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Developing an IIR Robust Adaptive Algorithm in the Modified Filtered-x RLS Form for Active Noise and Vibration Control Systems

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Abstract— **In this paper, a robust adaptive algorithm for active noise and vibration control applications is proposed and the robust stability of the algorithm is analyzed using a combination of the small gain theorem and Popov's hyperstability theorem. The algorithm is developed based on the so-called Filtered-x RLS algorithm in the modified form. In design and analysis of the algorithm, it is assumed that the estimated model of the secondary path is associated with a set of uncertainties of additive structure; and sufficient conditions for stability of the algorithm are derived. In fact, by introducing a stabilizing filter, the aim is to design this filter in a way that the achieved sufficient conditions for robust stability are satisfied. The employed method is to transform the proposed control structure to an equivalent output error identification problem, and then formulate the governing adaptive algorithm in a way that is representable as a feedback control problem. In view of this approach, sufficient conditions for robust stability of the adaptive algorithm will be equivalent to find the conditions for the stability of the established feedback control system. The technique applied here to this end is established on the energy conservation relation that is valid for the general data models in adaptive filters.**

I. Introduction

Active noise and vibration control (ANVC) systems usually deal with a large amount of dominant, weakly damped, resonance modes and need controllers with a large impulse response to obtain a good disturbance rejection. Furthermore, sampling rates are often in the range of 1–10 kHz to have sufficient control bandwidth. The controller not only should be adaptive to be able to follow the time-varying characteristics of the incident undesired noise, but also it has to be robust against variations in the system transfer function. In fact, because of the non-stationary nature of the environment where active noise and vibration control systems (ANVC) operate, the estimated models of the system are prone to changes. Depending on the application this could be due to different phenomena, such as aging, temperature variations, movement of persons, etc. Since in most of the adaptive algorithms developed for ANVC systems an estimation of the model of the secondary path is required, these changes will degrade or even destabilize the adaptive algorithm when it exceeds a predetermined threshold. These constraints make ANVC a challenging control problem even in the time of fast increasing computer power [1].

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There are two main control strategies to design an effective control system which reduces noise and vibration actively: 1. adaptive feedforward methods when a reference signal which is a measure of the incident noise is available; and 2. feedback control systems when this reference signal is not producible [2]. Each of these methods has their own pros and cons when applied in a particular application, and it is on the designer to select the suitable control structure.

Most of the adaptive algorithms designed for ANVC systems belong to the category of indirect adaptive control methods. In almost all of these methods it is assumed that an estimated model of the mechanical or acoustical plant, the so-called secondary path is available. However, due to the time-varying nature of these systems, they are prone to large changes and the designed algorithm has to be made robust against these changes. In the literature, two approaches are proposed to improve the robustness of the designed control algorithm: online secondary path modeling [3], [4] and increasing the robustness of the control algorithm by considering uncertainties in the design of control systems [5], [6], [7]. A group of adaptive feedforward methods which belongs to the larger category of indirect adaptive control techniques, and also well-known in ANVC applications, is the family of *Filtered LMS* algorithms. In these methods, the reference input signal or the error signal which has to be fed back to the LMS algorithm, is filtered linearly before applying it to the adaptive algorithm. One of the algorithms in this family which is widely exploited in different ANVC applications is the so-called FxLMS algorithm [8], [9]. In this algorithm the FIR filter structure is used for the controller and the reference signal is filtered with an estimation of the secondary path to update the coefficients of adaptive filters. Replacing the FIR filter structure with the IIR counterpart results in the so-called FuLMS algorithm. However, instability problems and multi-modal nature of the performance surface of this algorithm, unlike the FxLMS algorithm, has hindered its usage in different applications [10], [11].

Analysis of the FxLMS and FuLMS algorithms reveals that they have some intrinsic robustness against errors in the estimation of the secondary path, and the stability of these algorithms will be maintained despite of some errors in the estimation within the secondary paths [12], [13]. This phenomenon is better explained in [14] by formulating the LMS algorithm as *a priori* $\mathcal{H}_\infty$ optimal filter which has a close optimal solution in the $\mathcal{H}_\infty$ form. In light of this, one may assume that the characteristics are inherited to the LMS algorithm because it is a minimax algorithm or, more precisely, is optimal in the $\mathcal{H}_\infty$ sense. Following

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this concept, but using the *energy conservation* idea, the robustness characteristics of different classes of adaptive algorithms are investigated within a purely deterministic framework in [15], [16]. Specifically, a time domain feedback analysis is performed for the LMS algorithm in [15] and extended to Filtered LMS algorithms for FIR filters. In [16], this type of analysis is also developed and repeated for RLS type adaptive FIR filters.

Most of the approaches proposed in the literature to increase the robustness of adaptive filters and make it less sensitive to modeling errors in ANVC applications are applied to the FxLMS algorithm, e.g. as suggested in [17], a simple way is to add leakage to the FxLMS update rule. Another method, quite similar to leakage, is adding a control effort weighting to the cost function that reduces the control effort. Besides the necessity of tuning a scalar parameter, a drawback of leakage and control effort weighting is that there is no frequency selectivity in the regularization, resulting in too much conservatism. To reduce the control effort, especially in the frequency band where the model uncertainty is large, a robust FxLMS algorithm has been proposed in [18], in which the model uncertainty is considered as a frequency dependent stochastic variable with zero mean and known covariance. In [19], a general robust FxLMS algorithm structure in which the aforementioned regularization methods can be viewed as special case is proposed. The general robust FxLMS algorithm is obtained by adding to the cost function a term where the control signal filtered by a filter is to be designed. By using the concept of $\mathcal{L}_2$-stability and the time domain analysis in [15], two robust variants of the FxLMS algorithm are proposed in [20], in which the error signal is passed through a time-varying filter that has to be designed properly.

In this paper, a robust adaptive IIR filter in the modified form and based on the RLS algorithm is proposed. This is in continuation of previous works [21], [22], [23], in which the algorithm has been improved in different steps. As a matter of fact, the algorithm which in [21] is proposed under the condition of slowly time-varying adaptation of filter weights, is improved in [22], [23] by transforming it to the modified form, and then the robust stability of the algorithm is analyzed when a variable step size coefficient [22] and a nonlinear static function is introduced in the update algorithm [23]. Here it is attempted to increase the robust stability of this algorithm, by filtering the error signal with a linear filter which has to be designed appropriately. By using the energy conservation concept introduced in [16], sufficient conditions for the stability of the algorithms in front of uncertainties in the secondary path, are derived.

The paper is organized as follows. After this introduction, first some important quantities are introduced in Section II. Then the proposed algorithm and some preliminary formulations required for the stability analysis are exposed in Section III. In Section IV, by deriving the relations among different quantities in the feedforward and feedback path, sufficient conditions for robust stability of the algorithm based on the equivalent feedback representation of the algorithm are derived. Conclusions are drawn in Section V.

II. Definitions and Notations

The block diagram of a typical ANVC system in the modified form is depicted in Fig. 1.

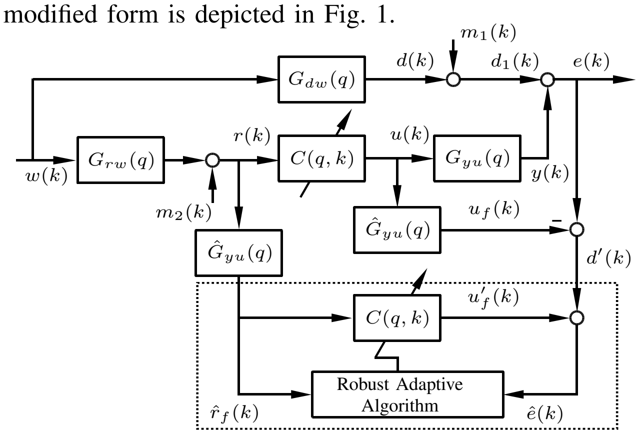

Fig. 1. Block diagram of adaptive IIR filter in the modified form

The primary disturbance $w(k) \in \mathcal{R}$, and the additive noises $m_1(k) \in \mathcal{R}$ and $m_2(k) \in \mathcal{R}$ are zero mean (possibly colored) processes which are uncorrelated with each other. Besides, in this figure $G_{dw}(q) \in \mathcal{RH}_\infty$ corresponds to the primary path, $G_{yu}(q) \in \mathcal{RH}_\infty$ corresponds to the secondary path, and the detector path is defined by $G_{rw}(q) \in \mathcal{RH}_\infty$. The aim is to adapt the coefficients of the controller

$$C(q,k) = \frac{\hat{b}_0(k) + \hat{b}_1(k)q^{-1} + \cdots + \hat{b}_{n_B}(k)q^{-n_B}}{1 + \hat{a}_1(k)q^{-1} + \cdots + \hat{a}_{n_A}(k)q^{-n_A}} \qquad (1)$$

so that the sum of squares of *a posteriori* errors in the error microphone $\varepsilon(i)$ is minimized, that is

$$\varepsilon(i) = d'(i) + G_{yu}(q)\left\lfloor \boldsymbol{\varphi}(i)\hat{\boldsymbol{\theta}}(k)\right\rfloor, i = 1,\ldots,k\,. \qquad (2)$$

The vectors in (2) are created by putting the coefficients of the numerators and denominators of the controller $C(q,k)$ in a vector and are defined in Table I. In this table, $\hat{\boldsymbol{\varphi}}_f(i)$ represents the filtered regression vector whose elements are filtered with the estimated transfer function of the secondary path. The vector of deviations of the controller coefficients from their optimal value is defined as

$$\tilde{\boldsymbol{\theta}}(k) = \boldsymbol{\theta} - \hat{\boldsymbol{\theta}}(k), \qquad (3)$$

where $\boldsymbol{\theta}$ is the vector of the optimal controller coefficient values. After multiplying both sides of the parameter error vector with $\hat{\boldsymbol{\varphi}}_f(k)$ four errors, as described in Table I, can be defined and will be used in the sequel. For brevity, the whole procedure required for developing the primary version of the algorithm is not given here. For a more complete description and also preliminary attempts to improve the robustness of this algorithm in active noise and vibration control systems, the reader may refer to [21], [22], [23].

III. The Proposed Algorithm

A. Algorithm in the modified filtered-x form

It can be shown that the block diagram separated by the dashed-box in Fig. 1 can be represented by an equivalent block diagram in which the parameters of the controller will be adapted with the performance of an RLS type algorithm and the stability of the algorithm for this block diagram

TABLE I

PARAMETERS AND DEFINITIONS

Symbol	Defininition
$\hat{\boldsymbol{\theta}}(k)$	$\left(\hat{a}_1(k), \hat{a}_2(k), \ldots, \hat{a}_{n_A}(k), \hat{b}_0(k), \hat{b}_1(k), \ldots, \hat{b}_{n_B}(k)\right)^{\mathrm{T}}$
$\boldsymbol{\varphi}(i)$	$(-u'(i-1), \ldots, -u'(i-n_A), r(i), \ldots, r(i-n_B))^{\mathrm{T}}$ where $u'(i) = C(q,i)r(i)$
$\hat{\boldsymbol{\varphi}}_f(i)$	$\left(-\hat{u}'_f(i-1), \ldots, -\hat{u}'_f(i-n_A), \hat{r}_f(i), \ldots, \hat{r}_f(i-n_B)\right)^{\mathrm{T}}$ where $\hat{r}_f(i) = \hat{G}_{yu}(q)r(i),\ \hat{u}'_f(i) = C(q,i)\hat{r}_f(i)$
$e_0(k)$	unperturbed *a priori* error $\tilde{\boldsymbol{\theta}}^{\mathrm{T}}(k-1)\hat{\boldsymbol{\varphi}}_f(k)$
$\varepsilon_0(k)$	unperturbed *a posteriori* error $\tilde{\boldsymbol{\theta}}^{\mathrm{T}}(k)\hat{\boldsymbol{\varphi}}_f(k)$
$\hat{e}(k)$	perturbed *a priori* error $e_0(k) + n(k)$
$\hat{\varepsilon}(k)$	perturbed *a posteriori* error $\varepsilon_0(k) + n'(k)$
$n(k)$	defined in (5)
$n'(k)$	defined in (6)

TABLE II

SUMMARY OF THE PROPOSED ALGORITHM

Steps	Computations
1. Updating the regression vector by filtering the new samples	$\hat{r}_f(i) = \hat{G}_{yu}(q)r(i),\ u'_f(i) = C(q,i)\hat{r}_f(i)$ $\hat{\boldsymbol{\varphi}}_f(k) = \big(-\hat{u}'_f(k-1), \ldots, -\hat{u}'_f(k-n_A), \hat{r}_f(k), \ldots, \hat{r}_f(k-n_B)\big)^{\mathrm{T}}$
2. Calculating the control signal	$u(k+1) = C(q,k)r(k+1)$ $u_f(k+1) = \hat{G}_{yu}(q)u(k+1)$
3. Calculating the control signal	$\hat{e}(k+1) = d'(k+1) + \hat{\boldsymbol{\varphi}}_f^{\mathrm{T}}(k+1)\hat{\boldsymbol{\theta}}(k)$ $d'(k+1) = e(k+1) - u_f(k+1)$
4. Updating the covariance of the parameter estimation error	$F(k+1) = \frac{1}{\lambda_1(k)}\left(F(k) - \frac{F(k)\hat{\boldsymbol{\varphi}}_f(k+1)\hat{\boldsymbol{\varphi}}_f^{\mathrm{T}}(k+1)F(k)}{\frac{\lambda_1(k)}{\lambda_2(k)} + \hat{\boldsymbol{\varphi}}_f^{\mathrm{T}}(k+1)F(k)\hat{\boldsymbol{\varphi}}_f(k+1)}\right)$
5. Calculating the *a posteriori* error	$\hat{\varepsilon}(k+1) = \frac{\hat{e}(k+1)}{1+\hat{\boldsymbol{\varphi}}_f^{\mathrm{T}}(k+1)F(k)\hat{\boldsymbol{\varphi}}_f(k+1)}$
6. Calculating the filtered *a posteriori* error	$\nu(k+1) = H(q)\,\hat{\varepsilon}(k+1)$
7. Updating the IIR filter weights	$\hat{\boldsymbol{\theta}}(k+1) = \hat{\boldsymbol{\theta}}(k) - F(k)\hat{\boldsymbol{\varphi}}_f(k+1)\nu(k+1)$

guarantees the stability for the original block diagram in Fig. 1. This is proved by Lemma 1 in [23]. As can be followed in [23], in the next step, the obtained equivalent block diagram is transformable to an output error identification problem represented in Fig. 2. In this figure, $\Delta(q) \in \mathcal{RH}_\infty$ is the additive uncertainty due to estimation of the secondary path, and $m'_1(k) \in \mathcal{R}$ is the total independent additive noise modeled at the output of the system. The transfer function in Fig. 2 can be defined as

$$\frac{B(q)}{A(q)} = G_{dw}(q)G_{rw}^{-1}(q)\hat{G}_{yu}^{-1}(q) = \frac{b_0 + b_1q^{-1} + \cdots + b_{n_B}q^{-n_B}}{1 + a_1q^{-1} + \cdots + a_{n_A}q^{-n_A}}\,. \quad (4)$$

Remark 1: In (4), it is assumed that the transfer functions $G_{rw}(q)$ and $G_{yu}(q)$ are minimum phase, and hence their inverses exist. However, in reality these transfer functions are non-minimum phase, and by using the approach of [21], the algorithm can be extended to the general case.

Remark 2: By writing $n(k) \in \mathcal{R}$ as the total disturbances added at the output of the controller in Fig. 2, it is reasonable to assume an upper bound and define $n'(k)$ alike $n(k)$:

$$n(k) = \Delta(q)u(k) + m'_1(k), \quad \|n(k)\|_\infty < \beta \quad (5)$$

$$u(k) = C(q,k-1)r(k) = \boldsymbol{\varphi}^{\mathrm{T}}(k+1)\hat{\boldsymbol{\theta}}(k)\,. \quad (6)$$

$$n'(k) = \Delta(q)u'(k) + m'_1(k) \quad (7)$$

$$u'(k) = C(q,k)r(k) = \boldsymbol{\varphi}^{\mathrm{T}}(k)\hat{\boldsymbol{\theta}}(k)\,.$$

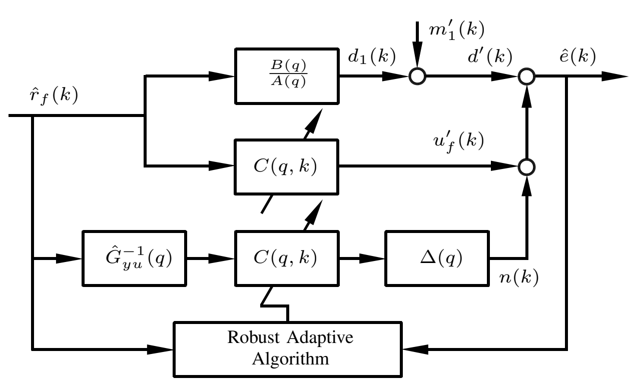

Fig. 2. Output-error identification problem equivalent to diagram of Fig. 1

The proposed algorithm which adapts the parameters of the controller in the dashed-box in Fig. 1 is summarized in Table II. Here, $H(q)$ is a linear filter that should be designed properly to enhance the robustness of the adaptive algorithm with respect to the possible uncertainties in the secondary path. Besides, $0 \ll \lambda_1(k) \leq 1$ and $0 \ll \lambda_2(k) < 2$ are two scalar parameters which determine the variations of the adaptation gain through the time.

B. Preliminary Analysis

Considering the block diagram of Fig. 2, the *a posteriori* error, i.e. $\hat{\varepsilon}(k+1)$, can be calculated as

$$\hat{\varepsilon}(k) = d'(k) + \hat{u}'_f(k) + n(k) = d_1(k) + \hat{u}'_f(k) + \delta(k) \quad (8)$$

where

$$\delta(k) = m'_1(k) + \Delta(q)u'(k),\ \hat{u}'_f(k) = \hat{\boldsymbol{\varphi}}_f^{\mathrm{T}}(k)\boldsymbol{\theta}(k)\,. \quad (9)$$

Then by replacing the output of the recursive IIR filter based on the previous value, and adding and subtracting $A^\star(q)\hat{u}'_f(k-1)$ to the right hand side of (7), i.e.

$$\begin{aligned}\hat{\varepsilon}(k) =& - A^\star(q)d_1(k-1) + B(q)\hat{r}_f(k) + \hat{u}'_f(k) + \\ & A^\star(q)\hat{u}'_f(k-1) - A^\star(q)\hat{u}'_f(k-1) + \delta(k) \\ =& - A^\star(q)\left(d_1(k-1) + \hat{u}'_f(k-1)\right) + \\ & B(q)\hat{r}_f(k) + \hat{u}'_f(k) + A^\star(q)\hat{u}'_f(k-1) + \delta(k) \\ =& - A^\star(q)\left(\hat{\varepsilon}(k-1) - \delta(k-1)\right) + B(q)\hat{r}_f(k) + \\ & \hat{u}'_f(k) + A^\star(q)\hat{u}'_f(k-1) + \delta(k)\end{aligned}$$

Hence,

$$\begin{aligned}A(q)\hat{\varepsilon}(k) =& A^\star(q)\hat{u}'_f(k-1) + B(q)\hat{r}_f(k) + \hat{u}'_f(k) + A(q)\delta(k) \\ =& - \hat{\boldsymbol{\varphi}}_f^{\mathrm{T}}(k)\boldsymbol{\theta} + \hat{\boldsymbol{\varphi}}_f^{\mathrm{T}}(k)\boldsymbol{\theta}(k) + A(q)\delta(k) \\ \Rightarrow\ \hat{\varepsilon}(k) =& - \frac{1}{A(q)}\left(\hat{\boldsymbol{\varphi}}_f^{\mathrm{T}}(k)\tilde{\boldsymbol{\theta}}(k)\right) + \delta(k) \\ =& - \frac{1}{A(q)}\varepsilon_0(k) + \delta(k) \quad (10)\end{aligned}$$

On the other hand, following the energy conservation relation of adaptive algorithms and its equivalent feedback representation as in [16], we will analyze the stability and robustness condition of the proposed adaptive RLS-based algorithm for active noise and vibration control systems. To this end, the first step to establish the equivalent feedback representation of the proposed algorithm is to derive the relation between unperturbed *a posteriori* error, i.e. $\varepsilon_0(k+1)$, and unperturbed *a priori* error $e_0(k+1)$ and in the second step the forward lossless mapping T will be derived. By subtracting both sides of parameter update equation in step 7 of Table II from the desired parameter vector θ, the update equation can be written in terms of the weight-error vector $\tilde{\theta}(k)$ along

$$\tilde{\theta}(k+1) = \tilde{\theta}(k) + F(k)\hat{\varphi}_f(k+1)\nu(k+1) \tag{11}$$

and by multiplication of both sides of (10) with $\hat{\varphi}_f(k+1)$ we will get:

$$\varepsilon_0(k+1) = e_0(k+1) + \hat{\varphi}_f^{\mathrm{T}}(k+1)F(k)\hat{\varphi}_f(k+1)\nu(k+1) \tag{12}$$

Furthermore, multiplying both sides of the parameter update equation by $\hat{\varphi}^{\mathrm{T}}(k+1)$ results in:

$$u'(k+1) = u(k+1) - \hat{\varphi}^{\mathrm{T}}(k+1)F(k)\hat{\varphi}_f(k+1)\nu(k+1) \tag{13}$$

The relations (12) and (13) will be used later to establish the feedback equivalent representation of the algorithm.

IV. Stability Analysis

A. Establishing the feedback path

By substituting $\nu(k+1)$ from step 6 of Table II into (12) and (13), and considering the relation obtained for $\hat{\varepsilon}(k)$ in (10) we will get

$$\begin{aligned}\varepsilon_0(k+1) &= e_0(k+1) + \mu_1(k)H(q)\\ &\left(-\frac{1}{A(q)}\varepsilon_0(k+1) + \Delta(q)u'(k+1) + m_1'(k+1)\right)\end{aligned} \tag{14}$$

$$\begin{aligned}u'(k+1) &= u(k+1) - \mu_2(k)H(q)\\ &\left(-\frac{1}{A(q)}\varepsilon_0(k+1) + \Delta(q)u'(k+1) + m_1'(k+1)\right)\end{aligned} \tag{15}$$

where

$$\mu_1(k) = \hat{\varphi}_f^{\mathrm{T}}(k+1)F(k)\hat{\varphi}_f(k+1)$$
$$\mu_2(k) = \hat{\varphi}^{\mathrm{T}}(k+1)F(k)\hat{\varphi}_f(k+1)$$

By simplifying the relations (14) and (15) we have

$$\begin{aligned}\varepsilon_0(k+1) &= \mu_1(k)H_1(q,k)H(q)\Delta(q)u'(k+1)+\\ H_1(q,k)e_0(k+1) &+ \mu_1(k)H_1(q,k)H(q)m_1'(k+1)\end{aligned} \tag{16}$$

and

$$\begin{aligned}u'(k+1) &= \mu_2(k)H_2(q,k)\frac{H(q)}{A(q)}\varepsilon_0(k+1)+\\ H_2(q,k)u(k+1) &- \mu_2(k)H_2(q,k)H(q)m_1'(k+1)\end{aligned} \tag{17}$$

where

$$H_1(q,k) = \frac{1}{1+\mu_1(k)\frac{H(q)}{A(q)}},\ H_2(q,k) = \frac{1}{1+\mu_2(k)H(q)\Delta(q)}.$$

Then substitution of (17) into (16) yields

$$\begin{aligned}&H_3(q,k)\varepsilon_0(k+1) = H_1(q,k)e_0(k+1)+\\ &\quad\mu_1(k)H_1(q,k)H(q)\Delta(q)H_2(q,k)u(k+1)+\\ &\mu_1(k)H_1(q,k)H(q)\left(1-\mu_2(k)\Delta(q)H_2(q,k)H(q)\right)m_1'(k+1)\end{aligned} \tag{18}$$

and

$$H_3(q,k) = 1-\mu_1(k)H_1(q,k)H(q)\Delta(q)\mu_2(k)H_2(q,k)\frac{H(q)}{A(q)}$$

Simplification of $H_3(q,k)$ will turn it into

$$\begin{aligned}H_3(q,k) &= H_1(q,k)H_2(q,k)\\ &\left(1+\mu_1(k)\frac{H(q)}{A(q)} + \mu_2(k)H(q)\Delta(q)\right).\end{aligned} \tag{19}$$

Remark 3: To be able to change the order of the transfer functions in (18) and (19) it is necessary to assume that $\mu_1(k)$ and $\mu_2(k)$ are slowly varying or constant with time. Since this assumption might not hold in a general sense, it is possible to design the tuning parameters $\lambda_1(k)$ and $\lambda_2(k)$ such that this condition becomes true, e.g. by assuming

$$\lambda_1(k) = \lambda_2(k) = \left(1+\hat{\varphi}_f^{\mathrm{T}}(k+1)F(k)\hat{\varphi}_f(k+1)\right)^{-1} \tag{20}$$

we will have $\mu_1(k) = \mu_2(k) = \mu$.

In order to find the conditions for the stability of the proposed algorithm, it is necessary that the obtained recursive formula in (18) operates in a stable manner. For this purpose it is necessary that the transfer functions $H_1(q,k)$, $H_2(q,k)$, $H_3(q,k)$ be stable. The conditions for the stability of these transfer functions are stated in the following Lemma.

Lemma 1: A sufficient condition to assure the stability of the transfer functions $H_1(q,k)$, $H_2(q,k)$, and $H_3(q,k)$ in (18) is to design $H(q)$ so that $\frac{H(q)}{A(q)}$ becomes strictly positive real, and also $\|H(q)\Delta(q)\|_\infty < \frac{1}{|\mu_2(k)|}$.

Proof: Since the matrix $F(k)$ is positive definite for all k, the parameter $\mu_1(k)$ in the denominator of $H_1(q,k)$ is always positive, and hence by using *Popov's hyper-stability theorem* a sufficient condition for the stability of $H_1(q,k)$ is that $\mathrm{Re}\left(\frac{H(q)}{A(q)}\right) > 0$. After applying the small gain theorem to the transfer function $H_2(q,k)$ a sufficient stability condition reads $\|\mu_2(k)H(q)\Delta(q)\|_\infty < 1$, which by extracting $\mu_2(k)$ from the norm turns to $\|H(q)\Delta(q)\|_\infty < \frac{1}{\mu_2(k)}$. Now it is enough to prove that these two stability conditions will assure the stability of the transfer function $H_3(q,k)$. Considering the fact that $\mu_1(k)$ and $\mu_2(k)$ are slowly varying, the stability of $H_3(q,k)$ can be proved by using the Nyquist criterion. In this case, if we assume $\Gamma(q) = \mu_1(k)\frac{H(q)}{A(q)} + \mu_2(k)H(q)\Delta(q)$ as the loop gain of the system then $1+\mu_1(k)\frac{H(q)}{A(q)} + \mu_2(k)H(q)\Delta(q)$ will not have any poles in the right half plane when the plot of $\Gamma(q)$ will not encircle the point $-1+j\,0$. Since $\|H_2(q,k)\|_\infty < 1$ we have $-1 < \mathrm{Re}\left(H_2(q,k)\right) < 1$ and since $\mathrm{Re}\left(H_1(q,k)\right) > 0$ we conclude that the real part of their summation is always larger than -1 and the Nyquist curve will not encircle the point $-1+j\,0$. Hence, $H_3(q,k)$ is also stable. ■

B. Establishing the feedforward path

It is possible to find the relation among different variables in the forward path by equation (11). In fact, by multiplying both sides of (11) by $F^{-\frac{1}{2}}(k)$ and rearranging it, we obtain

$$F^{-\frac{1}{2}}(k)\tilde{\theta}(k) = F^{-\frac{1}{2}}(k)\tilde{\theta}(k+1) - F^{\frac{1}{2}}(k)\hat{\varphi}_f(k+1)\nu(k+1). \tag{21}$$

Then by calculating the square norm of both sides we get

$$\tilde{\theta}^{\mathrm{T}}(k)F^{-1}(k)\tilde{\theta}(k) = \tilde{\theta}^{\mathrm{T}}(k+1)F^{-1}(k)\tilde{\theta}(k+1) - 2\varepsilon_0(k+1)\nu(k+1) + \hat{\varphi}_f^{\mathrm{T}}(k+1)F(k)\hat{\varphi}_f(k+1)\nu^2(k+1).$$

Replacing $\nu(k+1)$ from (12) into the equation above and doing some simplifications yields

$$\begin{aligned}\tilde{\theta}^{\mathrm{T}}(k)F^{-1}(k)\tilde{\theta}(k) + \bar{\mu}(k)\varepsilon_0^2(k+1) = \\ \tilde{\theta}^{\mathrm{T}}(k+1)F^{-1}(k)\tilde{\theta}(k+1) + \bar{\mu}(k)e_0^2(k+1)\end{aligned} \tag{22}$$

where $\bar{\mu}(k) = \left(\hat{\varphi}_f^{\mathrm{T}}(k+1)F(k)\hat{\varphi}_f(k+1)\right)^{-1}$. Now by replacing $F^{-1}(k)$ in terms of $F^{-1}(k+1)$, from equation

$$F^{-1}(k+1) = \lambda_1(k)F^{-1}(k) + \lambda_2(k)\hat{\varphi}_f(k+1)\hat{\varphi}_f^{\mathrm{T}}(k+1) \tag{23}$$

the energy conservation relation in (22) transforms to

$$\begin{aligned}\tilde{\theta}^{\mathrm{T}}(k)F^{-1}(k)\tilde{\theta}(k) + \left(\bar{\mu}(k) + \frac{\lambda_2(k)}{\lambda_1(k)}\right)\varepsilon_0^2(k+1) = \\ \lambda_1^{-1}(k)\tilde{\theta}^{\mathrm{T}}(k+1)F^{-1}(k)\tilde{\theta}(k+1) + \bar{\mu}(k)e_0^2(k+1).\end{aligned} \tag{24}$$

Eqn. (24) shows the relation among the quantities $\tilde{\theta}(k+1)$, $\tilde{\theta}(k)$, $\varepsilon_0(k+1)$, and $e_0(k+1)$. The first term on the left-hand side of (24) is the weighted energy of current coefficients, the second term is the weighted energy of the *a posteriori* error. The first term in the right-hand side of (24) is the weighted energy of the updated coefficients and the second term is the weighted energy of the *a priori* error. The relation (24) can be represented as a mapping between two vectors as follows:

$$\begin{pmatrix} \bar{\mu}^{\frac{1}{2}}(k)e_0(k+1) \\ \lambda_1^{-\frac{1}{2}}(k)F^{-\frac{1}{2}}(k+1)\tilde{\theta}(k+1)\end{pmatrix} = T\begin{pmatrix} F^{-\frac{1}{2}}(k)\tilde{\theta}(k) \\ \left(\bar{\mu}(k) + \frac{\lambda_2(k)}{\lambda_1(k)}\right)^{\frac{1}{2}}\varepsilon_0(k+1)\end{pmatrix} \tag{25}$$

where

$$T = \begin{pmatrix} T_{11} & T_{12} \\ T_{21} & T_{22}\end{pmatrix}$$

is a unitary matrix so that $T^{\mathrm{T}}T = I$, given as

$$T_{11} = \bar{\mu}^{\frac{1}{2}}(k)\hat{\varphi}_f^{\mathrm{T}}(k+1)F^{\frac{1}{2}}(k), \quad T_{12} = 0$$

$$T_{21} = F^{-\frac{1}{2}}(k)\left(I + \bar{\mu}(k)F(k)\hat{\varphi}_f(k+1)\hat{\varphi}_f^{\mathrm{T}}(k+1)\right)F^{\frac{1}{2}}(k)$$

$$T_{22} = \bar{\mu}(k)\left(\bar{\mu}(k) + \frac{\lambda_2(k)}{\lambda_1(k)}\right)^{-\frac{1}{2}}F^{\frac{1}{2}}(k)\hat{\varphi}_f(k+1)$$

By multiplying both sides of (18) with the inverse of $H_3(q,k)$ and defining

$$\alpha(k) = \bar{\mu}(k) + \frac{\lambda_2(k)}{\lambda_1(k)} \tag{26}$$

$$G(q,k) = H_3^{-1}(q,k)H_1(q,k) \tag{27}$$

$$H(q,k) = \sqrt{\alpha(k)}\mu_1(k)H_3^{-1}(q,k)H_1(q,k)H(q)\Delta(q)H_2(q,k) \tag{28}$$

$$\begin{aligned}F(q,k) = &\sqrt{\alpha(k)}\mu_1(k)H_3^{-1}(q,k)H_1(q,k)H(q) \\ &(1-\mu_2(k)\Delta(q)H_2(q,k)H(q))\end{aligned} \tag{29}$$

the relation among the variables in the feedback path will be obtained when putting the feedforward and the feedback block together. The equivalent block diagram of the algorithm can be represented as shown in Fig. 3.

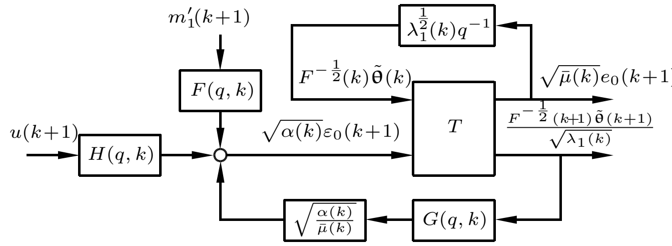

Fig. 3. Equivalent feedback representation of the output-error identification problem represented by the block diagram of Fig. 1

C. Stability conditions

To analyze the stability of the algorithm proposed in Table II, we will resort to the developed equivalent feedback representation in Fig. 3. Since the forward path is a unitary map with infinity norm equal to one, based on the small gain theorem, the stability of the feedback system will be guaranteed if the infinity norm of the feedback path remains less than one. For this purpose a sufficient condition for stability of this system will be obtained if the infinity norm of the transfer function in the feedback path is less than one. This issue is expressed in the following theorem.

Theorem 1: The algorithm proposed in Table II to update the coefficients of the adaptive IIR filter in Fig. 1 is asymptotically stable in the presence of uncertainty in the estimation of the secondary path, and will converge to the desired filter weights if the filter $H(q)$ and the parameters $\lambda_1(k)$ and $\lambda_2(k)$ are such that the following conditions are satisfied:

I) $\mathrm{Re}\left(\frac{H(q)}{A(q)}\right) > 0$

II) $\left\|\frac{H(q)}{A(q)} - \frac{2\lambda_2(k)}{\lambda_1(k)}\right\|_\infty > 4\bar{\mu}(k)$

III) $\|H(q)\Delta(q)\|_\infty < \bar{\mu}'(k)$

with

$$\bar{\mu}(k) = \left(\hat{\varphi}_f^{\mathrm{T}}(k+1)F(k)\hat{\varphi}_f(k+1)\right)^{-1}$$
$$\bar{\mu}'(k) = \left(\hat{\varphi}^{\mathrm{T}}(k+1)F(k)\hat{\varphi}_f(k+1)\right)^{-1}$$

and $A(q)$ denominator of the transfer function defined in (4).

Proof: Conditions I and III are clear by Lemma 1. In order to derive condition II we follow the formulation used to establish the equivalent feedback representation of the proposed algorithm. By continuing this point, the sufficient condition that guarantees robust stability of the adaptive algorithm reads

$$\left\|G(q,k)\sqrt{\frac{\alpha(k)}{\bar{\mu}(k)}}\right\|_\infty < 1\,. \tag{30}$$

Using the definition of infinity norm and substituting $G(q,k)$ and $\alpha(k)$ from (26) and (27) we have $\forall\omega \in [-\pi,\pi]$ that

$$\left|H_3^{-1}(e^{j\omega},k)H_1(e^{j\omega},k)\right| < \sqrt{\frac{\bar{\mu}(k)}{\bar{\mu}(k) + \frac{\lambda_2(k)}{\lambda_1(k)}}}. \tag{31}$$

We change the inequality (31) to another inequality in two steps such that by the new inequality it is easy to extract the sufficient condition for stability. In the first step, it is easy to prove that for the right hand-side of (31) holds

$$\frac{\bar{\mu}(k)}{\bar{\mu}(k)+\frac{\lambda_2(k)}{\lambda_1(k)}} < \sqrt{\frac{\bar{\mu}(k)}{\bar{\mu}(k)+\frac{\lambda_2(k)}{\lambda_1(k)}}} \tag{32}$$

and in the second step, by writing the left-hand side of (31) in another form and taking into account the result of Lemma 1, at the same time, we will obtain

$$\left|\frac{1+\mu_2(k)H(e^{j\omega})\Delta(e^{j\omega})}{1+\mu_2(k)H(e^{j\omega})\Delta(e^{j\omega})+\mu_1(k)\frac{H(e^{j\omega})}{A(e^{j\omega})}}\right| \leq \left|\frac{2}{1+\mu_2(k)H(e^{j\omega})\Delta(e^{j\omega})+\mu_1(k)\frac{H(e^{j\omega})}{A(e^{j\omega})}}\right|. \tag{33}$$

Therefore, instead of (31), it is enough to find conditions s.t.

$$\frac{2}{\left|1+\mu_2(k)H(e^{j\omega})\Delta(e^{j\omega})+\mu_1(k)\frac{H(e^{j\omega})}{A(e^{j\omega})}\right|} < \frac{1}{1+\mu_1(k)\frac{\lambda_2(k)}{\lambda_1(k)}} \tag{34}$$

is satisfied. By inverting both sides of (34) we have

$$\left|1+\hat{\boldsymbol{\varphi}}^{\mathrm{T}}(k+1)F(k)\hat{\boldsymbol{\varphi}}_f(k+1)H(e^{j\omega})\Delta(e^{j\omega})+ \hat{\boldsymbol{\varphi}}_f^{\mathrm{T}}(k+1)F(k)\hat{\boldsymbol{\varphi}}_f(k+1)\frac{H(e^{j\omega})}{A(e^{j\omega})}\right| > 2\left(1+\mu_1(k)\frac{\lambda_2(k)}{\lambda_1(k)}\right) \tag{35}$$

which then by applying the triangular inequality to (35),

$$\Big||a|-|b|\Big| < |a+b| < |a|+|b|,$$

yields the sufficient condition for equation (35) to hold

$$\hat{\boldsymbol{\varphi}}_f^{\mathrm{T}}(k+1)F(k)\hat{\boldsymbol{\varphi}}_f(k+1)\left|\frac{H(e^{j\omega})}{A(e^{j\omega})}-\frac{2\lambda_2(k)}{\lambda_1(k)}\right| > 2+\left|1+\hat{\boldsymbol{\varphi}}^{\mathrm{T}}(k+1)F(k)\hat{\boldsymbol{\varphi}}(k+1)H(e^{j\omega})\Delta(e^{j\omega})\right|. \tag{36}$$

Since

$$\left|1+\hat{\boldsymbol{\varphi}}_f^{\mathrm{T}}(k+1)F(k)\hat{\boldsymbol{\varphi}}_f(k+1)H(e^{j\omega})\Delta(e^{j\omega})\right| < 2 \tag{37}$$

inequality (36) holds if

$$\hat{\boldsymbol{\varphi}}_f^{\mathrm{T}}(k+1)F(k)\hat{\boldsymbol{\varphi}}_f(k+1)\left|\frac{H(e^{j\omega})}{A(e^{j\omega})}-\frac{2\lambda_2(k)}{\lambda_1(k)}\right| > 4. \tag{38}$$

Dividing both sides of (38) by the positive scalar term on the left-hand side, we finally obtain condition II. ■

V. Conclusions

A robust adaptive IIR filter based on the recursive least square technique has been developed. To avoid the assumption of slowly varying adaptation of the adaptive filter coefficients the algorithm is proposed in the modified form. Due to the time-varying nature of the secondary path, the algorithm is always exposed to eventual uncertainties and becomes vulnerable in terms of stability. Therefore, it is necessary to find methods that make it robust in the secondary path. To this end, in this study the possibility of designing a general linear filter for shaping the error signal generated by the algorithm is investigated. By analysis of the algorithm, sufficient conditions assuring robust stability of the algorithm in front of uncertainties in the secondary path are derived.

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