbb{R}^n$ is a state vector, $u \in \mathbb{R}$ is a control function, A, C and b are the matrix and the vector of appropriate dimensions, $\sigma \in \mathbb{R}$ is a sliding variable.

It is assumed that

(A1) the pair (A,b) is completely controllable,

(A2) the sliding variable $\sigma \in \mathbb{R}$ is properly designed so that its dynamics

$$\dot{\sigma} = CAx + Cb(u + f(x,t)),\ \det(Cb) \neq 0,\ Cb > 0 \quad (2)$$

are of relative degree 1,

(A3) system's dynamics (1) are stable in the sliding mode $\sigma = 0$.

(A4) the function $f(x,t)$ is bounded, i.e. $|f(x,t)| \le L_1$, and system's (1) dynamics are considered in a domain so that $|f(x,t) + CAx| \le L > 0$.

(A5) Without loss of generality, assume that $Cb = 1$

Then the control function [6], [7]

$$u = -U_m sign(\sigma),\quad U_m = \rho + L,\ \rho > 0 \quad (2a)$$

drives $\sigma \to 0$ in finite time t_r that is estimated as

$$t_r \le |\sigma(0)| / \rho \quad (3)$$

and keeps $\sigma = 0$ thereafter.

It is worth noting that in the sliding mode $\sigma = 0$

(a) there exists a limit cycle in system (1), (2), in which the sliding variable σ exhibits self sustained oscillations with zero amplitude and infinity frequency.

(b) system's (1)-(3) dynamics are insensitive to the bounded disturbance $f(x,t)$.

The problem is to define and identify the stability margins for system (1), (2), i.e. what dynamical perturbations system (1), (2) can sustain prior to destruction of the aforementioned limit cycle.

B. Methodology

Firstly, let us perform a limit cycle analysis in system (1) (2) using the DF technique. For this purpose the block-diagram of the unperturbed system is presented in Fig. 1, where the transfer function $G(s)$ is computed as

$$G(s) = \frac{\sigma(s)}{u(s)} = C(sI - A)^{-1} b \quad (4)$$

Assume that there exists periodic motion (self-sustained oscillations or a limit cycle) with the amplitude A_c and the oscillation frequency ω_c

$$-\sigma = A_c \sin(\omega_c t) \quad (5)$$

in system with classical SMC of Fig. 1.

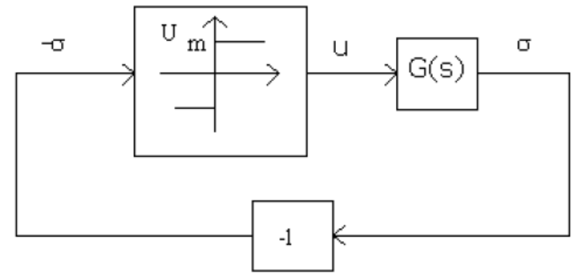

Fig. 1 Block-diagram of system with SMC

Then, in accordance with the DF technique, the amplitude A_c and the oscillation frequency ω_c have to satisfy harmonic balance equation [5], [16], [17].

$$G(j\omega) = -1 / N(A,\omega) \quad (6)$$

where the describing function $N(A,\omega)$ of the relay nonlinearity can be easily identified as

$$N(A,\omega) = 4U_m / (\pi A) \quad (7)$$

Since the transfer function (4) is of relative degree one,

$$\lim_{\omega\to\infty} \arg G(j\omega) = -\pi/2,\quad \lim_{\omega\to 0} \arg G(j\omega) = 0 \quad (8)$$

Taking into account eq. (8) we assume that the vector C in eq. (1) is selected so that the transfer function (4) satisfies the strict passivity condition

$$|\arg G(j\omega)| < \frac{\pi}{2} \quad \forall \omega \in [0,\infty) \quad (9)$$

Then there exists a unique stable limit cycle with the parameters [17]

$$A_c = 0,\quad \omega \to \infty \quad (10)$$

This fact is illustrated in Fig. 2. For clarity, only half of the Nyquist plot is shown in Fig. 2.

Definition 1. System (1)-(3) is understood to be finite time (asymptotically) stable if it exhibits a stable limit cycle with the parameters in eq. (10) that is reached in finite time (as time increases).

Traditionally, stability margin is a measure of the system's equilibrium point (here a limit cycle) stability.

It is worth noting that the DF technique [5] tells nothing about reaching time for the limit cycle. Further analysis of stability of the limit cycle in system (1)-(3) shows that in the case of the 1st order lag parasitic dynamics (Fig. 3) the aforementioned limit cycle can be reached only asymptotically.

Definition 2. We define the *ideal phase margin* (IPM) in SMC system (1)-(3) as the phase angle that the frequency response $G(j\omega)$ would have to gain in order to start crossing the negative part of the real axis to the left from the origin. At marginal stability, the solution of eqs. (6), (7) becomes

$$A_c = \varepsilon > 0,\quad \omega_c = \Omega < \infty \quad (11)$$

Apparently, based on Definition 2 and Fig. 2, in system with classical SMC (1), (2) and properly selected vector C the ideal phase margin is

$$IPM = \pi / 2 \quad (12)$$

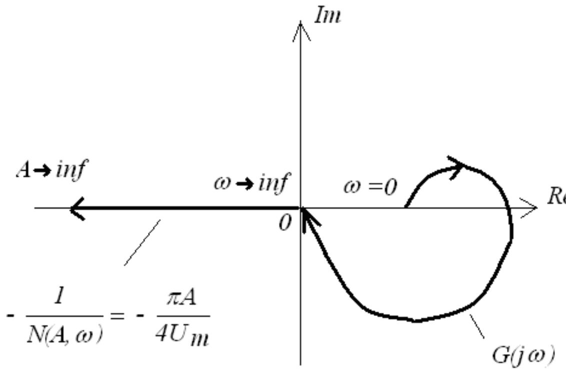

Fig. 2 Limit cycle analysis in SMC system

Remark 1. Apparently, in practical SMC systems the *ideal limit cycle* in eq. (1)-(4) can barely exist due to different imperfections in switching element, including hysteresis, parasitic dynamics, and time delay. Therefore, it makes sense to give a definition of the *practical phase margin* in SMC system (1)-(3). If the limit cycle in a perturbed system, described by the amplitude $A_c > 0$ and the corresponding oscillation frequency $\omega_c < \infty$, is "acceptable", we call the compensated system stable. If it is not acceptable, we declare the system *practically unstable*.

Definition 3. We define the *practical phase margin* (PPM) in SMC system (1)-(3) as the phase angle that the frequency response $G(j\omega)$ would have to gain in order to cross the negative part of the real axis to the left from the origin, whereby the solution of eqs. (6), (7) becomes

$$A_c = \varepsilon \le \varepsilon^* > 0, \quad \omega_c = \Omega \ge \Omega^* > 0 \qquad (13)$$

where ε^* is a maximal *practically acceptable* amplitude and Ω^* is the *minimal acceptable* frequency of the self sustained oscillations.

PPM can be determined (see example problems) by successive addition of phase to the frequency response curve of the unperturbed system until the Nyquist criterion indicates marginal stability or the extended Nyquist method predicts a limit cycle with marginally acceptable properties.

Apparently, based on Definition 3 and Fig. 2, in system with classical SMC (1)-(3) and properly selected vector C the practical phase margin is

$$PPM > \frac{\pi}{2} \qquad (14)$$

C. Parasitic cascade dynamics

The *ideal* and *practical phase margins* can be also defined in terms of parasitic cascade dynamics. In this work we consider the 1st order, $G_p(s) = \frac{\omega_0}{s+\omega_0}$, 2nd order, $G_p(s) = \frac{\omega_0^2}{s^2 + 1.4\omega_0 s + \omega_0^2}$, and time-delay $G_p(s) = e^{-s/\omega_0}$ cascade parasitic dynamics. All of them are parameterized in terms of the parameter ω_0.

Definition 4. We define the *ideal phase margin* (IPM) in SMC system (1)-(3) as the 1st order, 2nd order, or time-delay *cascade parasitic dynamics*, which system (1)-(3) can tolerate until a loss of stability in a sense of Definition 1.

Definition 5. We define the *practical phase margin* (PPM) in SMC system (1)-(3) as the 1st order, 2nd order, or time-delay *cascade parasitic dynamics* with $0 < \omega_0 \le \bar{\omega}_0$, which system (1)-(3) can tolerate while exhibiting an acceptable limit cycle with the parameters given in eq. (13).

It is clear, based on Figs. 2 and 3, that SMC system (1)-(3) can tolerate any strictly passive lag parasitic cascade dynamics of relative degree 1 [17].

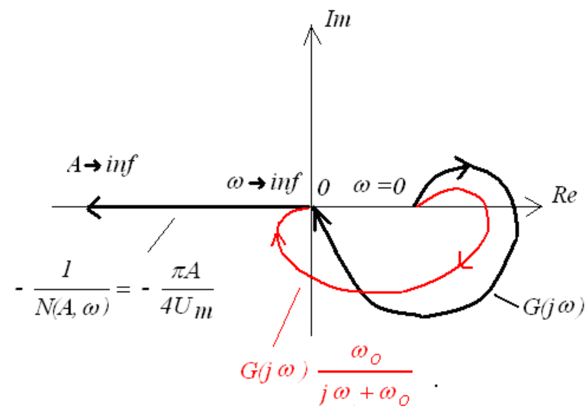

Fig. 3 Limit cycle analysis in SMC system with 1st order parasitic dynamics $G_p(s) = \frac{\omega_0}{s+\omega_0}$

This fact does not contradict to eq. (12), since

$$\lim_{\omega\to\infty} \arg G_p(j\omega) = -\pi/2 \qquad (15)$$

for the 1st order lag parasitic dynamics given by

$$G_p(s) = \frac{\omega_0}{s+\omega_0} \quad \forall \omega_0 > 0 \qquad (16)$$

It means that in this case IPM is 1st order lag cascade parasitic dynamics given in eq. (16).

It becomes clear, based on Fig. 3, that any 2nd order parasitic dynamics given by a transfer function $G_p(s) = \frac{\omega_0^2}{s^2 + 1.4\omega_0 s + \omega_0^2}$ destroy the ideal limit cycle in eq. (10), and $IPM = 0$. This fact is illustrated in Fig. 4.

However, given parameters of the acceptable limit cycle in eq. (13) the PPM can be identified in terms of parameters ω_0 of tolerable 2nd order parasitic dynamics given by a transfer function

$$G_p(s) = \frac{\omega_0^2}{s^2 + 1.4\omega_0 s + \omega_0^2} \qquad (17)$$

that satisfy the harmonic balance equation

$$G(j\omega)G_p(j\omega) = -1/N(A,\omega), \quad N(A,\omega) = 4U_m/(\pi A) \quad (18)$$

with $A = A_c = \varepsilon \le \varepsilon^* > 0, \quad \omega = \omega_c = \Omega \ge \Omega^* > 0$.

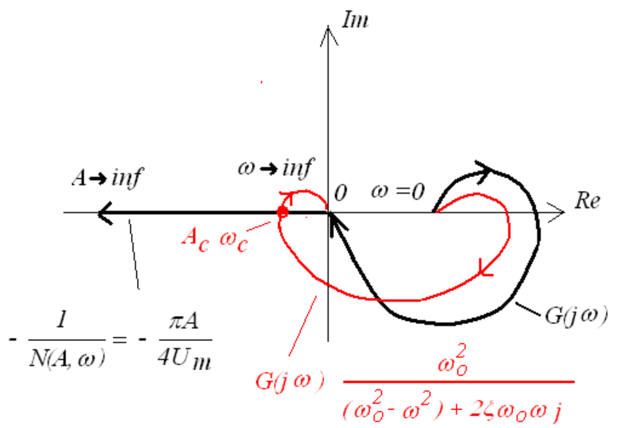

Fig. 4 Limit cycle analysis in SMC system with 2nd order parasitic dynamics

Also, it becomes clear from Figs. 2 and 3 that any gain increase or decrease of the gain of the transfer function (4), even augmented by 1st order cascade parasitic dynamics, will not affect the existence of the *ideal limit cycle*.

However, the *ideal gain margin* can be introduced directly based on the sliding mode (ideal limit cycle with parameters in eq. (10)) existence condition

$$\sigma\dot{\sigma} \le -\rho|\sigma| \quad (19)$$

taking into account the bounded disturbance $|f(x,t)+CAx| \le L$. This condition will be

$$U_m \ge L + \rho \quad \rightarrow \quad U_m > L \quad (20)$$

Therefore, the following definition of the gain margin in SMC system (1),(2) is proposed

Definition 6. We define the *ideal gain margin* (*GM*) in SMC system (1)-(3) as the possible *decrease* of the control amplitude U_m until the sliding mode (limit cycle) existence condition (13) starts violating. This is

$$GM = U_m / L \quad (21)$$

The *practical gain margin* (PGM) can be also introduced in terms of *tolerable parasitic cascade dynamics.* For instance, if the parasitic dynamics is characterized by a transfer function of relative degree 2

$$G_p(s) = k\frac{\omega_0^2}{s^2 + 1.4\omega_0 s + \omega_0^2}, \quad k \ge 1 \quad (22)$$

and given a value of the control (3) amplitude U_m, the *practical gain margin* (PGM), is defined as the possible increase of the gain $k \ge 1$ until the parameters of resulting limit cycle stop satisfying the conditions (13).

III. Stability Marins in Systems with Super-Twisting (2-SMC) Sliding Mode Control

The super-twisting algorithm [8] is one of the popular 2-SMC algorithms. It is used for the sliding variable dynamics of relative degree 1 given by eq. (2). Assume for simplicity $Cb = 1$, and

$$|\dot{F}(x,t)| \le L_2, \quad F(x,t) = CAx + f(x,t) \quad (23)$$

The *continuous* super-twisting control is designed as [8]

$$\begin{cases} u = -\lambda|\sigma|^{1/2}\,\mathrm{sign}(\sigma) - z, \\ \dot{z} = \alpha\,\mathrm{sign}(\sigma) \end{cases} \quad (24)$$

and drives $\sigma, \dot{\sigma} \rightarrow 0$ in the presence of bounded disturbance (23). Parameters λ and α are defined as [8]

$$\alpha = 1.1L_2, \quad \lambda = 1.5\sqrt{L_2} \quad (25)$$

In this section the limit cycle analysis and stability margins are studied in the 2-SMC system given by (1)-(3), (23)-(25). The block-diagram of this system is given in Fig. 5.

The limit cycle analysis is performed in accordance with the works [13], [16]. The describing function of a nonlinear dynamics block in Fig. 5 is identified [13], [16]:

$$N(A,\omega) = \frac{4\alpha}{\pi A}\cdot\frac{1}{j\omega} + 1.1128\frac{\lambda}{\sqrt{A}} \quad (25a)$$

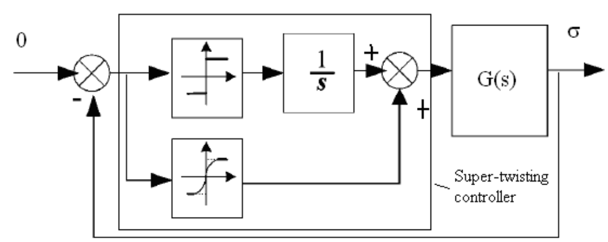

Fig. 5 Block-diagram if system with super-twisting control

The negative reciprocal of the DF in eq. (23) is given by the following formula [13], [16]:

$$-\frac{1}{N(A,\omega)} = -\frac{0.8986\frac{\sqrt{A}}{\lambda} + j\cdot 1.0282\frac{\alpha}{\lambda^2\omega}}{1 + 1.3092\frac{\alpha^2}{\lambda^2}\frac{1}{A\omega^2}} \quad (25b)$$

The function $-1/N(A,\omega)$ in eq. (25b) is a function of two variables: the amplitude A and the frequency ω. It can be depicted as a family of plots representing the amplitude dependence, with each of those plots corresponding to a certain frequency. Also, it is shown [13], [16] that for $\omega = const$

$$\lim_{A\to 0}\left[\arg\left(-1/N(A,\omega)\right)\right] = -\pi/2 \quad (25c)$$

The plots of function $-1/N(A,\omega)$ and the 1/2 Nyquist plot $G(j\omega)$ are depicted in Fig. 6. Based on Fig. 6, it is clear that the unique limit cycle exists with the parameters in eq. (10) in 2-SMC system given by (1)-(3), (23)-(25).

Remark 2. The Definitions 1-5 are valid for the 2-SMC systems given by eqs. (1), (2), and (23)-(25a).

In accordance with Definition 2, *ideal phase margin* does not exists for this 2-SMC system, since any additional phase shift added to $G(j\omega)$ destroys ideal limit cycle (10).

However, in accordance with Definition 3, the practical phase margin can exist.

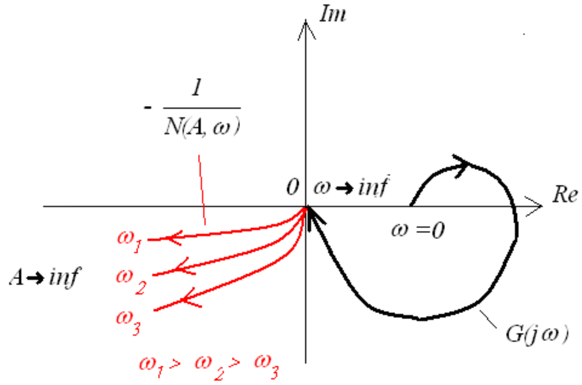

Fig. 6 Limit cycle analysis in 2-SMC system

In accordance with Definition 3, the following two-fold algorithms are proposed for computing PPM

Algorithm 1 (illustrated in Fig. 7)

Step 1. Let's assume that the **frequency** of the self-sustained oscillations of the practically acceptable limit cycle is given

$$\omega_c = \omega_2 > 0 \qquad (26)$$

Step 2. Then the PPM is the maximum phase shift of the Nyquist plot $G(j\omega)$ so that it crosses the plot $-1/N(A,\omega_2)$ corresponding to the frequency $\omega = \omega_2$.

Step 3. The corresponding value of the amplitude of the oscillations can be read as $A_c = |G(j\omega_2)|$.

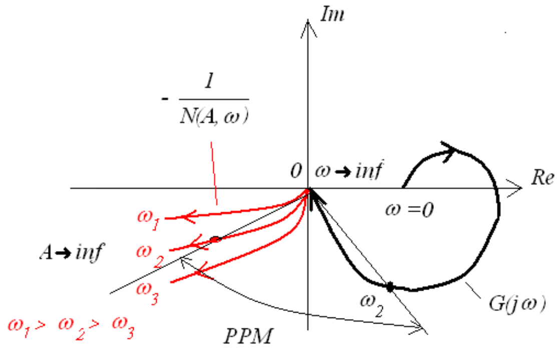

Fig. 7 Practical Phase Margin in 2-SMC system

Algorithm 2

Step 1. Let's assume that the **amplitude** of the self-sustained oscillations of the practically acceptable limit cycle is given $A = A_c > 0$

Step2. Read the corresponding value of the frequency of the self-sustained oscillations from the Nyquist plot $G(j\omega)$ at $|G(j\omega)| = A_c$

Step 3. Then the PPM is the maximum phase shift of the Nyquist plot $G(j\omega)$ so that it crosses the plot $-1/N(A_c,\omega)$ corresponding to the frequency $\omega = \omega_c$.

Graphical illustration of this algorithm is similar to the plot in Fig. 7.

Remark 3. The solutions can be obtained analytically, solving the harmonic balance equations

$$\begin{aligned} \mathrm{Re}\left[-1/N(A_c,\omega)\right] &= \mathrm{Re}\left[G(j\omega)G_p(\varpi_0, j\omega)\right] \\ \mathrm{Im}\left[-1/N(A_c,\omega)\right] &= \mathrm{Im}\left[G(j\omega)G_p(\varpi_0, j\omega)\right] \end{aligned} \qquad (27)$$

Remark 4. The PPM and PGM in 2-SMC systems can be computed in terms of parameters of tolerable cascade parasitic dynamics in a sense of Definitions 4 and 5 similar to as in Section II.

IV. Examples

Use describing function technique to estimate ideal and practical gain and phase margins for the following classical SMC system:

$$\ddot{x} + 2\dot{x} + x + u = \varphi(t),\ \sigma = \dot{x} + 2x,\quad |\varphi| \le L < 3 \qquad (28)$$

In the manner of section II, assume that there exist a limit cycle in (5). The problem is to identify IPM and PPM in system (28) driven by SMC and 2-SMC.

A. Example 1: Stability margins in SMC system

Traditional SMC is given by

$$u = -3\,sign(\sigma) \qquad (29)$$

The dynamically perturbed open-loop transfer functions are thus, respectively:

$$\text{(a)}\quad \frac{s+2}{\left(s^2+2s+1\right)}\frac{\omega_0}{\left(s+\omega_0\right)} \qquad (30a)$$

$$\text{(b)}\quad \frac{s+2}{\left(s^2+2s+1\right)}\frac{\omega_0^{\,2}}{\left(s^2+1.4\omega_0 s+\omega_0^{\,2}\right)} \qquad (30b)$$

$$\text{(c)}\quad \frac{1}{\left(s^2+2s+1\right)}e^{-s/\omega_0} \qquad (30c)$$

$$\text{(d)}\quad \frac{1}{\left(s^2+2s+1\right)}e^{-\theta j} \qquad (30d)$$

Suppose that a practical limit cycle (in the feedback variable σ) with amplitude $A_c = 0.02$ is acceptable. This translates to $-1/N(A,\omega) = -1/N(A) = -0.0052$. The extended ½ Nyquist plots near the origin are shown in Fig. 8, and the results of the predicted IPM and PPM are summarized in Table 1. Simulation results are not plotted for reasons of brevity, but agree well with describing function estimates.

B. Example 2: Stability margins in 2-SMC system

For the second example, we repeat the problem, but using the 2-SMC (supertwist) control (24) in place of traditional SMC. We use gains: $\alpha = 1.1(3) = 3.3$, $\lambda = 1.5\sqrt{3} = 2.6$. Retaining the limit cycle amplitude tolerance at $A_c = 0.02$, and substituting these into (24) yields: $-1/N(A_c,\omega) = -\omega\left(0.0489\omega + j\cdot 3.74\right)/\left(\omega^2 + 105.5\right)$.

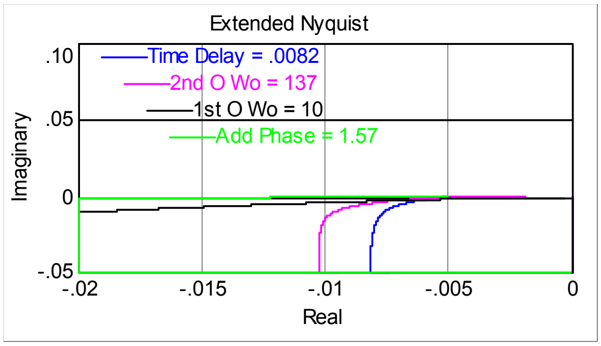

Fig. 8 SMC: Extended Nyquist Plot near the Origin

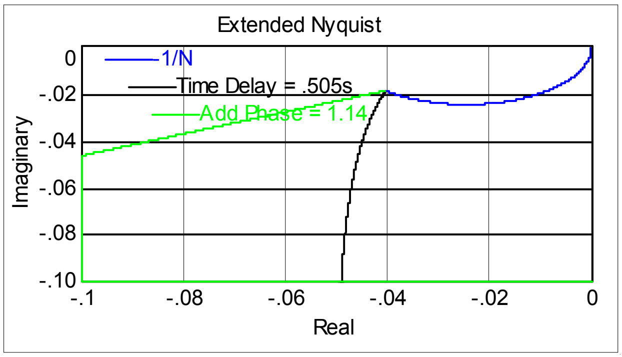

Fig. 9 2-SMC: Extended Nyquist Plot near Origin for added phase and time delay

Table 1: Phase Margin Results from Example #1

	1st Order	2nd Order	Time Delay
A_c	0	.02	.02
Simulated A_c	0	.02	.021
ω_c	∞	140 1/sec	191 1/sec
Simulated ω_c	1000 1/s	137 1/sec	209 1/sec
Min. ω_0	∞	137 1/sec	122 1/sec
IPM	1.57	0	0
PPM	>1.57	>1.57	>1.57

PPM from (30d) is 1.62 1/sec

Figure 9 shows the extended Nyquist plots for the perturbed systems with time delay and added phase such that this limit cycle amplitude is estimated to occur. The cases with the 1st and 2nd-order parasitic dynamics were also considered. The results are summarized in Table 2.

Remark 5: It appears that, with the possible exception of 1st-order dynamics, PPM is insensitive to the characterization of the parasitic dynamics.

V. Conclusions

A new concept of stability margins is introduced for classical SMC and 2-SMC systems. The corresponding ideal and practical phase and gain margins are introduced. They characterize the conditional stability/existence of limit cycles in classical SMC and 2-SMC (super-twisting control) systems. These newly introduced stability margins will help to certify classical and 2-SMC algorithms in different applications, especially in flight guidance and control systems [18].

Table 2: Phase Margin Results from Example #2

	1st Order	2nd Order	Time Delay
A_c	.02	.02	.02
Sim. A	.024	.022	.022
ω_c	14.6 1/sec	20.7 1/sec	22.5 1/sec
Sim. ω	14.9 1/sec	20.1 1/sec)	22.2 1/sec
Min. ω_0	10.4 1/sec)	29.1 1/sec	19.8 1/sec
IPM	0	0	0
PPM	.903	1.11	1.14

PPM from (30d) is 1.14

VI. References

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