2011 50th IEEE Conference on Decision and Control and
European Control Conference (CDC-ECC)
Orlando, FL, USA, December 12-15, 2011

Trustable Autonomous Systems Using Adaptive Control

M. Matsutani, A. M. Annaswamy, T. E. Gibson, and E. Lavretsky

***Abstract*— A long standing problem in adaptive control is the derivation of robustness properties in the presence of unmodeled dynamics, a necessary and highly desirable property for designing adaptive flight control for systems with trustable autonomy. We provide a solution to this problem in this paper for linear time-invariant plants whose states are accessible for measurement. This is accomplished by using a Lipschitz continuous projection algorithm that allows the utilization of properties of a linear system when the adaptive parameter lies on the projection boundary. This in turn helps remove the restriction on plant initial conditions, as opposed to the currently existing proofs of semi-global stability. A direct implication of this result is the robustness of adaptive control systems to time-delays, and the guarantee that the underlying adaptive system will have a delay margin.**

I. Introduction

Rapid development, technical transition, and insertion of autonomous platforms into aerospace applications enabled end-users with unique capabilities. Yet at the same time, a variety of technical challenges arose. Standing amongst them is the notion of trustable autonomous systems. The word trustable implies that there is an inherent requirement for any autonomous platform to resiliently operate in a realistic environment. Moreover, performance of an autonomous vehicle must be predictable, repeatable, and verifiable. In order to operate autonomously, these platforms are equipped with Guidance, Navigation & Control (GN&C) architectures that are designed to generate the required input signals using available online measurements (the system outputs) in a feedback loop. Given an autonomous platform, its control system must be designed to satisfy specific requirements related to the vehicle stability, robustness, and performance. These design specifications can be achieved and enforced through a systematic usage of formal methods from control and dynamics.

One of the key enablers for predictable stable operation of autonomous platforms is the adaptive control technology, which provides a prescription for stable on-line adjustment of control parameters. This feature of adaptive flight controllers becomes even more attractive in autonomous platforms due to the inevitable need for self-governance especially in the wake of unforeseen anomalies that can occur during the flight operation. Any control candidate for these flight systems, however, needs to undergo Verification and Validation (V&V), in order to be flight certified. That is, the underlying adaptive control system must have guaranteed performance in the presence of perturbations in gains and delays, both of which are inevitable in closed-loop, model-based, control implementations.

Adaptive control theory is a mature control discipline, evolved over the past four decades, and rigorously synthesized. Major milestones of adaptive control of linear plants are stability in 1980 [1], and robustness to bounded disturbances in 1986 [2]. Several attempts have been made since then to extend the robustness properties of adaptive systems to the case when unmodeled dynamics are present. The most general result to date in this direction can be found in [3], [4] where semi-global stability is guaranteed for a certain class of unmodeled dynamics with a small parameter μ (see section 9.3 in [4], section 8.7 in [3]) and several papers published in the '90s (see [4] for example). Given the nature of the semi-global result in [3], [4] and other published results in the area of robust adaptive control [5], it is clear that there is a gap between the needs of the flight control application and theory.

In this paper, we show that this gap can be closed. We demonstrate that boundedness can be guaranteed for linear plants whose states are accessible for measurement, when subjected to parameter uncertainties and unmodeled dynamics, for arbitrary initial conditions of the plant states. It is assumed that the parameter uncertainties lie in a bounded hypercube, enabling the use of an adaptive law with a parameter projection [6] using which the robustness result is established. This is used to a computable delay margin for adaptive systems with full state measurements.

II. Robust Adaptive Control Revisited

One of the very first problems where stable adaptive control was solved was for the case when states are accessible [7], with the plant given by[1]

$$\dot{x}_p = A_p x_p + b\lambda u \tag{1}$$

where $A_p \in \mathbb{R}^{n\times n}$ and the scalar λ are unknown parameters with b and the sign of λ known, and (A_p, b) controllable. It is well known that an adaptive controller of the form

$$u = \theta_x^T(t) x_p + \theta_r(t) r, \tag{2}$$

adaptive laws

$$\dot{\theta} = -\Gamma \omega b^T P e, \tag{3}$$

This work was supported by NASA IRAC project, Grant No. NRA NNH07ZEA001N and the Boeing Strategic University Initiative.

M. Matsutani is with the Department of Aeronatutics and Astronautics, MIT, Cambridge, MA 02139.

E. Lavretsky is with the Boeing Company, Huntington Beach, CA 92647.

A. M. Annaswamy and T. E. Gibson are with the Department of Mechanical Engineering, MIT, Cambridge, MA 02139.

[1]The argument t is suppressed for the sake of convenience, except for emphasis.

where $\Gamma = \Gamma^T > 0$, $\omega = \begin{bmatrix} x_p & r \end{bmatrix}^T$, $\theta = \begin{bmatrix} \theta_x^T & \theta_r \end{bmatrix}^T$, $e = x_p - x_m$, and x_m is the state of a reference model

$$\dot{x}_m = A_m x_m + br \tag{4}$$

with A_m Hurwitz, and P is the solution of the Lyapunov equation $A_m^T P + P A_m = -Q$, $Q > 0$, guarantees stability when the matching conditions

$$A_p + b\lambda\theta_x^{\star T} = A_m, \qquad \lambda\theta_r^* = 1 \tag{5}$$

are satisfied for some $\theta^* = [\theta_x^{*T}, \theta_r^*]^T$. The controller in (2) and (3) also ensures that $x(t)$ tracks $x_m(t)$. The underlying Lyapunov function is quadratic in e and the parameter error $\tilde{\theta} = \theta - \theta^*$, with a negative semi-definite time-derivative $\dot{V}$ [3].

When a bounded disturbance d is present, with the plant dynamics changed as

$$\dot{x}_p = A_p x_p + b\lambda(u + d(t)) \tag{6}$$

robust adaptive laws need to be designed that modify (3) as

$$\dot{\theta} = -\Gamma\omega b^T P e - \sigma g(\theta, e) \tag{7}$$

where $g(\theta, e) = \theta$, $||e||\theta$, or of the form[2]

$$g(\theta, e) = \theta \left(1 - \frac{||\theta||}{\theta_{\max}}\right)^2 \tag{8}$$

where $\theta_{\max}$ is a known bound on the parameter θ. While the boundedness of the overall adaptive systems is well known and was established several decades ago, we briefly describe it below. Without loss of generality, we assume that $\lambda > 0$.

A quadratic positive definite function is chosen as

$$V = \frac{1}{2}e^T P e + \frac{1}{2}\lambda\tilde{\theta}^T\Gamma^{-1}\tilde{\theta} \tag{9}$$

which yields a time-derivative

$$\dot{V} \leq -\frac{1}{2}e^T Q e + k_1\|e\|\|d\| - \frac{1}{2}\lambda\sigma\|\tilde{\theta}\| g(\theta, e), \;\; k_1 > 0. \tag{10}$$

The property of $g(e, \theta)$, together with the fact that d is bounded, ensures that $\dot{V} < 0$ outside a compact set Ω in the $(e, \tilde{\theta})$ space. This ensures the global boundedness of both e and $\tilde{\theta}$. Boundedness of x_p follows.

In all of the above methods, the idea behind adding the term $g(e, \theta)$ is this: The parameter θ can drift away from the correct direction due to the term $k_1\|e\|\|d\|$, and the construction of $g(e, \theta)$ is such that it counteracts this drift and keeps the parameter in check, by adding a negative quadratic term in $\tilde{\theta}$. The boundedness of both e and θ are simultaneously assured in the above since V has a time-derivative $\dot{V}$ that is non-positive outside a compact set in the $(e, \tilde{\theta})$ space. It should be noted however that this was possible to a large extent because d was bounded and as a result, the sign-indefinite term remained linear in $\|e\|$.

An alternate procedure, originally proposed in [9] and revised and refined in [10], [6] proceeds in a slightly different manner. Here, the boundedness of θ is first established, independent of the error equation. It should be noted that a similar approach is adopted in the context of output feedback in plants with higher relative degree by using normalization and an augmented error approach[3]. In [10] and [6], no normalization is used but a projection algorithm. In the following section, we briefly present this algorithm, which is the adaptive law of interest in this paper, as well as the proof of boundedness for the sake of completeness.

A. Robust adaptive control in the presence of a projection algorithm

We first state a few definitions and two lemmas. Let sets Ω_0 and Ω_1 be defined as

$$\begin{aligned} \Omega_0 &= \{\theta \in \mathbb{R}^n | f(\theta) \leq 0\} \\ \Omega_1 &= \{\theta \in \mathbb{R}^n | f(\theta) \leq 1\}, \end{aligned} \tag{11}$$

and a Projection function, denoted as Proj, be defined as follows:

$$\text{Proj}(\theta, y) = \begin{cases} y - \frac{\nabla f(\theta)(\nabla f(\theta))^T}{\|\nabla f(\theta)\|^2} y f(\theta) \\ \qquad\qquad \text{if} \;\; [f(\theta) > 0 \wedge y^T \nabla f(\theta) > 0] \\ y \qquad\qquad \text{otherwise.} \end{cases} \tag{12}$$

Lemma 1: Let $f(\theta)$ be a convex function. Let $\theta \in \Omega_1$ and $\theta^\star \in \Omega_1$. Then for any vector y, the following inequality holds:

$$(\theta - \theta^\star)^T(\text{Proj}(\theta, y) - y) \leq 0. \tag{13}$$

Lemma 2: Let $f(\theta)$ be a convex function. For any time varying piecewise continuous vector y, if

$$\begin{aligned} &\theta(0) \in \Omega_0 \\ &\dot{\theta} = \Gamma\text{Proj}(\theta, y) \end{aligned} \tag{14}$$

where $\text{Proj}(\theta, y)$ is given by Eq. (12), then $\theta(t) \in \Omega_1$, for all $t \geq 0$.

The reader is referred, for the proofs of Lemmas 1 and 2 to [9] and to [6] for simpler versions applicable to linear plants.

The implications of Lemma 2 on robust adaptive control are now obvious. If the adaptive law is chosen as in Lemma 2 with $y = -e^T P b\omega$, then irrespective of the boundedness of e, it follows that $\theta(t)$ is bounded and lies in Ω_1 if $\theta(0)$ and θ^* belong to Ω_0, Ω_1 respectively. This is formally stated in Lemma 3 below:

Lemma 3: Consider the IVP in (14) with

$$y = -e^T P b\omega \tag{15}$$

$$f(\theta) = \frac{\|\theta\|^2 - \theta_{\max}'^2}{\varepsilon^2 + 2\varepsilon\theta'_{\max}} \tag{16}$$

where $\theta'_{\max}$ and ε are arbitrary positive constants, and Ω_0 and Ω_1 are defined as in (11). Then,

$$\theta(0) \in \Omega_0 \implies \theta(t) \in \Omega_1.$$

In addition,

$$\begin{aligned} \Omega_0 &= \{\theta \in \mathbb{R}^n | \|\theta\| \leq \theta'_{\max}\} \\ \Omega_1 &= \{\theta \in \mathbb{R}^n | \|\theta\| \leq \theta_{\max}\} \\ &\text{where} \quad \theta_{\max} = \theta'_{\max} + \varepsilon. \end{aligned} \tag{17}$$

[2]One can choose to set γ to zero if $\|\theta\| \leq \theta_{\max}$, as is done in [[4],[8]] and many other references in the literature.

The proof of Lemma 3 follows immediately from Lemma 2.

Remark 1: We can also apply the projection algorithm to an adaptive law in a slightly different way. Instead of treating the vector θ as a whole, it is also possible to implement the projection algorithm by parts, for θ_x and θ_r independently. The design parameters in this case will be $\theta'_{x,\max}$, ε_x, and $\theta'_{r,\max}$, ε_r, respectively. The boundedness of the norm of θ_x by $\theta_{x,\max} = \theta'_{x,\max} + \varepsilon_x$ and that of θ_r by $\theta_{r,\max} = \theta'_{r,\max} + \varepsilon_r$ are guaranteed as in Lemma 3.

With the boundedness of θ established using Lemma 3, boundedness of e follows by the application of the Gronwall-Bellman Lemma. This is summarized in the Theorem below. Throughout the paper, we use the following notations. Let

$$\underline{s}_A = \min_i \left|\Re\big(\lambda_i(A)\big)\right|$$

$$\bar{s}_A = \max_i \left|\Re\big(\lambda_i(A)\big)\right|$$

where λ_i is ith eigenvalue of a matrix A and $\Re(\lambda_i)$ is its real part.

Theorem 1: Consider the closed-loop adaptive system given by (6), the control law (2), the adaptive law (14) with y and $f(\theta)$ chosen as in (15) and (16). If the reference model in (4) and $\theta_{\max}$ are such that θ^* in (5) belongs to Ω_1, then for any initial conditions $x_p(0)$ and $x_m(0)$, and $\theta(0) \in \Omega_0$, the closed-loop adaptive system has bounded solutions, with $\theta(t)$ remaining in Ω_1 for all $t \geq 0$.

Proof: Lemma 3 implies that $\theta(t) \in \Omega_1$ with a bound as in (17). With a V as in (9), we obtain

$$\begin{aligned}\dot{V} = &-\frac{1}{2}e^T Q e + 2e^T P b \lambda d \\ &+\big(e^T P b \lambda \tilde{\theta}^T \omega + \lambda \tilde{\theta}^T \mathrm{Proj}(\theta, -\omega b^T P e)\big).\end{aligned} \tag{18}$$

Equation (13) in Lemma 1 together with (15) implies that the term within the parantheses in Eq. (18) is non-positive. This in turn implies that

$$\dot{V} \leq -e^T Q e + k_1 \|e\| \|d\|. \tag{19}$$

From (9) and (19) and the fact that $\theta(t)$ is bounded, it can be shown that

$$\dot{V} \leq -k_2 \left(V - k_3\right) + k_4 \sqrt{V} \tag{20}$$

where

$$k_1 = 2P_b\lambda, \quad k_2 = \frac{\underline{s}_Q}{\bar{s}_P}, \quad k_3 = \frac{\lambda \theta_{\max}^2}{\underline{s}_\Gamma}, \quad k_4 = \frac{k_1 d_{\max}}{\sqrt{\underline{s}_P}}. \tag{21}$$

For positive constants Δ_1, Δ_2 such that $\Delta_1 < k_2$ and $4\Delta_1\Delta_2 \geq k_4^2$, it can be shown that for any V,

$$\Delta_1 V + \Delta_2 \geq k_4 \sqrt{V} \tag{22}$$

through a straight forward completion of squares. Inequalities (20) and (22) imply that

$$\dot{V} \leq -\underbrace{(k_2 - \Delta_1)}_{K_0} V + \underbrace{k_2 k_3 + \Delta_2}_{K_1}. \tag{23}$$

From the application of the Gronwall-Bellman Lemma [11] [12] to (23), it follows that

$$V(t) \leq \left(V(0) - \frac{K_1}{K_0}\right) exp(-K_0 t) + \frac{K_1}{K_0} \tag{24}$$

and therefore $V(t)$ is bounded. This in turn implies the boundedness of $e(t)$ and therefore $x(t)$ for any initial conditions in $e(0)$. ∎

III. Robustness of Adaptive Systems to Unmodeled Dynamics

We now consider an LTI plant in the presence of a disturbance that may not be known to be bounded apriori, such as a state-dependent disturbance η given by

$$\dot{\zeta} = A_\eta \zeta + b_\eta u, \quad \eta = c_\eta^T \zeta \tag{25}$$

where A_η is a Hurwitz matrix. For ease of exposition, we assume that the plant has the form

$$\dot{x}_p = A_m x_p + b\lambda(u - \theta_x^{*T} x_p + \eta) \tag{26}$$

where λ and θ_x^* are unknown, and A_m and b are known. With the same reference model and definitions as in Section II, we obtain the error dynamics

$$\dot{e} = A_m e + b\lambda(\tilde{\theta}^T \omega + \eta), \tag{27}$$

We now show that an identical result as in Theorem 1 can be derived in this case even though the disturbance η is not known to be bounded apriori.

We introduce a few definitions and a key lemma. P and P_η are the solutions of the Lyapunov equations

$$\begin{aligned}A_m^T P + P A_m &= -q_1 I \\ A_\eta^T P_\eta + P_\eta A_\eta &= -q_2 I\end{aligned} \tag{28}$$

where q_1 and q_2 are positive scalars. Since A_m and A_η are Hurwitz, P and P_η exist and are positive definite and symmetric. Let

$$\begin{aligned}&x_{m_0} = \theta_{x,\max} \max_{t\geq 0}\left(x_m(t)\right), \\ &c_1 = x_{m_0}, \quad c_2 = \theta_{r,\max} \max_{t\geq 0}\left(r(t)\right) \\ &p_b = \|Pb\|, \quad p_\eta = ||P_\eta b_\eta|| \\ &c_3 = \left(\lambda p_b \|c_\eta\| + p_\eta \theta_{x,\max}\right), \quad c_4 = 2p_\eta(c_1 + c_2)\end{aligned} \tag{29}$$

$$F(e, \zeta) = q_1 \|e\|^2 + q_2 \|\zeta\|^2 - 2c_3 \|e\| ||\zeta|| - c_4 \|\zeta\|. \tag{30}$$

Theorem 2: Consider the closed-loop adaptive system given by (26), the unmodeled dynamics by (25), the control law (2), the adaptive law (14) with y and $f(\theta)$ chosen as in (15) and (16). If the reference model in (4) and $\theta_{\max}$ are such that θ^* in (5) belongs to Ω_1, then for any initial conditions $x_p(0)$, $x_m(0)$, and $\theta(0) \in \Omega_0$, the closed-loop adaptive system has bounded solutions, with $\theta(t)$ remaining in Ω_1 for all $t \geq 0$ if

$$q_1 q_2 > c_3^2. \tag{31}$$

Proof: Let a Lyapunov function candidate be chosen as

$$V = e^T P e + \lambda \tilde{\theta}^T \Gamma^{-1} \tilde{\theta} + \zeta^T P_\eta \zeta. \tag{32}$$

Taking the time derivative

$$\dot{V} \le -q_1||e||^2 - q_2||\zeta||^2 + 2e^T Pb\lambda\eta + 2\zeta^T P_\eta b_\eta u$$

with some simplifications leads to

$$\begin{aligned}\dot{V} \le & -q_1||e||^2 - q_2||\zeta||^2 + 2||e||p_b\lambda||c_\eta||\ ||\zeta|| \\ & +2p_\eta||\zeta||(\theta_{\text{x,max}}||x|| + \theta_{\text{r,max}} r_{\text{max}}).\end{aligned} \tag{33}$$

Noting that $e = x_p - x_m$ and x_m is bounded, using the definitions in (29) and (30), (33) can be simplified as

$$\dot{V} \le -F(e,\zeta).$$

It can be shown that $F(e,\zeta) = 0$ is an ellipse in the $(\|e\|, \|\zeta\|)$ space if (31) holds. Defining $z = [e^T, \zeta^T]^T$, and

$$M = \begin{bmatrix} q_1 & -c_3 \\ -c_3 & q_2 \end{bmatrix},$$

(33) can be rewritten as

$$\dot{V} \le -z^T Mz + 2c_4\|z\| \tag{34}$$

where M is positive definite due to (31), and $\|\zeta\| \le \|z\|$. We note that the form of the inequality (34) is identical to that of (19), and that V is a function of z and θ with θ bounded. Therefore, identical arguments to that of Theorem 1 can be used to conclude the boundedness of z for any initial conditions $e(0)$ and $\zeta(0)$. Boundedness of $x_p(t)$ follows in a straight forward manner. ■

Remark 2: It should be noted that the global nature of the above result was possible primarily because boundedness of the parameter was established independent of the error dynamics. The former allowed the sign-definiteness terms to be bounded by a quadratic function, thereby leading to boundedness of all signals in the system with arbitrary initial conditions in the state. In other words, the parameter projection algorithm allowed the overall adaptive system, by virtue of Lemma 2, to be treated as a linear time-varying system, thereby leading to a global result. This could not have been accomplished by other adaptive laws with robustness–based modifications than the projection algorithm discussed above.

We now show that a class of unmodeled dynamics (A_η, b_η, c_η) as in (25) exists for any A_m, b, λ, and θ^* in (26). The following lemma is useful in this regard.

Lemma 4: Let P be the solution of the Lyapunov equation $A^T P + PA = -qI$ for a matrix A that is Hurwitz. Then

$$\bar{s}_P = \frac{q}{2\underline{s}_A}. \tag{35}$$

Proof: Since A is Hurwitz,

$$P = \int_0^\infty e^{A^T t} Q e^{At} dt. \tag{36}$$

If λ_i and v_i are ith eigenvalue and corresponding normalized eigenvector of A, respectively, it follows that

$$\begin{aligned}Pv_i &= \Big(q\int_0^\infty e^{\lambda_i^* t} e^{\lambda_i t} dt\Big)v_i \\ &= \frac{q}{2|\Re(\lambda_i)|} v_i\end{aligned} \tag{37}$$

since $Av_i = \lambda_i v_i$, $A^T v_i = \lambda_i^* v_i$, and $e^{At}v_i = e^{\lambda_i t}v_i$. Therefore we can derive (35). ■

We note using Lemma 4 that we can express c_3 in (29) as

$$c_3 \le q_1 \frac{\|b\|\|c_\eta\|\lambda_{\max}}{2\underline{s}_{A_m}} + q_2 \frac{\|b_\eta\|\theta_{x,\max}}{2\underline{s}_{A_\eta}}. \tag{38}$$

Defining

$$\alpha = \sqrt{\frac{q_1}{q_2}}, \qquad \beta_m = \frac{\|b\|\|c_\eta\|}{2\underline{s}_{A_m}}, \qquad \beta_\eta = \frac{\|b_\eta\|}{2\underline{s}_{A_\eta}} \tag{39}$$

and

$$g(\alpha, \beta_m, \beta_\eta) = \beta_m \lambda_{\max}\alpha + \frac{\beta_\eta \theta_{x,\max}}{\alpha}, \tag{40}$$

it follows that the sufficient condition (31) is satisfied if

$$g(\alpha, \beta_m, \beta_\eta) < 1 \tag{41}$$

or equivalently, since $\alpha > 0$, if

$$\beta_m \lambda_{\max}\alpha^2 - \alpha + \beta_\eta \theta_{x,\max} < 0. \tag{42}$$

For ease of exposition, we set $\|c_\eta\| = 1$. This implies that the known parameters A_m and b determine β_m and the parameters of the unmodeled dynamics determine β_η. The question that needs to be answered can be posed as follows: Given β_m and $\theta_{x,\max}$, does a β_η exist such that (42) is satisfied? The answer is affirmative, since it can be derived that there exists $\alpha > 0$ with which (42) is satisfied if

$$4\beta_m\beta_\eta\theta_{x,\max}\lambda_{\max} < 1, \tag{43}$$

and α, defined in (39), is a free parameter. The above discussions are summarized in the following proposition:

Proposition 1: If β_η satisfies the inequality (43), then the sufficient condition (31) in Theorem 1 is satisfied.

Proposition 1 implies that for any A_m, b, λ, and θ^*, a class of unmodeled dynamics always exists for which the sufficient condition (31) is satisfied. This conclusively demonstrates that the closed-loop adaptive system described in this section is robust with respect to a class of unmodeled dynamics that satisfies (43) with the relevant quantities defined in (39).

IV. Robustness of Adaptive Systems to Time-delay

Suppose the input into the plant is delayed so that the plant equations are given by

$$\dot{x}_p = A_m x_p + b\lambda(u(t-\tau) - \theta_x^{*T} x_p). \tag{44}$$

Equation (44) can be rewritten as

$$\dot{x}_p = A_m x_p + b\lambda\big((u(t) + \eta(t)) - \theta_x^{*T} x_p\big) \tag{45}$$

where

$$\eta(t) = [G(s)]u(t), \tag{46}$$

and $G(s)$ is an operator defined by $G(s) = [e^{-\tau s} - 1]$, whose rational approximation of order $2N$(where $N \in \mathbb{Z}_{>0}$) can be obtained by using the Pade approximation of $e^{-\tau s}$:

$$e^{-\tau s} \approx \frac{\sum_{k=0}^{2N}(-1)^k c_k \tau^k s^k}{\sum_{k=0}^{2N} c_k \tau^k s^k} \tag{47}$$

where the coefficients are

$$c_k = \frac{(4N-k)!(2N)!}{(4N)!k!(2N-k)!}, \quad k = 0, 1, \ldots, 2N. \tag{48}$$

It is easy to see that the rational approximation, $G_{Pade}(s)$, of $G(s)$ admits a state-space representation (25), with the parameters

$$A_\eta = \frac{1}{\tau} \underbrace{\begin{bmatrix} -w_1 & 1 & 0 & \cdots \\ -w_2 & 0 & 1 & \cdots \\ -w_3 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & \ddots \end{bmatrix}}_{A_N},$$
$$b_\eta = \frac{1}{\tau} \underbrace{\begin{bmatrix} -v_1 \\ -v_2 \\ -v_3 \\ \vdots \end{bmatrix}}_{b_N}, \quad c_\eta^T = \begin{bmatrix} 1 & 0 & 0 & \cdots \end{bmatrix}. \tag{49}$$

It is important to note that in (49), while the $2N \times 2N$ dimensional matrix A_η and the $2N \times 1$ dimensional matrix b_η depend on τ, the matrix A_N, b_N, and c_η are independent of τ, with A_N Hurwitz. This allows us to conclude from Theorem 2 that there exists a family of the adaptive controller given by (2) and the projection algorithm in (14) with y and $f(\theta)$ chosen as in (15) and (16) which guarantees boundedness for A_η, b_η, and c_η. This is summarized in Theorem 3, with the introduction of additional parameters $\beta_m^\tau, \beta_\eta^\tau$ as

$$\beta_m^\tau = \frac{\|b\|}{2\underline{s}_{A_m}}, \qquad \beta_\eta^\tau = \frac{\|b_N\|}{2\underline{s}_{A_N}}.$$

Theorem 3: Consider the closed-loop adaptive system given by the plant (45), the disturbance η due to time delay which is represented by (25) with parameters (49), the control law (2), the adaptive law (14) with y and $f(\theta)$ chosen as in (15) and (16). If the reference model in (4) and $\theta_{\max}$ are such that θ^* in (5) belongs to Ω_1, then for any initial conditions $x_p(0)$, $x_m(0)$, and $\theta(0) \in \Omega_0$, the closed-loop adaptive system has bounded solutions, with $\theta(t)$ remaining in Ω_1 for all $t \geq 0$, if

$$\beta_\eta^\tau < \frac{1}{4\beta_m^\tau \theta_{x,\max} \lambda_{\max}}. \tag{50}$$

Proof: From the definitions of β_m^τ, β_η^τ and since $A_\eta = (1/\tau)A_N$, $b_\eta = (1/\tau)b_N$, it follows that $\beta_m = \beta_m^\tau$ and $\beta_\eta = \beta_\eta^\tau$. Therefore condition (50) immediately implies that (43) holds. Theorem 2 and Proposition 1 imply that if (43) is satisfied, then boundedness of the overall adaptive system follows, which proves Theorem 3.

Remark 3: As in Section III, whether it exists a class of Pade approximations for which β_η^τ satisfies (50) remains to be shown. Unlike (43), we note that β_η^τ depends on b_N and A_N both of which are independent of τ. In other words, β_η^τ is a fixed constant. Therefore the class of reference models and θ^* that satisfy the matching condition (5) are more limited in this case compared to those in Section III, for a given Pade approximant $G_{Pade}(s)$. In fact, it is possible to show that the sufficient condition (50) essentially requires the uncertain open-loop plant to be stable. The main reason for this limitation is the nature of "unmodeled dynamics" of $G_{Pade}(s)$, where both the zeros and poles diverge as τ becomes smaller, which makes the condition (50) quite restrictive.

Remark 4: Another point to note is that the sufficient condition is independent of τ. That is, if for a given A_m and θ^*, condition (50) is satisfied, then it continues to hold for any τ. However, this sufficient condition is a function of the order of $G_{Pade}(s)$, since $\|b_N\|$ increases with N, which imposes a limit on N for a given A_m and $\theta_{x,max}$. This in turn imposes a limit on the maximum τ that results in boundedness. This is because a fixed N ensures that the corresponding $G_{pade}(s)$ is a good approximation of the actual time-delay over the frequency range of interest only for a certain range of delays. This is formally stated in the following Proposition.

Proposition 2: Let the frequency range of interest be $\Omega = [0, \omega_f]$. Then given a positive constant ϵ and $N \in \mathbb{Z}_{>0}$, there exists a $\tau_N^* \in \mathbb{R}^+$ such that

$$|\angle G_{Pade(2N,2N),\tau}(j\omega) - \angle G(j\omega)| \leq \epsilon \qquad \forall \omega \in \Omega$$

for all $\tau \in [0 \;\; \tau_N^*)$. The time-delay margin τ^* can then be defined as

$$\tau^* = \max_{\mathcal{N}}(\tau_N^*)$$

where $\mathcal{N}$ is the set of all N such that β_η^τ satisfies (50).

References

[1] K. S. Narendra, Y. H. Lin, and L. S. Valavani, "Stable adaptive controller design - part II: proof of stability," *IEEE Transactions on Automatic Control*, vol. 25, pp. 440–448, 1980.

[2] K. S. Narendra and A. M. Annaswamy, "Robust adaptive control in the presence of bounded disturbances," *IEEE Transactions on Automatic Control*, vol. 31, pp. 306–315, 1986.

[3] ——, *Stable Adaptive Systems*. Dover Publications, New York, NY, 2004.

[4] P. A. Ioannou and J. Sun, *Robust Adaptive Control*. Prentice Hall, 1996.

[5] P. R. Naik, S. M. Kumar and B. E. Ydstie, "Robust continuous-time adaptive control by parameter projection," *IEEE Transactions on Automatic Control*, vol. 37, pp. 182–197, 1992.

[6] E. Lavretsky, "The projection operator," in *Personal Notes*, 2006.

[7] K. S. Narendra and P. Kudva, "Stable adaptive schemes for system identification and control - parts I & II," *IEEE Transactions on Systems, Man and Cybernetics*, vol. 4, pp. 542–560, Nov 1972.

[8] G. Kreisselmeier and K. S. Narendra, "Stable model reference adaptive control in the presence of bounded disturbances," *IEEE Transactions on Automatic Control*, vol. 27, pp. 1169–1175, Dec 1982.

[9] J. B. Pomet and L. Praly, "Adaptive nonlinear regulation: estimation from the lyapunov equation," *IEEE Transactions on Automatic Control*, vol. 37, no. 6, pp. 729 –740, jun 1992.

[10] S. Seshagiri and H. K. Khalil, "Output feedback control of nonlinear systems using rbf neural networks," *IEEE Transactions on Neural Networks*, vol. 11, no. 1, pp. 69 –79, jan 2000.

[11] J. K. Hale, *Ordinary Differential Equations*. Wiley-Interscience, 1969.

[12] R. Bellman, "The stability of solutions of linear differential equations," *Duke Math J.*, 1943.