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An interpolation approach for robust constrained output feedback

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Abstract— **In this paper, we consider the regulation problem for uncertain linear discrete time systems with bounded disturbance, bounded input and bounded output. Based on the input-output representation, an extended state space model is constructed via the delayed inputs and outputs of the systems and hence there is no need for any estimate of the unmeasured states. The control is obtained using an interpolation technique, which assures feasibility and a robustly asymptotically stable closed loop behavior.**

I. Introduction

The output feedback synthesis problem for linear systems with output and input constraints, subject to parametric uncertainty in the model description and to bounded additive disturbance has been a longstanding in the literature. We mention here two main approaches which aim at solving the state feedback constrained control problems. The first one is based on the optimal control principles and earned a good reputation under the name of Model Predictive Control (MPC) [1].Most traditional MPC are based on mathematical models which invariably present a mismatch with respect to the physical systems. The robust MPC is meant to address both model uncertainty and disturbances. However, the nominal MPC extension to the robust case presents great conservativeness and/or on-line computational burden [2], [3]. A second constrained control design methodology for linear systems is based on the explicit control action for the *extreme points* of a certain region in the state space. The approach is known in the literature as *vertex control* [4], [5]. A weakness of this method is that the full control range is exploited only on the border of the feasible invariant set and hence the control action is poorly employed when the trajectory leaves the region of constraint activation. For this reason the regulation performance for a linear plant of vertex control is poor compared to the classical linear quadratic regulator.

Recently, a novel approach for state feedback control of linear systems with output and input constraints, subject to parametric uncertainty was proposed in [6]. The main idea is to interpolate between the vertex control action u_v and the local control action u_o around the equilibrium. For the current state x, this strategy leads to a control action in the form $u(x) = cu_v + (1-c)u_o$, $0 \leq c \leq 1$, whereby $c(x)$ is minimized in order to obtain a control action as close as possible to the unconstrained optimal one. It was shown that with c as the objective function there exists a Lyapunov function for the system controlled by the interpolated controller u, thus guaranteeing stability.

The first aim of this paper is to extend the result in [6] to the case of both model uncertainty and bounded disturbances. Two interpolation schemes will be proposed. The first interpolation scheme uses the global vertex control and local unconstrained robust optimal control, while the second one interpolates between the optimal contractive control at the frontier of a controlled invariant set and local unconstrained robust optimal control.

The second aim of this paper is to address the *robust* peak to peak gain minimization using LMIs. Instead of minimizing the robust induced $L_\infty-$norm, we minimize its upper bound along the lines in [7], [8] and [5] with a contribution towards the parametric uncertainty handling. The optimal state feedback controller will be obtained by solving line search + semidefinite (convex) optimizations.

The third feature of the current paper is the use of an extended (non-minimal) state space, such that the state variables correspond to the measured plant input, output and their past measured values, as defined by the structure of the system transfer function model. Since in that representation, all the states are accessible, this approach eliminates the need of using an observer and the proposed scheme can answer the constrained output-feedback design demands.

II. Problem statement

Consider the problem of regulating to the origin the following linear discrete-time system, described by the input-output relationship:

$$\begin{aligned} &y(t+1) + D_1 y(t) + D_2 y(t-1) + \ldots + D_n y(t-n+1) \\ &= N_1 u(t) + N_2 u(t-1) + \ldots + N_m u(t-m+1) + w(t) \end{aligned} \tag{1}$$

where $y(t) \in \mathbb{R}^q$, $u(t) \in \mathbb{R}^p$ and $w(t) \in \mathbb{R}^q$. For simplicity, it is assumed that $m = n$.

The matrices D_i and N_i, $i = 1, \ldots, n$ have suitable dimensions and satisfy:

$$\Gamma = \begin{pmatrix} D_1 & \ldots & D_n \\ N_1 & \ldots & N_n \end{pmatrix} = \sum_{i=1}^{s} \alpha_i \Gamma_i \tag{2}$$

where $\alpha_i \geq 0$ and $\sum_{i=1}^{s} \alpha_i = 1$ and

$$\Gamma_i = \begin{pmatrix} D_1^i & \ldots & D_n^i \\ N_1^i & \ldots & N_n^i \end{pmatrix}$$

are the vertices of the polytope. The output and control are subject to the following hard constraints:

$$y(t) \in Y, \quad u(t) \in U \tag{3}$$

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where $Y = \{y : F_y y \leq g_y\}$ and $U = \{u : F_u u \leq g_u\}$ are polyhedral sets and containing the origin in their interior.

It is assumed that, the disturbance $w(t)$ is unknown, but contained in a bounded polytopic set W, i.e. $w(t) \in W$, where $W = \{w : F_w w \leq g_w\}$.

III. State space description

The measured plant input, output and their past measured values are used to represent the states of the plant:

$$\begin{array}{rl} x(t) = & [y(t)^T \quad y(t-1)^T \quad \dots \quad y(t-n+1)^T \\ & u(t-1)^T \quad u(t-2)^T \quad \dots \quad u(t-n+1)^T]^T \end{array} \tag{4}$$

The state space model is then defined as follows:

$$x(t+1) = Ax(t) + Bu(t) + B_w w(t) \tag{5}$$

$$y(t) = Cx(t) \tag{6}$$

$$A = \begin{bmatrix} -D_1 & -D_2 & \dots & -D_n & N_2 & \dots & N_{n-1} & N_n \\ I & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ 0 & I & \dots & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & I & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & I & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & I & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} N_1^T & 0^T & 0^T & \dots & 0^T & I_{p\times p}^T & 0^T & \dots & 0^T \end{bmatrix}^T$$

$$B_w = \begin{bmatrix} I & 0 & 0 & \dots & 0 & 0^T & 0^T & \dots & 0^T \end{bmatrix}^T$$

$$C = \begin{bmatrix} I & 0 & 0 & \dots & 0 & 0 & 0 & \dots & 0 \end{bmatrix}$$

where I and 0 denote the identity and zero matrices.

From the equation (2), it is clear that matrices A and B belong to a polytope:

$$(A, B) \in \Delta \tag{7}$$

with $\Delta = \text{Convex hull}\{(A_1, B_1), (A_2, B_2) \dots, (A_s, B_s)\}$

The vertices (A_i, B_i) are obtained from the vertices of (2).

Although the obtained representation is non-minimal, it has the merit that the original output-feedback problem for the uncertain plant has been transformed into a state-feedback problem where the matrices A and B lie in the polytope defined by (7) and any state-feedback control which is designed for this representation $u = Kx$ can be translated into a dynamic output feedback controller.

Using (3), it is clear that $x(t) \in X \subset \mathbb{R}^{n_x}$ with $n_x = n(q+p)$. Explicitly X is given by:

$$\begin{aligned} X &= \underbrace{Y \times Y \times \ldots \times Y}_{\text{n times}} \times \underbrace{U \times U \times \ldots \times U}_{\text{n times}} \\ &= \{x \in \mathbb{R}^{n_x} : F_x x \leq g_x\}. \end{aligned}$$

IV. The robust peak-to-peak controller [7], [8]

In this section, we address the following problem:

The robust peak to peak control problem: For a prescribed scalar ρ, find a controller that asymptotically stabilizes the system and satisfies for all $w(t)$, such that $w(t) \in W$:

$$J = \frac{\|y\|_\infty}{\|w\|_\infty} \leq \rho$$

over the uncertainty polytope Δ and zero initial condition.

Here we make use of a linear matrix inequalities (LMI). The main idea of the method is that, instead of minimizing the robust induced L_∞-norm, we minimize its upper bound. The optimal state feedback controller is obtained by solving a semidefinite convex optimization problem in conjunction with line search. We extend the previously published results [7], [8] to the case of uncertain systems. In addition, for reducing the conservativeness, we adopt the parameter dependent Lyapunov approach [9].

Before proceeding, consider the following autonomous polytopic discrete time invariant system:

$$\begin{cases} x(t+1) = H(\alpha)x(t) + B_w w(t) \\ \qquad y(t) = Cx(t) \end{cases} \tag{8}$$

where $x(t)$, $y(t)$, $w(t)$ are respectively the state, the output and the disturbance input. The matrix $H(\alpha)$ satisfies:

$$H(\alpha) = \sum_{i=1}^{s} \alpha_i H_i \tag{9}$$

where $\alpha_i \geq 0$ and $\sum_{i=1}^{s} \alpha_i = 1$, and the matrices H_i are given extreme realizations of $H(\alpha)$. Note that the time invariant nature of the model (8) implies an unknown but fixed parameter α.

It is assumed that, $\rho(H(\alpha)) < 1$, where $\rho(H(\alpha))$ is the spectral radius of matrix $H(\alpha)$. This condition implies that the system (8) is robustly stable.

The non-degenerate E_x ellipsoid in $\mathbb{R}^{n_x}$ with the center at the origin is defined as follows:

$$E_x = \{x \in \mathbb{R}^{n_x} : x^T P^{-1} x \leq 1\}, \quad P \geq 0.$$

The disturbance $w(t)$ satisfies $w(t) \in W$. By calculating the maximal distance from the vertices of the polytope W to the origin, one can find the minimal outer circle[1] with radius R_w that contains W. Furthermore by scaling the matrix B_w, i.e. $B_w = B_w R_w$, it can be always assumed that $R_w = 1$.

Definition 1: The ellipsoid E_x is robustly positively invariant (RPI) with respect to (8) if and only if:

$$\forall x \in E_x \Rightarrow H_i x + B_w w \in E_x, \forall i = 1, 2, \dots, s \text{ and } \forall w \in W$$

If E_x is a RPI ellipsoid, then for $x \in E_x$, the output $y = Cx$ belongs to the ellipsoid:

$$E_y = \{y \in R^p : y^T (CPC^T)^{-1} y \leq 1\}$$

Remark 1: In the following we will be interested in the description of an RPI set with respect to the system (8) starting from ellipsoidal positive invariant sets for the extreme realizations of the polytopic system in (8). We will concentrate on the minimization of the norm of this ellipsoid as it will be defined in the following result. Subsequently, whenever using *minimal* ellipsoids is in the sense of this optimization problem. Note however that the resulting RPI set is not preserving an ellipsoid structure but rather the convex hull of finite number of ellipsoidal sets.

[1] In general terms we can employ ellipsoidal description of the additive uncertainty, but for simplicity of the presentation we choose to work with a scaled unitary norm 2 ball.

Theorem 1: If there exist symmetric matrices P_i, a matrix G and a number $0 < r < 1$ such that:

$$\begin{pmatrix} P_i & H_i G & B_w \\ G^T H_i^T & (1-r)(G+G^T-P_i) & 0 \\ B_w^T & 0 & rI \end{pmatrix} \succeq 0 \quad (10)$$

for all $i = 1, 2, \ldots, s$ then there exists $0 \le \alpha_i \le 1$, $i = 1, \ldots, s$ and $\sum_{i=1}^{s} \alpha_i = 1$ such that the set

$$E_h = \{x \in \mathbb{R}^n : x^T (\sum_{i=1}^{s} \alpha_i P_i)^{-1} x \le 1\} \quad (11)$$

is a collection of positively invariant (and attractive) sets for all possible realizations of polytopic system (8).

Proof: Consider the quadratic Lyapunov function

$$V(x) = x^T Q(\alpha) x$$

for the uncertain system (8) with $Q(\alpha) = (\sum_{i=1}^{s} \alpha_i P_i)^{-1}$ and $P_i \succ 0$, $\forall i = 1, \ldots, s$.

For the invariant property of the set $\{x : V(x) \le 1\}$, we require that $V(x(t+1)) \le 1$ for all x and w such that $x^T Q(\alpha) x \le 1$ and $w^T w \le 1$.That is

$$(H(\alpha)x(t) + B_w w(t))^T Q(\alpha)(H(\alpha)x(t) + B_w w(t)) \le 1$$

or equivalently

$$\begin{pmatrix} x(t) \\ w(t) \end{pmatrix}^T \begin{pmatrix} H(\alpha)^T Q(\alpha) H(\alpha) & H(\alpha)^T Q(\alpha) B_w \\ B_w^T Q(\alpha) H(\alpha) & B_w^T Q(\alpha) B_w \end{pmatrix} \begin{pmatrix} x(t) \\ w(t) \end{pmatrix} \le 1$$

for all x and w such that $x^T Q(\alpha) x \le 1$ and $w^T w \le 1$.

By using the $S-$theorem [10] with two quadratic constraints, one can rewrite the condition (10) as follows:

$$\begin{pmatrix} H(\alpha)^T Q(\alpha) H(\alpha) - \tau_1 Q(\alpha) & H(\alpha)^T Q(\alpha) B_w \\ B_w^T Q(\alpha) H(\alpha) & B_w^T Q(\alpha) B_w - \tau_2 I \end{pmatrix} \preceq 0$$

for some values of $\tau_1 \ge 0$, $\tau_2 \ge 0$, such that $\tau_1 + \tau_2 \le 1$.

It is clear that $H(\alpha)^T Q(\alpha) H(\alpha) \succ 0$, $B_w^T Q(\alpha) B_w \succ 0$ as a consequence of the fact that $Q(\alpha) \succ 0$. Hence τ_1 and τ_2 must be strictly positive. It is also well known that it is nonrestrictive to use $\tau_1 = 1 - \tau_2$ [10].

Using the Schur complements, from the preceding condition one obtains:

$$\begin{array}{c} H(\alpha)^T Q(\alpha) H(\alpha) - (1-r) Q(\alpha) \\ -H(\alpha)^T Q(\alpha) B_w (B_w^T Q(\alpha) B_w - rI)^{-1} B_w^T Q(\alpha) H(\alpha) \preceq 0 \end{array} \quad (12)$$

where $r = \tau_2$.

From the Woodbury formula [11], one has:

$$\begin{array}{c} Q(\alpha) B_w (B_w^T B_w - rI)^{-1} B_w^T Q(\alpha) = \\ = Q(\alpha) - (Q(\alpha)^{-1} - r^{-1} B_w B_w^T)^{-1} \end{array}$$

Then the condition (12) can be rewritten as:

$$-(1-r) Q(\alpha) + H(\alpha)^T (Q(\alpha)^{-1} - r^{-1} B_w B_w^T)^{-1} H(\alpha) \preceq 0$$

By using the Schur complements, one obtains:

$$\begin{pmatrix} (1-r) Q(\alpha) & H(\alpha)^T \\ H(\alpha) & Q(\alpha)^{-1} - r^{-1} B_w B_w^T \end{pmatrix} \succeq 0$$

or equivalently:

$$\begin{array}{c} Q(\alpha)^{-1} - r^{-1} B_w B_w^T - (1-r)^{-1} H(\alpha) Q(\alpha)^{-1} H(\alpha)^T \succeq 0 \\ \Leftrightarrow P(\alpha) - r^{-1} B_w B_w^T - (1-r)^{-1} H(\alpha) P(\alpha) H(\alpha)^T \succeq 0 \end{array}$$

where $P(\alpha) = Q(\alpha)^{-1} = \sum_{i=1}^{s} \alpha_i P_i$. By using Schur complements, the above condition can be written as:

$$\begin{pmatrix} (1-r) P(\alpha) - \frac{(1-r)}{r} B_w B_w^T & H(\alpha) P(\alpha) \\ P(\alpha) H(\alpha)^T & P(\alpha) \end{pmatrix} \succeq 0 \quad (13)$$

Lemma: Condition (13) is satisfied if and only if there exists a matrix G such that

$$\begin{pmatrix} (1-r) P(\alpha) - \frac{(1-r)}{r} B_w B_w^T & H(\alpha) G \\ G^T H(\alpha)^T & G + G^T - P(\alpha) \end{pmatrix} \succeq 0 \quad (14)$$

Note that, from (13) one can easily recover (14) by choosing $G = G^T = P(\alpha) \succ 0$, hence (13) implies (14).

Conversely, by multiplying (14) by $\Xi = \begin{pmatrix} I & -H(\alpha) \end{pmatrix}$ on the left and by Ξ^T on the right one gets (13), hence (14) implies (13) and concludes the proof of the *Lemma*.

It is worth noticing that, by introducing a new additional matrix G, we obtain an LMI in which the Lyapunov matrix $P(\alpha)$ is not involved in any product with the matrix H.

From (14) by using the Schur complements, one can obtain:

$$\begin{pmatrix} P(\alpha) & H(\alpha) G & B_w \\ G^T H(\alpha)^T & (1-r)(G + G^T - P(\alpha)) & 0 \\ B_w^T & 0 & rI \end{pmatrix} \succeq 0 \quad (15)$$

The condition (15) can be treated as a linear function of α and which reaches the minimum on one of the vertices of $H(\alpha)$ and $P(\alpha)$, so the set of LMI conditions to be satisfied to check an invariant property is the following:

$$\begin{pmatrix} P_i & H_i G & B_w \\ G^T H_i^T & (1-r)(G+G^T-P_i) & 0 \\ B_w^T & 0 & rI \end{pmatrix} \succeq 0 \quad (16)$$

for $i = 1, 2, \ldots, s$. □

Theorem 1 states that, for a given realization of $0 \le \alpha_i \le 1$, $i = 1, \ldots, s$, $\sum_{i=1}^{s} \alpha_i = 1$, the set $E_x(\alpha_1, \ldots, \alpha_s) = \{x : x^T (\sum_{i=1}^{s} \alpha_i P_i)^{-1} x \le 1\}$, where P_i is a solution of (15), is a positively invariant set for the respective realization of the system (8).

It is very useful to know the shape of the set E_h in (11). By denoting $Q_i = P_i^{-1}$, it is well known that E_h is the convex hull of the ellipsoids $E_i = \{x : x^T Q_i x \le 1\}$ [12].

If the system is robustly stable, then there exists a unique invariant ellipsoid, which minimizes some convex objective $J(P(\alpha))$. In this paper this objective function is chosen as $J(P(\alpha)) = \sum_{i=1}^{s} \text{trace}(C P_i C^T)$. Here the trace of a square matrix is defined to be the sum of the elements on the main diagonal of the matrix. Minimization of the trace of matrices corresponds to the search for the minimal sum of eigenvalues of matrices. It is important to note that when r is fixed, the conditions (15) are LMIs, for which nowadays, there exist several effective solvers (see for example [14]).

For the synthesis problem, given the system (5), a linear controller structure in the form

$$u(t) = Kx(t) \quad (17)$$

which yields the following closed loop matrices:

$$H(\alpha) = A + BK, \quad B_w = B_w, C = C$$

The following linearizing change of variable $L = KG$ is able to preserve the linearity of the condition given in the Theorem 1 with respect to the synthesis variables P, G and L, providing the set of LMI given in the following corollary:

Corollary 2: Let P, G and L be solutions of the following semidefinite problem:

$$\min \text{trace} \sum_{i=1}^{s} CP_iC^T,$$
$$\text{subject to} \begin{pmatrix} P_i & A_iG + B_iL & B_w \\ G^TA_i^T + L^TB_i^T & (1-r)(G + G^T - P_i) & 0 \\ B_w^T & 0 & rI \end{pmatrix} \succeq 0$$

in the matrix variables $P_i = P_i^T$, G, L and the scalar parameter $r \in (0,\ 1)$. Then the control $u = Kx$, $K = LG^{-1}$ is stabilizing and among all stabilizing control it provides the minimum of invariant ellipsoid size (in the sense of trace criterion) for the output of the closed loop system.

V. Interpolation based controller with linear programming

It is assumed that, the set $E_h = \{x \in R^n : x^T(\sum_{i=1}^{s} \alpha_i P_i)^{-1}x \leq 1\}$ is inside the constrained polyhedral set X. With this assumption, based on procedures in [6], [5] one can find a maximal robustly positively invariant (MRPI) set O_∞ in the form $O_\infty = \{F_o x \leq g_o\}$. Furthermore, with some given and fixed N, one can find a controlled invariant set $P_N = \{F_N x \leq g_N\}$ such that all $x \in P_N$ can be steered into O_∞ in no more than N steps when a suitable control is applied [6].

A. Vertex control law [4]

Given a positve invariant polytope $P_N \in R^n$, this polytope can be decomposed in a sequence of simplices P_N^k each formed by n vertices $x_1^{(k)}, x_2^{(k)}, \ldots, x_n^{(k)}$ and the origin. These simplices have following properties:

- P_N^k has nonempty interior,
- $\text{Int}(P_N^k \cap P_N^l) = \emptyset$ if $k \neq l$,
- $\bigcup_k P_N^k = P_N$,

Denote by $X^{(k)} = (x_1^{(k)}\ x_2^{(k)}\ \ldots\ x_n^{(k)})$ the square matrix defined by the vertices generating P_N^k. Since P_N^k has nonempty interior, $X^{(k)}$ is invertible. Let $U^{(k)} = (u_1^{(k)}\ u_2^{(k)}\ \ldots\ u_n^{(k)})$ be the matrix defined by the admissible control values at these vertices. For $x \in P_N^k$ consider the following linear gain K^k:

$$K^k = U^{(k)}(X^{(k)})^{-1} \tag{18}$$

Theorem 3: The piecewise linear control $u = K^k x$ is constraint admissible and asymptotically stable $\forall x \in P_N$.

Proof: The proof of this theorem is not reported here. The reader is referred to [4] or [5] for more details. □

B. Interpolation via linear programming - Algorithm 1

Any state $x(t)$ in P_N can be decomposed as follows:

$$x(t) = cx_v(t) + (1-c)x_o(t) \tag{19}$$

where $x_v(t) \in P_N$, $x_o(t) \in O_\infty$ and $0 \leq c \leq 1$.

Consider the following control law:

$$u(t) = cu_v(t) + (1-c)u_o(t) \tag{20}$$

where $u_v(t)$ is obtained by applying the vertex control law and $u_o(t) = Kx_o(t)$ is the feasible control law in O_∞.

Theorem 4: The control law (20) is feasible $\forall x \in P_N$.

Proof: Corresponding to the decomposition, the control law is given by (20).

One has to prove that $F_u u(t) \leq g_u$ and $x(t+1) = Ax(t) + Bu(t) + B_w w(t) \in P_N$ for all $x(t) \in P_N$ and all $w(t) \in W$.

One has

$$\begin{aligned} &F_u u(t) = F_u(cu_v(t) + (1-c)u_o(t)) \\ &= cF_u u_v(t) + (1-c)F_u u_o(t) \leq cg_u + (1-c)g_u = g_u \end{aligned}$$

and

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) + B_w w(t) \\ &= A(cx_v(t) + (1-c)x_o(t)) + \ldots \\ &\quad + B(t)(cu_v(t) + (1-c)u_o(t)) + B_w w(t) \\ &= c(Ax_v(t) + Bu_v(t) + B_w w(t)) + \ldots \\ &\quad + (1-c)(Ax_o(t) + Bu_o(t) + B_w w(t)) \end{aligned}$$

We have $Ax_v(t) + Bu_v(t) + B_w w(t) \in P_N$ and $Ax_o(t) + Bu_o(t) + B_w w(t) \in O_\infty \subset P_N \Rightarrow x(t+1) \in P_N$. □

Referring to the discussion in the introduction to give a maximal control action, one would like to minimize c, so the following program is optimized:

$$c^*(x) = \min_{c, x_v, x_o} c, \quad \text{s.t.} \begin{cases} F_N x_v \leq g_N, \\ F_o x_o \leq g_o, \\ cx_v + (1-c)x_o = x, \\ 0 \leq c \leq 1 \end{cases} \tag{21}$$

Denote $r_v = cx_v$, $r_o = (1-c)x_o$. It is clear that $r_v \in cP_N$ and $r_o \in (1-c)O_\infty$ or equivalently $F_N r_v \leq cg_N$ and $F_o r_o \leq (1-c)g_o$. The above non-linear program is translated into a linear program (LP) as follows.

Algorithm 1: Interpolation based on LP

$$c^*(x) = \min_{c, r_v} c, \quad \text{s.t.} \begin{cases} F_N r_v \leq cg_N \\ F_o(x - r_v) \leq (1-c)g_o \\ 0 \leq c \leq 1 \end{cases} \tag{22}$$

Remark 2: If instead of minimization, one maximizes c in (22), it is obvious that $c = 1$ for all $x \in P_N$. In this case the controller turns out to be the vertex controller.

Theorem 5: The control law using interpolation based on linear programming (19), (20), (22) guarantees robustly asymptotic stability for all initial state $x(0) \in P_N$.

Proof: First of all we will prove that all solutions starting in P_N will reach the set O_∞ in finite time.

For this purpose, consider the positive function $V(x) = c^*$ for all $x(t) \in P_N \setminus O_\infty$. $V(x)$ is the Lyapunov function candidate.

For any $x(t) \in P_N$, one has $x(t) = c^*(t)x_v(t) + (1 - c^*(t))x_o(t)$ and $u(t) = c^*(t)u_v(t) + (1 - c^*(t))u_o(t)$. It follows that:

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t) + B_w w(t) \\ &= c^*(t)x_v(t+1) + (1 - c^*(t))x_o(t+1) \end{aligned}$$

where $x_v(t+1) = Ax_v(t) + Bu_v(t) + B_w w(t) \in P_N$ and $x_o(t+1) = Ax_o(t) + Bu_o(t) + B_w w(t) \in O_\infty$.

By using the interpolation based on linear programming, one can obtain $x(t+1) = c^*(t+1)x_v^o(t+1) + (1 - c^*(t+$

$1))x_o^o(t+1)$, where $x_v^o(t+1) \in P_N$ and $x_o^o(t+1) \in O_\infty$. It follows that $c^*(t+1) \leq c^*(t)$ and $V(x)$ is a non-increasing Lyapunov function.

The asymptotically stable property of the vertex control law and the feasible controller over O_∞ assures that there is no initial condition $x(0) \in P_N \setminus O_\infty$ such that $c^*(t) = c^*(0), \forall t \geq 0$. It follows that $V(x) = c^*$ is a Lyapunov function for $x \in P_N \setminus O_\infty$.

Using the vertex controller, an interpolation between the vertices of the feasible invariant set and the origin is obtained. Conversely using the controller (19), (20), (22) an interpolation is constructed between the vertices of the feasible invariant set and those of the MRPI which contains the origin as an interior point. This last property proves that the vertex controller is a feasible choice for the interpolation based technique. From these facts we conclude that the closed sets defined by the Lyapunov function level curves for the closed loop system with the controller (19), (20), (22) are subsets of the closed sets defined by the corresponding Lyapunov function level curves for the closed loop with vertex control. The latter ones are, in fact, homothetical polyhedra with respect to the border of the vertex control feasible invariant set.

The proof is complete by noting that inside the set O_∞ the feasible asymptotically stable controller $u = Kx$ has contractive properties and thus the interpolation-based controller is assuring asymptotic stability for all $x \in P_N$. □

C. Interpolation via linear programming - Algorithm 2

The following properties can be exploited at the construction stage:

1) For $x \in O_\infty$ the result of the optimal interpolation problem has a trivial solution $x_0^* = x$ and thus $c^* = 0$ in (22).
2) Let $x \in P_N \setminus O_\infty$ with a particular convex combination $x = cx_v + (1-c)x_o$, where $x_v \in P_N$ and $x_o \in O_\infty$. If x_o is strictly inside O_∞, one can set $x_o^* = \mathrm{Fr}(O_\infty) \cap \overline{x, x_o}$ (the intersection between the frontier of O_∞ and the line connecting x and x_o). Using convexity arguments $x = c^* x_v + (1-c^*)x_o^*$ with $c^* \leq c$. In general terms, the optimal interpolation process leads to a solution $(x_v, x_o)^*$ with $x_o^* \in \mathrm{Fr}(O_\infty)$.
3) On the other hand, if x_v is strictly inside P_N by setting $x_v^* = \mathrm{Fr}(P_N) \cap \overline{x, x_v}$(the intersection between the frontier of P_N and the line connecting x and x_v) one can obtain $x = c^* x_v^* + (1-c^*)x_o$ with $c^* \leq c$ leading to the conclusion that for the optimal solution $(x_v, x_o)^*$ we have $x_v^* \in \mathrm{Fr}(P_N)$.

From the previous remark we conclude that c will reach a minimum in (22) if x is written as a convex combination of two points, one belonging to the frontier of O_∞ and the other on the frontier of P_N.

As a consequence of the above remark, the vertex control law is only one of several candidates of the controller at the frontier of P_N. It is clear that, any control law that steers the states on the frontier of P_N go inside P_N will make the interpolated control (19), (20, (22) asymptotically stable.

An intuitive approach is to devise a controller, that steers the state at the frontier of the controlled invariant set P_N with the maximal contraction factor.

In order to give a precise definition for *far* points, the following definition is introduced [5].

Definition 6: Given a C-set[2] Θ, the Minkowski functional Γ_Θ of Θ is defined by:

$$\Gamma_\Theta(x) = \min\{\mu \geq 0 : x \in \mu\Theta\} \tag{23}$$

So we will try to minimize the Minkowski functional for the state x_v at the frontier of the feasible set. This can be done by solving the following program:

$$J = \min_{\mu, u} \mu$$
$$\text{s.t.} \begin{cases} F_N(A_i x_v + B_i u) \leq \mu g_N - \max F_N B_w w, \\ F_u u \leq g_u, \\ 0 \leq \mu \leq 1. \end{cases} \tag{24}$$

for all $i = 1, \ldots, s$ and for all $w \in W$.

In summary, the interpolation based controller involves two steps:

Algorithm 2:

1) For any state $x \in P_N$, solve the program (22). In the result one get x_v, x_o and c^* with $x_v \in \mathrm{Fr}(P_N)$, $x_o \in \mathrm{Fr}(O_\infty)$ and $x = c^* x_v + (1-c^*)x_o$.
2) For $x_v \in \mathrm{Fr}(P_N)$, one gets the control value u_v by solving the program (24).
3) The control u is defined as a convex combination of u_v and u_o: $u = c^* u_v + (1-c^*)u_o$.

It is worth noticing that, for the algorithm 2, one has to solve two linear programs at each time instant, hence this algorithm is more computationally demanding than the algorithm 1. However if the number of vertices of the feasible set P_N exceeds the number of facets, the algorithm 2 is preferable, due to the complexity of the global vertex controller of the algorithm 1. In order to guarantee the stability of the control scheme over the entire set P_N, this has to enjoy contractiveness properties with respect to the control degree of freedom

VI. Examples

Consider the following discrete time system:

$$\begin{aligned} y(t+1) - (2-0.1k_2)y(t) + (1-0.1k_2)y(t-1) = \\ = 0.1k_1 u(t) + (0.01-0.1k_1)u(t-1) + w(t) \end{aligned}$$

with $k_1 = 0.787$, $0.1 \leq k_2 \leq 3$ and the sampling time $1sec$.

The constraints are $-10 \leq y(t) \leq 10$ and $-8 \leq u(t) \leq 8$ and $-0.01 \leq w \leq 0.01$ and the state vector $x(t)$ is:

$$x(t) = (y(t) \quad y(t-1) \quad u(t-1))^T$$

with the state space model (5) given by:

$$A = \begin{bmatrix} (2-0.1k_2) & -(1-0.1k_2) & (0.01-0.1k_1) \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
$$B = \begin{bmatrix} 0.1k_1 & 0 & 1 \end{bmatrix}^T, B_w = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}^T$$
$$C = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$$

[2]A C-set is a convex and compact set containing the origin in its interior.

Using the polytopic uncertainty description, one has

$$A = \alpha A_1 + (1-\alpha)A_2, \text{ where}$$
$$A_1 = \begin{bmatrix} 1.99 & -0.99 & -0.0687 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$
$$A_2 = \begin{bmatrix} 1.7 & -0.7 & -0.0687 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$0 \leq \alpha \leq 1$ is a fixed and random number. At each time instant $-0.01 \leq w \leq 0.01$ is an uniformly distributed pseudo-random number.

Using the robust peak to peak controller, the feedback gain is obtained without knowledge of α:

$$K = \begin{pmatrix} -22.7894 & 10.6006 & 0.8729 \end{pmatrix}$$

It is worth noticing that, this controller can be described in the output-feedback form

$$K(z) = \frac{-22.7894 + 10.6006z^{-1}}{1 - 0.8729z^{-1}}$$

Overall the control scheme is described by a second order plant and a first order local controller which provides a reduced order solution for the stabilization problem. By using procedures in [5], [6], one obtains the set O_∞ and P_N. Due to the limited space, the Figure, presenting the set O_∞ and P_N is not shown here. Using algorithm 2 for an initial condition $x_0 = \begin{pmatrix} 3.8890 & 5.8079 & -8.0000 \end{pmatrix}^T$ and the realization of $\alpha = 0.1016$, we obtain the input and output trajectories in Figure 1.

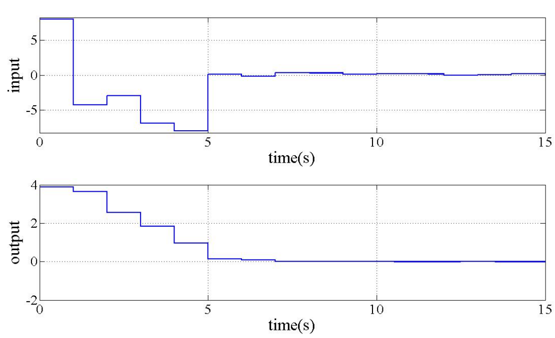

Fig. 1. The input and output trajectory for example 1.

Figure 2 shows the interpolating coefficient $c(t)$ and the realization of $w(t)$. As expected, $c(t)$ is a positive and non-increasing function, assimilated to the decrease of a Lyapunov function for the closed-loop system.

VII. Conclusion

This paper discussed the output-feedback constrained control law design. The invariant set techniques are used to derive a robust *peak to peak* controller by the optimal rejection of unknown but bounded disturbances for an uncertain linear time-invariant discrete time systems. As a second contribution, an interpolation scheme based on linear programming is introduced in two formulations. Both schemes use the local peak to peak controller around the origin. The other term of the interpolation is given either by a predefined global vertex control action or by the optimal (on-line computed) *contractive* control action at the frontier of the controlled invariant set. The resulting interpolation based control assures the asymptotic stability in presence of constraints.

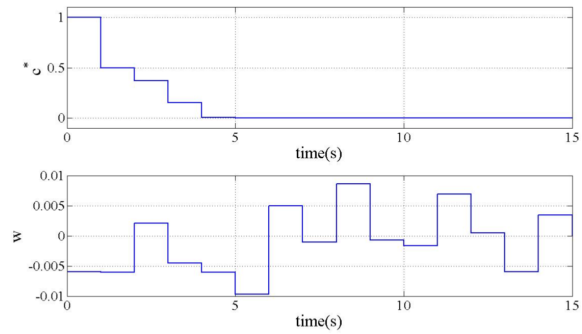

Fig. 2. The interpolating coefficient $c(t)$ and the realization of $w(t)$ for example 1.

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