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Robust stability of time-delay systems with structured uncertainties: a μ-analysis based algorithm

Fabien Lescher and Clément Roos

Abstract— **This paper considers robust stability of uncertain time-delay systems affected by structured uncertainties and multiple constant delays. The objective is to compute the maximum value $\overline{\tau}$ the delays can reach without destabilizing the system over the uncertainty domain. A suitable modeling of the phase variations induced by the delays along the frequency range first allows to obtain an equivalent μ-analysis problem, where the bounds on some uncertain parameters depend on frequency. An algorithm is then proposed to solve this specific problem and to compute upper and lower bounds on $\overline{\tau}$. It is finally shown that the gap between both bounds can be reduced to any positive value in case of purely real uncertainties. The computational efficiency of the method and its ability to analyze large-scale systems are demonstrated on a numerical example, which aims at computing the MIMO time-delay margin of a high-fidelity parameter-dependent flexible aircraft model.**

I. Introduction

Uncertain time-delay systems arise in many fields such as engineering, biology, physics, operations research and economics ([21]). They have notably received much attention in the automatic control community during the last two decades (see for instance [17] for a survey). Indeed, time-delays and uncertainties appear in most control engineering problems, and the presence of sensors and actuators connected in feedback loops is generally sufficient to induce some non-negligible time-delay effects. Time-domain approaches, which generally use Lyapunov-Krasovskii or Lyapunov-Razumikhin functionals ([23], [13]), or the quadratic separation principle ([7]), have a predominant place in the literature. But frequency domain approaches have also been proposed. They are mainly based on the small-gain theorem, and use for example the IQC ([9]) or the μ-analysis ([8]) framework. Research on time-delay systems is still very active, with recent works from Niculescu, Olgar, Sipahi and many others (see [12], [3] and references therein).

The main issue in many papers is to transform the time-delay expression $e^{-\tau s}$ into a more regular form that can be handled by the different analysis techniques. This point is crucial, since whatever the approach used, the conservatism of the analysis is mainly related to the modeling of the time-delay effects. A first rough modeling consists in eliminating the time-delay elements by covering their value sets with unit disks, which leads to conservative results, since all the phase information and some gain information are lost. This step appears in a transparent form in the Small Gain Theorem based approaches. Moreover, it is stated in [25] that the modeling strategy of most time-domain methods (such as Lyapunov-based methods) is equivalent to this covering by the unit disk. In the case of constant (*i.e.* time-invariant) delays, different rational approximations have been introduced, such as Padé approximations ([2]) or dynamic transfer functions, using the $\mathcal{H}_\infty$ framework ([8]). However, most of these modelings introduce some conservatism, which can be reduced by increasing the complexity of the function representing the time-delay effects, to the detriment of the computational cost. The Rekasius transformation is also worth being mentioned as a mean to get rid of the exponential term $e^{-\tau s}$ (see for example [10], [15]).

An original modeling of the time-delay effects is proposed in this paper through the use of a quite simple static function depending on a parameter whose variation range depends on frequency. It fully exploits the phase properties of the delay function, does not introduce any conservatism, and allows to consider multiple delays. A non-standard μ-analysis problem is obtained, whose particularity is that the variation range of some uncertain parameters depends on frequency. Some classical μ lower and upper bounds algorithms are then slightly modified to take this into account, exploiting some theoretical results developed in [22]. The proposed methodology benefits from the advantages of μ-analysis techniques, which are known to efficiently handle large-scale systems (see for instance [20]). Moreover, such techniques allow to consider a wide class of structured uncertainties representing for example unknown parameters and unmodeled dynamics.

A classical drawback of frequency domain robustness analysis methods (such as μ-analysis) is that the derived stability conditions hold only for a single frequency. An infinite number of conditions must thus be checked in order to guarantee robust stability over the whole frequency range. This drawback is here encompassed by the computation of some confidence frequency intervals, inside which the robust stability condition remains valid. Hence, the method is able to provide a rigorous guarantee of robust stability over the whole frequency domain. Moreover, the conservatism of the resulting algorithm, which is only due to the fact that μ upper bounds are computed instead of the exact values, can be mastered thanks to the use of an appropriate branch and bound procedure.

The paper is organized as follows. The analysis problem is stated in Section II. It is first reformulated in the μ-analysis framework in Section III thanks to a suitable model transformation, and then solved in Section IV. Three numerical examples are finally provided in Section V.

Notations: $\|z\|$ and $\angle z$ denote the modulus and the phase

Both authors are with the Systems Control and Flight Dynamics Department of ONERA (The French Aerospace Lab), Toulouse, France clement.roos@onera.fr

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angle of the complex scalar z respectively. $\overline{\sigma}(M)$ denotes the maximum singular value of the complex matrix M. I_n corresponds to the $n \times n$ identity matrix. $\mathcal{F}_u(M, \Delta)$, where $M = \begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}$, denotes the upper Linear Fractional Transformation obtained by closing Δ around the upper channels of M : $\mathcal{F}_u(M, \Delta) = M_{22} + M_{21}\Delta(I - M_{11}\Delta)^{-1}M_{12}$.

II. Problem Statement

Consider the following uncertain time-delay system:

$$\dot{x}(t) = A_0 x(t) + \sum_{i=1}^{N} A_i x(t - \tau_i) \tag{1}$$

The positive scalars τ_i, $i = 1, \ldots, N$ represent constant delays, while A_i, $i = 0, \ldots, N$ are $n \times n$ constant and uncertain matrices whose dependence to the uncertainty matrix $\Delta_u(s)$ is assumed to be rational. $\Delta_u(s)$ is a block-diagonal $n_u \times n_u$ LTI operator, which gathers all model uncertainties. It is composed of real scalars (corresponding to parametric uncertainties), as well as complex scalars and unstructured transfer matrices (representing neglected dynamics). Let $\mathbf{\Delta_u}$ be the set of all such structured operators and let $\mathbf{B\Delta_u} = \{\Delta_u(s) \in \mathbf{\Delta_u} : \overline{\sigma}(\Delta_u(j\omega)) \leq 1 \; \forall \omega \in \mathbb{R}^+\}$.

It is assumed that the following necessary condition for robust stability of system (1) holds.

Assumption 2.1: The system (1) is stable for all $\Delta_u(s) \in \mathbf{B\Delta_u}$ in the absence of delays, *i.e.* for $\tau_i = 0$, $i = 1, \ldots, N$.

In this context, the stability problem considered in this paper can be stated as follows:

Problem 2.1: Compute the maximum value $\overline{\tau}$ of τ such that the system (1) is stable $\forall \Delta_u(s) \in \mathbf{B\Delta_u}$ and $\forall \tau_i \leq \tau$, $i = 1, \ldots, N$.

III. Model Transformation

Without loss of generality, system (1) can be transformed into the standard interconnection structure $M(s) - \Delta(s)$ of Fig. 1. $M(s)$ is a stable real-valued LTI plant representing the system without uncertainties and $\Delta(s)$ is defined as:

$$\Delta(s) = \text{diag}\left(\Delta_u(s), e^{-\tau_1 s}, \ldots, e^{-\tau_N s}\right) \tag{2}$$

The dependence of $\Delta(s)$ on the parameters τ_i in (2) is non-rational. Thus, powerful robustness tools such as μ-analysis cannot be applied, unless the effects of the delays are modeled in a different way.

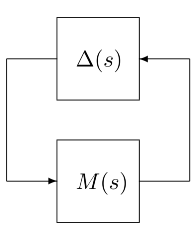

Fig. 1. Standard interconnection structure $M(s) - \Delta(s)$.

Remember that the introduction of a delay generates a phase variation whose value depends on the frequency ω.

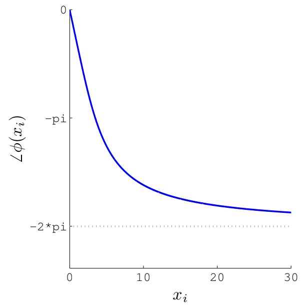

Fig. 2. Phase of $\phi(x_i)$.

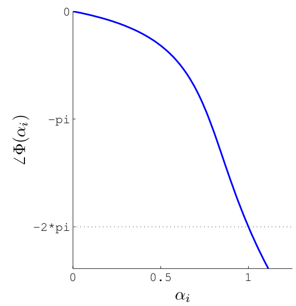

Fig. 3. Phase of $\Phi(\alpha_i)$.

More precisely, the system (1) is robust to the introduction of a delay τ_i if it remains stable despite any phase variation comprised between $-\tau_i\omega$ and 0. In this context, the idea is to replace each exponential term $e^{-j\tau_i\omega}$ by a rational expression $\Phi(\alpha_i)$ such that $\|\Phi(\alpha_i)\| = 1$ and $\angle\Phi(\alpha_i) \in [-\tau_i\omega,\, 0]$ when α_i belongs to a suitable set parameterized by ω.

In this perspective, let us first introduce the function:

$$x_i \mapsto \phi(x_i) = \frac{d(-x_i)}{d(x_i)}$$

where $d(.)$ is a polynomial of order 2 defined by:

$$d(y) = \sum_{q=0}^{2} \frac{4(4-q)!}{4!q!(2-q)!} j^q y^q$$

$\phi(x_i)$ integrally describes the complex unit circle when x_i covers $[0,\, +\infty)$: its modulus $\|\phi(x_i)\|$ is equal to 1 for all $x_i \in \mathbb{R}^+$, and its phase $\angle\phi(x_i)$ is illustrated in Fig. 2. Note that the polynomial $d(.)$ is chosen such that $\phi(\frac{\tau_i s}{j})$ corresponds exactly to the 2nd-order Padé approximation of the delay function $e^{-\tau_i s}$.

Let us then perform the following change of variable:

$$x_i = \frac{\alpha_i}{1 - \alpha_i}$$

and define the function $\alpha_i \mapsto \Phi(\alpha_i)$ such that $\Phi(\alpha_i) = \phi(x_i)$. The phase $\angle\Phi(\alpha_i)$ now covers the whole set $[-2\pi,\, 0]$ when α_i describes the *finite* interval $[0,\, 1]$. More precisely, as shown in Fig. 3, $\angle\Phi(.)$ performs a bijection from the interval $[0,\, 1]$ to the phase interval $[-2\pi,\, 0]$.

Let us finally introduce the frequency-dependent function $\omega \mapsto g_\tau(\omega)$ defined by:

$$\begin{cases} \angle\Phi(g_\tau(\omega)) = -\tau\omega & \text{if} \quad \omega \in [0,\, \frac{2\pi}{\tau}] \\ \angle\Phi(g_\tau(\omega)) = -2\pi & \text{if} \quad \omega > \frac{2\pi}{\tau} \end{cases} \tag{3}$$

Lemma 3.1: Problem 2.1 can be solved using either:

1) the interconnection $M(j\omega) - \Delta(j\omega)$, where $\Delta(j\omega)$ is obtained from (2) and $\tau_i \leq \tau$, $i = 1, \ldots, N$,
2) the interconnection $M(j\omega) - \tilde{\Delta}(j\omega)$, where $\tilde{\Delta}(j\omega) = \text{diag}(\Delta_u(j\omega), \Phi(\alpha_1), \ldots, \Phi(\alpha_N))$ and $\alpha_i \in [0,\, g_\tau(\omega)]$, $i = 1, \ldots, N$.

Proof: The phase variation generated by the delay $e^{-\tau_i s}$, $\tau_i \in [0,\, \tau]$ covers exactly, at each frequency ω, the phase range described by $\angle\Phi(\alpha_i)$ for $\alpha_i \in [0,\, g_\tau(\omega)]$. ■

As its dependence with respect to α_i is rational, $\Phi(\alpha_i)$ can be rewritten as a Linear Fractional Representation (LFR). Using for instance [14], a minimal representation (in terms of size of the LFR Δ-matrix) is obtained and corresponds to a static LFR with a 2×2 Δ-matrix (parameter α_i is repeated twice). Hence, $\Phi(\alpha_i)$ can be rewritten as $\Phi(\alpha_i) = \mathcal{F}_u(\tilde{\Phi}, \alpha_i I_2)$, where $\tilde{\Phi}$ is a constant matrix. Problem 2.1 can thus be solved using the interconnection $P(jw) - \Delta'(jw)$, where $P(s)$ is an LTI model obtained from $M(s)$ and $\tilde{\Phi}$, and $\Delta'(j\omega) = \text{diag}\,(\Delta_u(j\omega), \alpha_1 I_2, \ldots, \alpha_N I_2)$.

The initial problem has now been recast in the μ-analysis framework. Note that no conservatism has been introduced during the aforementioned model transformation. Moreover, modeling complexity has been kept as low as possible in order to reduce the computational load of the analysis (especially in the case of multiple delays). The proposed description of the time-delay effects can for instance be compared to the one employed in [2], which corresponds to a 5th-order parametrized Padé approximation (thus resulting in an LFR with a 5×5 Δ-matrix), and which induces a non-zero conservatism (about 0.4%).

IV. Robust stability analysis

The robustness analysis problem formulated in Section III is particular, since the variation range of the uncertain parameters α_i depends on the frequency. Consequently, it slightly differs from a standard μ-analysis problem, but it can be handled using an extension of the small-μ theorem to systems with frequency-dependent uncertainty bounds ([22]): robust stability is guaranteed for such systems if the classical structured singular value condition is satisfied on the whole frequency range, provided adequate bounds on the uncertainties are considered at each frequency. The following lemma is a direct application of this extension.

Lemma 4.1: Let $W(\omega) = \begin{pmatrix} I_{n_u} & 0 \\ 0 & g_\tau(\omega) I_{2N} \end{pmatrix}$. The uncertain time-delay system (1) is stable $\forall \Delta_u(s) \in \mathbf{B\Delta_u}$ and $\forall \tau_i \leq \tau$, $i = 1, \ldots, N$ if:

$$\mu_{\mathbf{\Delta}'} \left(W(\omega) P(j\omega) \right) \leq 1 \quad \forall \omega \in \mathbb{R}^+ \tag{4}$$

Problem 2.1 has thus been transformed, at each frequency, into a standard μ-analysis problem, which can be classically solved by computing (skew-)μ lower and upper bounds (computing the exact value of μ is indeed NP-hard). Condition (4) is, however, conservative: the sign constraints $\alpha_i \geq 0$ are not considered, which can lead to an under-evaluation of $\overline{\tau}$. Two solutions are proposed in the sequel to overcome this problem, depending on the nature of the computed μ bound. Note that μ-analysis is not broached as such in this paper due to space limitations, but a good introduction to this technique can be found in [4].

A. Computation of a $\overline{\tau}$ upper bound

A positive scalar $\overline{\tau}_u$ is a $\overline{\tau}$ upper bound if there exists a value $\tau_i \leq \overline{\tau}_u$ for some $i = 1, \ldots, N$ which brings system (1) to instability. It does not provide any guarantee about stability, but it is used to evaluate the accuracy of the $\overline{\tau}$ lower bound computed in Section IV-B.

Such a bound can be obtained by computing, for each point ω_k of a rough frequency grid, a matrix $\Delta'_k = \text{diag}\,(\Delta_{u,k}, \alpha_{1,k} I_2, \ldots, \alpha_{N,k} I_2)$ such that:

$$\det\left(I - P(j\omega_k)\Delta'_k\right) = 0 \tag{5}$$

where $\overline{\alpha}_k = \max_{i=1,\ldots,N} \alpha_{i,k}$ is minimal, $\alpha_{i,k} \geq 0$, $i = 1, \ldots, N$ and $\Delta_{u,k}$ is an admissible block-diagonal matrix such that $\overline{\sigma}(\Delta_{u,k}) \leq 1$. This is a skew-$\mu$ problem ([4]), but it cannot be directly solved using existing algorithms (see for example [19] and references therein). Indeed, these algorithms do not allow to consider the sign constraints $\alpha_{i,k} \geq 0$ associated to the delays.

This problem can be tackled by setting $\alpha_i = \xi_i^2$. The interconnection $P(j\omega) - \Delta'(j\omega)$ can then be equivalently rewritten as $\tilde{P}(j\omega) - \tilde{\Delta}'(j\omega)$, where $\tilde{\Delta}'(j\omega) = \text{diag}(\Delta_u(j\omega), \xi_1 I_4, \ldots, \xi_N I_4)$. Hence, for each frequency ω_k of the grid, a matrix $\tilde{\Delta}'_k = \text{diag}(\Delta_{u,k}, \xi_{1,k} I_4, \ldots, \xi_{N,k} I_4)$ is computed such that $\det\left(I - \tilde{P}(j\omega_k)\tilde{\Delta}'_k\right) = 0$, where $\overline{\xi}_k = \max_{i=1,\ldots,N} \xi_{i,k}$ is minimal and $\Delta_{u,k}$ is an admissible block-diagonal matrix such that $\overline{\sigma}(\Delta_{u,k}) \leq 1$. A combination of delays $(\tau_{i,k})_i$ which destabilizes system (1) at ω_k is then determined by computing $\tau_{i,k} = -\frac{\angle \Phi({\xi_{i,k}}^2)}{\omega_k}$, $i = 1, \ldots, N$. The $\overline{\tau}$ upper bound $\overline{\tau}_u$ is finally obtained as:

$$\overline{\tau}_u = \min_k \left(\max_{i=1,\ldots,N} \tau_{i,k} \right) \tag{6}$$

Remark 4.1: The change of variable $\alpha_i \leftarrow \xi_i^2$ does not introduce any conservatism. Nevertheless, it increases the number of repetitions of the parameters inside the uncertain matrix: α_i is repeated twice in $\Delta'(j\omega)$, whereas ξ_i is repeated 4 times in $\tilde{\Delta}'(j\omega)$. This leads to an increase in the computational time required to compute a $\overline{\tau}$ upper bound, which still remains much lower in most cases than the one required to compute a $\overline{\tau}$ lower bound in Section IV-B.

B. Computation of a $\overline{\tau}$ lower bound

Several μ-upper bound computations will allow to determine whether a given value $\overline{\tau}_l$ is a $\overline{\tau}$ lower bound, *i.e.* whether robust stability of system (1) can be guaranteed $\forall \tau_i \leq \overline{\tau}_l$, $i = 1, \ldots, N$.

As Lemma 4.1 does not consider the constraint on the sign of the uncertain parameters α_i, another theorem is derived in order to overcome this problem. For this purpose, the parameters α_i are first normalized at each frequency $\omega \in \mathbb{R}^+$ by performing the following change of variables:

$$\alpha_i = \frac{g_{\overline{\tau}_l}(\omega)}{2} \left(\zeta_i + 1 \right) \tag{7}$$

Note that α_i covers exactly the interval $[0, g_{\overline{\tau}_l}(\omega)]$ when the normalized parameters ζ_i vary between -1 and 1. Let us then define $P_\omega(j\omega) - \Delta''(j\omega)$, where $\Delta''(j\omega) = \text{diag}\,(\Delta_u(j\omega), \zeta_1 I_2, \ldots, \zeta_N I_2)$, the interconnection built from $P(j\omega) - \Delta'(j\omega)$ by incorporating the change of variables (7).

Theorem 4.1: The uncertain time-delay system (1) is stable $\forall \Delta_u(s) \in \mathbf{B\Delta_u}$ and $\forall \tau_i \leq \overline{\tau}_l$, $i = 1, \ldots, N$ iff:

$$\mu_{\mathbf{\Delta}''}(P_\omega(j\omega)) \leq 1 \quad \forall \omega \in \mathbb{R}^+ \tag{8}$$

Proof: Using the representation of the time-delays effects introduced in Section III and the small μ-theorem extended to systems with frequency-dependent uncertainty bounds, the two following statements are equivalent:

1) System (1) is stable $\forall \Delta_u(s) \in \mathbf{B\Delta_u}$ and $\forall \tau_i \leq \overline{\tau}_l$, $i = 1 \ldots N$.
2) The interconnection $P(j\omega) - \Delta'(j\omega)$ has no singularity on the imaginary axis whatever the frequency $\omega \in \mathbb{R}^+$, *i.e.* $\det(I - P(j\omega)\Delta'(j\omega)) \neq 0$ $\forall \Delta_u(s) \in \mathbf{B\Delta_u}$, $\forall \alpha_i \in [0, g_{\overline{\tau}_l}(\omega)]$, $i = 1 \ldots N$ and $\forall \omega \in \mathbb{R}^+$.

From the definition of the structured singular value μ and the change of variables (7), condition 2) is equivalent to (8). ■

Computing the exact value of μ is NP-hard, so an upper bound $\overline{\mu}$ is usually determined instead using some efficient polynomial-time algorithms ([24]). But the main drawback of Theorem 4.1 is that an infinite number of constraints must be satisfied in order to cover the whole frequency range. Most of the times, condition (8) is thus only checked on a finite frequency grid. Nevertheless, a critical frequency can be missed and the lower bound $\overline{\tau}_l$ can be over-evaluated, *i.e.* be larger than the real value of $\overline{\tau}$.

In the context of μ-analysis, various solutions exist to overcome this problem (see for instance [6], [11]). More recently, a numerically very efficient algorithm was presented in [20]. Consider a standard interconnection $M(j\omega) - \Delta(j\omega)$, where $\Delta(j\omega)$ is an admissible block-diagonal matrix such that $\overline{\sigma}(\Delta(j\omega)) \leq 1$ for all $\omega \in \mathbb{R}^+$. Assume that this interconnection satisfies the constraint $\overline{\mu}_{\mathbf{\Delta}}(M(j\omega_k)) \leq \beta < 1$ for a given frequency ω_k. The key idea of the algorithm of [20] is to compute a non-empty frequency segment $\mathcal{I}_M(\omega_k) = [\omega_k^-, \omega_k^+]$ such that $\omega_k \in \mathcal{I}_M(\omega_k)$ and:

$$\overline{\mu}_{\mathbf{\Delta}}(M(j\omega)) \leq 1 \quad \forall \omega \in \mathcal{I}_M(\omega_k) \tag{9}$$

This can be achieved by computing the eigenvalues of an appropriate Hamiltonian matrix. An iterative procedure is then implemented, which tries to determine a frequency grid $(\omega_k)_k$ such that $\bigcup_k \mathcal{I}_M(\omega_k)$ covers the whole frequency range. In case of success, it is guaranted that $M(s)$ is robustly stable and that no critical frequency has been missed.

Nevertheless, this algorithm only considers uncertain systems with frequency-independent uncertainty bounds. It must thus be adapted to the case where the bounds on some uncertainties depend on frequency. Indeed, it becomes necessary to consider a frequency-dependent operator $P_\omega(s)$ in condition (9).

Proposition 4.1: The interconnection $M(j\omega) - \Delta(j\omega)$ representing the uncertain time-delay system (1), where $\Delta(j\omega)$ is obtained from (2), satisfies:

$$\det(I - M(j\omega)\Delta(j\omega)) \neq 0 \quad \begin{cases} \forall \Delta_u(s) \in \mathbf{B\Delta_u} \\ \forall \tau_i \leq \overline{\tau}_l \\ \forall \omega \in [\omega_k^-, \omega_k^+] \end{cases} \tag{10}$$

if the interconnection $P_{\omega_k^+}(j\omega) - \Delta''(j\omega)$ satisfies:

$$\mu_{\mathbf{\Delta}''}(P_{\omega_k^+}(j\omega)) \leq 1 \quad \forall \omega \in [\omega_k^-, \omega_k^+] \tag{11}$$

$P_{\omega_k^+}(j\omega)$ is built such that the interconnection $P_{\omega_k^+}(j\omega) - \Delta''(j\omega)$ is equivalent to the interconnection $P(j\omega) - \Delta'(j\omega)$ with the change of variables:

$$\alpha_i = \frac{g_{\overline{\tau}_l}(\omega_k^+)}{2}(\zeta_i + 1) \tag{12}$$

Proof: Using change of variables (12), condition (11) is equivalent to:

$$\det(I - P(j\omega)\Delta'(j\omega)) \neq 0 \quad \begin{cases} \forall \Delta_u(s) \in \mathbf{B\Delta_u} \\ \forall \alpha_i \in [0, g_{\overline{\tau}_l}(\omega_k^+)] \\ \forall \omega \in [\omega_k^-, \omega_k^+] \end{cases} \tag{13}$$

Because $\omega \mapsto g_{\overline{\tau}_l}(\omega)$ is a strictly decreasing function, condition (13) also holds $\forall \alpha_i \in [0, g_{\overline{\tau}_l}(\omega)]$. Noting that $\alpha_i \in [0, g_{\overline{\tau}_l}(\omega)]$ is equivalent to $\tau_i \in [0, \overline{\tau}_l]$ for the interconnection $M(s) - \Delta(s)$ finally allows to claim that (13) implies (10), which concludes the proof. ■

A method is now proposed to determine an interval $[\omega_k^-, \omega_k^+]$ as large as possible around a given frequency ω_k, inside which condition (11) holds. ω_k being fixed, the operator $P_{\omega_k}(s)$ does not depend on frequency anymore. Neither do the uncertainties ζ_i, which all take their values inside the interval $[-1, 1]$. The algorithm of [20] can thus be applied to compute $\mathcal{I}_{P_{\omega_k}}(\omega_k) = [\omega^-, \omega^+]$, which means that $\overline{\mu}_{\mathbf{\Delta}''}(P_{\omega_k}(j\omega)) \leq 1$ $\forall \omega \in [\omega^-, \omega^+]$. Condition (11) thus holds with $\omega_k^- = \omega^-$ and $\omega_k^+ = \omega_k$, since $\omega_k \in [\omega^-, \omega^+]$.

The idea is now to increase as much as possible the value of ω_k^+. Indeed, ω_k^+ can be any frequency $\tilde{\omega} \geq \omega_k$ such that $\tilde{\omega} \in \mathcal{I}_{P_{\tilde{\omega}}}(\omega_k)$. An iterative procedure is proposed. At each step, a test frequency ω_t (initialized at ω_k) is slightly increased and the interval $\mathcal{I}_{P_{\omega_t}}(\omega_k) = [\omega_t^-, \omega_t^+]$ is computed, as long as the inequality $\omega_t \leq \omega_t^+$ remains true. When the procedure stops, ω_k^+ is set to the last value of ω_t satisfying $\omega_t \leq \omega_t^+$.

The aforementioned strategy is summarized in Algorithm 4.1.

Algorithm 4.1: 1) Select a frequency ω_k for which condition (10) has not been checked yet. Set $\omega_t \leftarrow \omega_k$.
2) If $\overline{\mu}_{\mathbf{\Delta}''}(P_{\omega_t}(j\omega_k)) > 1$, **stop**. Otherwise, compute $\mathcal{I}_{P_{\omega_t}}(\omega_k) = [\omega_t^-, \omega_t^+]$. Set $\omega_k^- \leftarrow \omega_t^-$ and $\omega_k^+ \leftarrow \omega_t$.
3) Repeat:
   - Set $\omega_t \leftarrow (1+\epsilon)\omega_t$, where ϵ is a small user-defined positive constant.
   - If $\overline{\mu}_{\mathbf{\Delta}''}(P_{\omega_t}(j\omega_k)) < 1$, compute $\mathcal{I}_{P_{\omega_t}}(\omega_k) = [\omega_t^-, \omega_t^+]$. Otherwise, **return to step** 1.
   - If $\omega_t \leq \omega_t^+$, set $\omega_k^+ \leftarrow \omega_t$. Otherwise, **return to step** 1.

Algorithm 4.1 provides a set of frequency intervals $[\omega_k^-, \omega_k^+]$ inside which $\det(I - M(j\omega)\Delta(j\omega)) \neq 0$ $\forall \Delta_u(s) \in \mathbf{B\Delta_u}$ and $\forall \tau_i \leq \overline{\tau}_l$. If $\bigcup_k [\omega_k^-, \omega_k^+]$ covers the whole frequency range, then robust stability of the uncertain time-delay system (1) is guaranteed for all $\tau_i \leq \overline{\tau}_l$ and $\overline{\tau}_l$ is a $\overline{\tau}$ lower bound.

C. Reduction of conservatism

Conservatism is defined in this paper as the relative gap η between the lower bound $\overline{\tau}_l$ and the upper bound $\overline{\tau}_u$ on $\overline{\tau}$:

$$\eta = \frac{\overline{\tau}_u - \overline{\tau}_l}{\overline{\tau}_l} \tag{14}$$

Remember that the modeling of the time-delays effects in Section III is non conservative, which means that the conservatism of the proposed method is only due to the fact that μ lower and upper bounds are computed instead of the exact values.

η sometimes reaches unacceptable values, notably in the presence of highly repeated real parametric uncertainties in Δ_u. A well-known technique to ensure that it remains below a specified threshold η_{tol} is to use a branch and bound algorithm ([1], [16]). The idea is to partition the real parametric domain in more and more subsets until the relative gap between the highest lower bound and the highest upper bound computed on all subsets becomes less than η_{tol}. This algorithm is known to converge for uncertain systems with only real uncertainties ([16]), *i.e.* conservatism can be reduced to an arbitrarily small value. However, it usually exhibits an exponential growth of computational complexity as a function of the number of real uncertainties. Specifying a threshold η_{tol} thus allows to handle the tradeoff between the accuracy of the bounds and the computational time.

Nevertheless, in order to alleviate the computational burden, a strategy based on the progressive validation of the frequency range is proposed here. Assume for example that a subset $\mathcal{D}_N$ and a frequency domain Ω_N are considered at step N of the branch and bound procedure. Algorithm 4.1 is applied to compute a frequency domain $\Omega_{v,N} \subset \Omega_N$ such that $\det(I - M(j\omega)\Delta) \neq 0$ holds $\forall \Delta_u \in \mathcal{D}_N$ and $\forall \omega \in \Omega_{v,N}$. During the next step, the analysis performed on each subset of $\mathcal{D}_N$ then only considers the frequencies $\Omega_N \backslash \Omega_{v,N}$ which have not been validated at step N. Consequently, after a few steps, the analysis is only restricted to very narrow frequency intervals corresponding to critical frequencies. This results in a drastic reduction of the computational load induced by a classical branch and bound procedure.

V. NUMERICAL EXAMPLES

In order to illustrate the effectiveness of the method described in Section IV, three numerical examples are provided. The implementation relies on some μ and skew-μ algorithms of the Skew-μ Toolbox [5], and LFR objects are generated using the LFR Toolbox [14]. The calculations are performed with Matlab on a 3 GHz PC with 3 Go RAM.

A. Example 1

The first example is taken from [7]. The considered uncertain time-delay system is described by:

$$\begin{aligned} \dot{x}(t) &= \begin{pmatrix} 0 & -0.12 + 0.42\delta \\ 1 & -0.465 - 0.035\delta \end{pmatrix} x(t) \\ &+ \begin{pmatrix} -0.1 & -0.35 \\ 0 & 0.3 \end{pmatrix} x(t-\tau) \end{aligned}$$

where the uncertain parameter $\delta \in \mathbb{R}$ satisfies $|\delta| \leq 1$, and both δ and τ are time-invariant. With $\eta_{tol} = 0.01$, the computed bounds on $\overline{\tau}$ are $\overline{\tau}_l = 0.8894$ sec and $\overline{\tau}_u = 0.8983$ sec. The value of $\overline{\tau}_l$ is similar to the one obtained in [7]. Moreover, the value of $\overline{\tau}_u$ shows that this result is almost non-conservative, and the value of δ, which destabilizes the system for $\tau = \overline{\tau}_u$, is $\delta = 1$.

B. Example 2

The uncertain time-delay system considered in the second example is described by the following equation:

$$\begin{aligned} \dot{x}(t) &= \begin{pmatrix} -2 + 1.6\delta_1 & 0 \\ 0 & -0.9 + 0.05\delta_2 \end{pmatrix} x(t) \\ &+ \begin{pmatrix} -1 + 0.1\delta_3 & 0 \\ -1 & -1 + 0.3\delta_4 \end{pmatrix} x(t-\tau) \end{aligned}$$

where the uncertain parameters $\delta_i \in \mathbb{R}$, $i = 1, \ldots, 4$ satisfy $|\delta_i| \leq 1$, and both the δ_i and τ are time-invariant. Fig. 4 shows the values of $\overline{\tau}_u$ and $\overline{\tau}_l$ for several values of η_{tol}, as well as the associated computational times. The same value $\overline{\tau}_u = 1.904$ sec is found whatever η_{tol}, and the values of the δ_i, which destabilize the system for $\tau = \overline{\tau}_u$, are $\delta_1 = -1$, $\delta_2 = -0.30$, $\delta_3 = 1$ and $\delta_4 = 0.54$. As expected, $\overline{\tau}_l$ increases as η_{tol} decreases. The computational time seems to grow exponentially in the same time, but it remains quite low for $\eta_{tol} \geq 0.03$. The parameter η_{tol} thus allows to efficiently handle the trade-off between accuracy and conservatism.

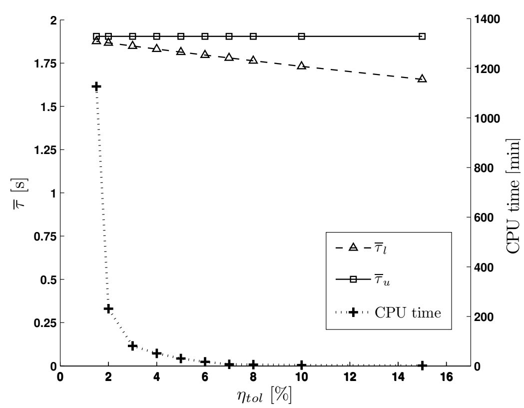

Fig. 4. Example 2 - Upper and lower bounds $\overline{\tau}_u$ and $\overline{\tau}_l$ on $\overline{\tau}$, and associated CPU time, for different values of the conservatism η_{tol}.

Assume now that $|\delta_i| \leq \epsilon$. $\overline{\tau}_u$ increases as ϵ tends to 0, and the value obtained for $\epsilon = 0$ (no uncertainty) is $\overline{\tau}_u = 6.1726$ sec. This result is very close to the exact value of $\overline{\tau}$, which can be computed analytically in this simple case and is equal to $\overline{\tau} = 6.1725$ sec (see [7]). Moreover, noting that only real parametric uncertainties are considered in this example, the gap between $\overline{\tau}_l$ and $\overline{\tau}_u$ can be reduced to any arbitrarily small value. Consequently, the value of $\overline{\tau}_l$ asymptotically tends to $\overline{\tau} = 6.1725$ sec when ϵ and η_{tol} tend to 0.

C. Example 3

The third example considers the longitudinal motion of a civilian passenger aircraft. It is treated to demonstrate the ability of the proposed algorithm to operate on large-scale systems and in presence of several time-delays. The open-loop model of the aircraft includes actuators and sensors models. It has one control input (the elevator deflection δp) and two outputs (the pitch rate q and the vertical load factor N_z at the center of gravity). It accurately describes both the rigid and the flexible dynamics of the aircraft for a given flight point and on a continuum of mass configurations (it is parametrized by the center and the outer tanks filling levels, the embarked payload and the position of the center of gravity). It is written as an LFR $\mathcal{F}_u(\Sigma(s), \Delta_\Sigma)$, where Δ_Σ is a block-diagonal matrix gathering all mass parameters. The whole LFR generation process is thoroughly described in [18]. A parameter dependent controller $\mathcal{F}_u(K(s), \Delta_K)$ is then designed to improve the rigid behavior of the aircraft without destabilizing the flexible modes. The closed-loop system is finally obtained as the feedback interconnection of $\Sigma(s)$ and $K(s)$. Note that its order (35 states), as well as the size of $\Delta_u = \text{diag}(\Delta_\Sigma, \Delta_K)$ (170×170), are quite high. Thus, LMI-based analysis techniques cannot be applied.

The objective consists in computing the MIMO time-delay margin at the sensors output. For this purpose, two fictitious time-delays τ_1 and τ_2 are introduced (see Fig. 5). The problem to be solved is thus as follows: compute the maximum value $\overline{\tau}$ such that the closed loop system remains stable on the whole set of admissible mass configurations and $\forall \tau_i \leq \overline{\tau}$, $i = 1, 2$.

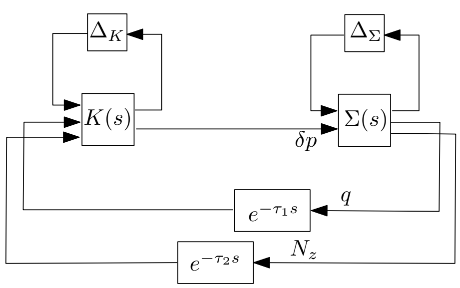

Fig. 5. Example 3 - Closed-loop interconnection for MIMO time-delay margin computation at the system output.

The algorithm described in Section IV is applied with $\eta_{tol} = 0.25$. An optimistic bound $\overline{\tau}_u = 0.392$ sec is obtained, as well as a guaranteed bound $\overline{\tau}_l = 0.313$ sec. The computational time is 6.72 hours. It is quite acceptable, considering the size of this realistic model, which is used in an industrial context for control laws validation.

VI. Conclusion

A μ-analysis based algorithm is proposed in this paper to analyze the stability of systems with both constant time-delays and structured uncertainties. A wide class of time-delay uncertain systems can be considered, since multiple delays, as well as parametric and dynamical uncertainties, are taken into account. Conservatism is efficiently mastered and can be reduced to any positive value in the case of purely real uncertainties. Three numerical examples demonstrate the accuracy of the results and the ability of the algorithm to handle some large-scale industrial systems.

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