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# Robust Fault Detection Filter Design for Uncertain Switched Systems with Adaptive Threshold Setting

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*Abstract*— **In this paper, the fault detection (FD) problem of uncertain discrete-time switched systems under the constrained switching law and external disturbances is addressed. In the multiple Lyapunov functions (MLF) framework with $\mathscr{H}_-/\mathscr{H}_\infty$ performance index, a robust fault detection filter (RFDF) is designed in order to generate a residual signal which is sensitive to the faults and robust against the disturbances. The design method is formulated to be solved by linear matrix inequality (LMI) technique. The main contribution in this paper is to improve the performance of the FD for switched systems by considering switch detection logics. The design of the RFDF is based on the local performance index of each subsystem. An adaptive threshold is set based on both, the bounds of disturbances and the individual performance index for each subsystem.**

## I. INTRODUCTION

In the last decade, the study on hybrid systems has received increasing attention, due to their ability in modeling and representing the high complex systems in multi-models, which are normally linear and simple. Such complex systems may exist in physical and practical technical systems as in the control of mechanical system, process control, automotive industry, power systems, aircrafts, traffic control and many other fields, see e.g. [1] and [2]. A particular class of the hybrid systems are linear switched systems, which are characterized by subsystems (models), and switching rules governing the active subsystem for each time instant.

One of the fundamental problems in studying the switched systems is the stability. Surveys on this problem can be found in [2], [3], [4] and references therein. The stability problem of the switched systems is usually studied in the Lyapunov function framework, such as, common Lyapunov function, switched Lyapunov functions and multiple Lyapunov functions. More details can be found in [4].

While a lot of works deal with designing a robust control law for switched systems, the fault detection and isolation (FDI) problem in switched systems is still an open issue. It is the aim of this paper to investigate the problem of robust residual generation, residual evaluation and threshold setting for FD in switched systems. To the best of authors' knowledge, the FD problem has not been intensively investigated for switched systems. Recently, some works have addressed the design of a robust residual generator for switched systems, for example, in [5], [6] and [7] the design of a robust hybrid observer for switched linear systems with unknown inputs is given based on the $\mathscr{H}_-/\mathscr{H}_\infty$ performance index. In [8] and [9] the $\mathscr{H}_\infty$-filtering problem is considered for design a robust fault detection for discrete and continuous-time switched systems with state delays. The $\mathscr{H}_\infty$ fault detection for continuous-time linear switched systems with its application to turntable systems is shown in [10]. In these works, the evaluation and threshold setting are defined as in the linear time invariant system, and it didn't adapt the behavior of the switched systems. Thus, a highlight on the evaluation and threshold setting of the residual signal will be given in this paper.

In this paper, the FD problem of discrete-time linear switched systems under modeling uncertainties and system disturbances with average dwell-time will be addressed. The residual signal will be generated using the fault detection filter (FDF). In order to make the residual signal more reliable for fault detection purpose, $\mathscr{H}_-/\mathscr{H}_\infty$ performance index will be used to minimize the effects of unknown inputs, and to maximize the faults effects. This residual signal will be evaluated, and then compared with switch detection logics. The MLF with average dwell-time will be used to ensure the stability of the FD system. The solution is obtained by solving a set of LMIs. The main tasks of this paper are:

- To design a robust residual generator for switched systems in the existence of disturbances and ploytopic model uncertainties by considering the local performance index for each subsystem.
- To find an appropriate evaluation function, fault decision logic and an adaptive threshold for FD of the switched systems by considering both, the individual performance index, and the bound values of the disturbances for each subsystem.

This paper is organized as follows. After the introduction, problem formulation is given in Section II. Some preliminaries are given in section III. Section IV contains the RFDF design theorem for switched systems. The residual evaluation and threshold setting is addressed in Section V.

Finally, Section VI shows the concluding remarks and the possible future work.

*Notations:* The notation used in this paper is generally standard. $X^T$ is the transpose of the matrix $X$. The star symbol $\{\ast\}$ in a symmetric matrix denotes the transposed block in the symmetric position. The notation $P>0$, $(P<0)$ means $P$ is real-symmetric and positive definite, (negative definite). 0 is a zero matrix of appropriate dimension. $I$ represent the identity matrix and $\mathscr{L}_2$-norm for each subsystem is defined as $||x_{\sigma(k)}(k)||_2 = \sqrt{\sum_{k=0}^{\infty} x^T_{\sigma(k)}(k)x_{\sigma(k)}(k)}$, where $\sigma(k)$ is the switching signal which will be defined in the next section.

## II. Problem Formulation

The following discrete-time linear switched system with polytopic uncertainties is considered:

$$x_{k+1} = \bar{A}_{\sigma(k)}x_k + \bar{B}_{\sigma(k)}u_k + \bar{E}_{d,\sigma(k)}d_{k,\sigma(k)} + E_{f\sigma(k)}f_k \quad (1)$$
$$y_k = \bar{C}_{\sigma(k)}x_k + \bar{D}_{\sigma(k)}u_k + \bar{F}_{d,\sigma(k)}d_{k,\sigma(k)} + F_{f\sigma(k)}f_k \quad (2)$$

where, $x \in \mathscr{R}^n$ is the system state vector, $y \in \mathscr{R}^m$ is the measurement output vector and $u \in \mathscr{R}^p$ is the input vector. $d_{\sigma(k)} \in \mathscr{R}^{k_d}$ represents the disturbance vector for each subsystem and $f \in \mathscr{R}^{k_f}$ is the vector of faults to be detected.
Switching signal $\sigma(k)$ can be classified as: time dependent, state dependent and control input or output signal. ***In this study, it is assumed that, the switching signal is unknown a priori but its value is real-time available***. The switching rule $\sigma(k)$, which decide the active linear vector field at a certain time instant, takes values in the finite set $i=\{1,2,...,N\}$, i.e. $\sigma(k)\in i$.
The indices of $i$ related to the subsystem in the switched systems, are follows: $\bar{A}_i \triangleq \{\bar{A}_1,\bar{A}_2,...,\bar{A}_N\}$, $\bar{B}_i \triangleq \{\bar{B}_1,\bar{B}_2,...,\bar{B}_N\}$, $\bar{C}_i \triangleq \{\bar{C}_1,\bar{C}_2,...,\bar{C}_N\}$, $\bar{D}_i \triangleq \{\bar{D}_1,\bar{D}_2,...,\bar{D}_N\}$. These matrices are known and in appropriate dimension.
In the switched systems, each linear model may be affected by different disturbances and faults. Therefore, the disturbance and fault matrices are known and defined as follows: $\bar{E}_{di} \triangleq \{\bar{E}_{d1},\bar{E}_{d2},...,\bar{E}_{dN}\}$, $E_{fi} \triangleq \{E_{f1},E_{f2},...,E_{fN}\}$, $\bar{F}_{di} \triangleq \{\bar{F}_{d1},\bar{F}_{d2},...,\bar{F}_{dN}\}$ and $F_{fi} \triangleq \{F_{f1},F_{f2},...,F_{fN}\}$.
Model uncertainties can be represented in different forms, [11]. For the switched systems the polytopic uncertainty is used, where:
$\bar{A}_i = A_i + \Delta A_i$, $\bar{B}_i = B_i + \Delta B_i$, $\bar{E}_{di} = E_{di} + \Delta E_{di}$,
$\bar{C}_i = C_i + \Delta C_i$, $\bar{D}_i = D_i + \Delta D_i$ and $\bar{F}_{di} = F_{di} + \Delta F_{di}$.
with

$$\begin{bmatrix} \Delta A_i & \Delta B_i & \Delta E_{di} \\ \Delta C_i & \Delta D_i & \Delta F_{di} \end{bmatrix} = \sum_{l=1}^{m} \eta^l \begin{bmatrix} A_i^l & B_i^l & \Delta E_{di}^l \\ C_i^l & D_i^l & \Delta F_{di}^l \end{bmatrix},$$
$$i\in\{1,2,...,N\}, \quad \sum_{l=1}^{m}\eta^l = 1, \quad \eta^l \geq 0, \quad l\in\{1,2,...,m\}.$$

Residual signal can be generated using a fault detection filter (FDF), see e.g. [11] and [12]. The FDF structure for discrete-time linear switched systems is as follows:

$$\hat{x}_{k+1} = A_{\sigma(k)}\hat{x}_k + B_{\sigma(k)}u_k + L_{\sigma(k)}(y_k - \hat{y}_k), \quad (3)$$
$$\hat{y}_k = C_{\sigma(k)}\hat{x}_k + D_{\sigma(k)}u_k, \quad (4)$$
$$r_{k,\sigma(k)} = G_{\sigma(k)}(y_k - \hat{y}_k). \quad (5)$$

where $\hat{x}_k \in \mathscr{R}^n$ is the estimation of the state vector $x_k$, $\hat{y}_k \in \mathscr{R}^m$ is the estimation of the output vector $y_k$ and $\sigma(k)\in i$. $L \in \mathscr{R}^{m\times m}$ is the observer gain and $L_i \triangleq \{L_1,L_2,...,L_N\}$, $r_{\sigma(k)} \in \mathscr{R}^m$ is the residual vector for each subsystem, and $G \in \mathscr{R}^{m\times m}$, is the post filter matrix, and $G_i \triangleq \{G_1,G_2,...,G_N\}$, the indices $\{1,2,...,N\}$ decide the active FDF.

To design a RFDF, $\mathscr{H}_-/\mathscr{H}_\infty$ performance index will be used in order to reduce the effects of the unknown inputs on the residual signal, and simultaneously increase the system sensitivity to the faults. It is assumed that, the unknown inputs $d$ and the faults $f$ are $\mathscr{L}_2 - norm$ bounded. The performance of switched systems can be expressed in two forms:

- The performance index for each subsystem can be taken individually. So if there is $\gamma$-performance index, then it will be defined for each local subsystem $\gamma_i$, $i\in\{1,2,...,N\}$.
- The maximum performance index among all subsystems can be considered for the whole switched systems, i.e. $\gamma = \max_{i\in\{1,...,N\}}\{\gamma_i\}$.

In this study, the local performance index for each subsystem is considered see e.g. [13] and [14]. Based on this property, $\mathscr{H}_-/\mathscr{H}_\infty$ performance index of system (1)-(2) can be written as follows:

$$||r_{f,\sigma(k)}||_2 > \alpha_{\sigma(k)}||f||_2 \quad (6)$$
$$||r_{d,\sigma(k)}||_2 < \beta_{\sigma(k)}||d_{\sigma(k)}||_2 \quad (7)$$

where $r_{d,\sigma(k)} = r_{\sigma(k)}|_{f=0}$ , $r_{f,\sigma(k)} = r_{\sigma(k)}|_{d=0}$, $\sigma(k)\in i$, and the scalars $\alpha_{\sigma(k)}>0$, $\beta_{\sigma(k)}>0$.

## III. Preliminaries

### A. Average Dwell-time

One of the constrains on the switched systems is slow switching criteria, which ensure the global stability if it is set to an appropriate value, see e.g. [3] and [15]. Dwell-time and average dwell-time can be defined with the concept of slow switching. In this paper, the average dwell-time will be considered.
*Definition 1:* [16] *Average dwell-time in the discrete-time domain*
For the switching signal $\sigma(k)$ and for any $k_t > k_s > k_0$, let $N_\sigma(k_s,k_t)$ be the switching numbers of variations of $\sigma(k)$ over the interval $[k_s,k_t]$. If for any given $N_0>0, \tau_a>0$, we have

$$N_\sigma(k_s,k_t) \leq N_0 + \frac{(k_t-k_s)}{\tau_a} \quad (8)$$

then $\tau_a$ and $N_0$ are called average dwell-time and chatter bound, respectively.

Based on this definition, the slow switched systems are denoted by the systems which have at least $\tau_a$ between any two consecutive switching interval.

To this end, consider the following lemma,

*Lemma 1:* [17]

*Consider the discrete-time switched systems given by (1)-(2), $\sigma(k) \in i$ and let $0 < \zeta < 1, \mu > 1$ be given constants. Suppose that there exists a candidate Lyapunov function $V(x) = \{V_{\sigma(k)}(x),\ \sigma_{(k)} \in i\}$, satisfying the following Properties:*

$$\Delta V_i(x_k) \triangleq V_i(k_{k+1}) - V_i(x_k) \leq -\zeta V_i(x_k) \quad (9)$$

$$V_j(x_k) \leq \mu V_i(x_k) \quad (10)$$

*then the system is globally asymptotically stable for any switching signals with average dwell-time*

$$\tau_a \geq \tau_a^* = -\frac{ln\mu}{ln(1-\zeta)} \quad (11)$$

*Remark 1:*

- $\tau_a$ must be rounded to the nearest integer, due to working in the discrete-time domain, i.e. it is related to fixing sample number.
- $i$ and $j$ in the previous lemma means that, at time $k$ the switched systems will be in mode $i$, and at time $k+1$ it will be in mode $j$, where $i \neq j$ and $(i,j) \in \mathscr{I} \times \mathscr{I}$, $\mathscr{I} = \{1,2,...,N\}$.

### B. Multiple Lyapunov Functions (MLF)

The existence of a common Lyapunov function for all the subsystems in the switched systems is too conservative (or it may not exist). Therefore, MLF are a useful tool to prove the stability of the switched systems, see e.g. [4], where the general form of MLF is defined as follows:

$$V(x_k) = x_k^T P_{\sigma(k)} x_k, \quad \sigma(k) \in i \quad i = \{1,2,...,N\}$$

$\{P_1, P_2, ..., P_N\}$ are symmetric positive definite matrices. The most distinct character of MLF is the decreasing behavior of Lyapunov function values during the active subsystems. Moreover, at each switching instant its value is not increasing.

For the stability of the switched systems, it is assumed that

- for each mode, the pairs $(A_i, B_i)$ is stabilzable.
- the MLF can be set as follows:

1) $P_i > 0$ ,
2) $V_i(x_k)$ is a positive definite and decreasing function,
3) and $\Delta V_i(x_k)$ is negative definite.
4) when switching occurs, the MLF should satisfy the following conditions:
   - $P_i > 0$ and $P_j > 0$,
   - $V_i$ and $V_j$ are positive definite,
   - $\Delta V_i$ and $\Delta V_j$ are negative definite,
   - switching condition: as set in Lemma 1, at switching time instance the following conditions must be satisfied, 1) $\Delta V_i(x_k) \leq -\zeta V_i(x_k)$. 2) $V_j(x_k) \leq \mu V_i(x_k)$, which can be written as $x_k^T P_j x_k \leq \mu x_k^T P_i x_k$.

In the following, the design of the RFDF for the switched systems will be analyzed in the discrete-time domain.

## IV. RFDF DESIGN

The dynamics of the residual generator (3)-(5) is governed by:

$$x_{k+1} = \bar{A}_{\sigma(k)} x_k + \bar{B}_{\sigma(k)} u_k + \bar{E}_{d,\sigma(k)} d_{k,\sigma(k)} + E_{f,\sigma(k)} f_k \quad (12)$$

$$\begin{aligned} e_{k+1} = {} & (A_{\sigma(k)} - L_{\sigma(k)} C_{\sigma(k)}) e_k + (\Delta A_{\sigma(k)} - L_{\sigma(k)} \Delta C_{\sigma(k)}) x_k \\ & + (\Delta B_{\sigma(k)} - L_{\sigma(k)} \Delta D_{\sigma(k)}) u_k + (\bar{E}_{d,\sigma(k)} - \\ & L_{\sigma(k)} \bar{F}_{d,\sigma(k)}) d_{k,\sigma(k)} + (E_{f,\sigma(k)} - L_{\sigma(k)} F_{f,\sigma(k)}) f_k \end{aligned} \quad (13)$$

$$\begin{aligned} r_{k,\sigma(k)} = {} & G_{\sigma(k)} (C_{\sigma(k)} e_k + \Delta C_{\sigma(k)} x_k + \Delta D_{\sigma(k)} u_k \\ & + \bar{F}_{d,\sigma(k)} d_{k,\sigma(k)} + F_{f,\sigma(k)} f_k) \end{aligned} \quad (14)$$

where $e_k$ is the estimation error $e_k = x_k - \hat{x}_k$.

Model uncertainties can be transformed into the unknown inputs as follows:

$$x_{k+1} = \bar{A}_{\sigma(k)} x_k + \underbrace{\begin{bmatrix} \bar{B}_{\sigma(k)} & 0 \\ 0 & \bar{E}_{d,\sigma(k)} \end{bmatrix}}_{E_{\sigma(k),u,d}} \underbrace{\begin{bmatrix} u_k \\ d_{k,\sigma(k)} \end{bmatrix}}_{d_{u,d}} + E_{f,\sigma(k)} f_k \quad (15)$$

$$\begin{aligned} e_{k+1} = {} & (A_{\sigma(k)} - L_{\sigma(k)} C_{\sigma(k)}) e_k + (\Delta A_{\sigma(k)} - L_{\sigma(k)} \Delta C_{\sigma(k)}) x_k \\ & + \underbrace{\begin{bmatrix} \Delta B_{\sigma(k)} - L_{\sigma(k)} \Delta D_{\sigma(k)} & 0 \\ 0 & \bar{E}_{d,\sigma(k)} - L_{\sigma(k)} \bar{F}_{d,\sigma(k)} \end{bmatrix}}_{F_{\sigma(k),u,d}} . \\ & \underbrace{\begin{bmatrix} u_k \\ d_{k,\sigma(k)} \end{bmatrix}}_{d_{u,d}} + \underbrace{(\bar{E}_{f,\sigma(k)} - L_{\sigma(k)} \bar{F}_{f,\sigma(k)})}_{F_{\sigma(k)}} f_k \end{aligned} \quad (16)$$

$$\begin{aligned} r_{k,\sigma(k)} = {} & G_{\sigma(k)} C_{\sigma(k)} e_k + \Delta G_{\sigma(k)} C_{\sigma(k)} x_k + G_{\sigma(k)} F_{f,\sigma(k)} f_k \\ & + \underbrace{\begin{bmatrix} G_{\sigma(k)} \Delta D_{\sigma(k)} & G_{\sigma(k)} \bar{F}_{d,\sigma(k)} \end{bmatrix}}_{D^*_{\sigma(k)}} d_{u,d} \end{aligned} \quad (17)$$

The dynamics of the residual generator can be rewritten as follows:

$$\begin{aligned} \underbrace{\begin{bmatrix} x_{k+1} \\ e_{k+1} \end{bmatrix}}_{x_{o,k+1}} = {} & \underbrace{\begin{bmatrix} A_{\sigma(k)} + \Delta A_{\sigma(k)} & 0 \\ \Delta A_{\sigma(k)} - L_{\sigma(k)} \Delta C_{\sigma(k)} & A_{\sigma(k)} - L_{\sigma(k)} C_{\sigma(k)} \end{bmatrix}}_{A^*_{\sigma(k)}} . \\ & \underbrace{\begin{bmatrix} x_k \\ e_k \end{bmatrix}}_{x_{o,k}} + \underbrace{\begin{bmatrix} E_{u,d} \\ F_{u,d} \end{bmatrix}}_{E^*_{\sigma(k)}} d_{u,d} + \underbrace{\begin{bmatrix} E_{f,\sigma(k)} \\ F \end{bmatrix}}_{F^*_{\sigma(k)}} f_k \end{aligned} \quad (18)$$

$$\begin{aligned} r_{k,\sigma(k)} = {} & \underbrace{\begin{bmatrix} \Delta C_{\sigma(k)} & G_{\sigma(k)} C_{\sigma(k)} \end{bmatrix}}_{C^*_{\sigma(k)}} \underbrace{\begin{bmatrix} x_k \\ e_k \end{bmatrix}}_{x_{o,k}} \\ & + D^*_{\sigma(k)} d_{u,d} + \underbrace{(G_{\sigma(k)} F_{f,\sigma(k)})}_{H^*_{\sigma(k)}} f_k \end{aligned} \quad (19)$$

Finally, it can be presented in the compact form:

$$x_{o,k+1} = A^*_{\sigma(k)} x_{o,k} + E^*_{\sigma(k)} d_{u,d} + F^*_{\sigma(k)} f_k \quad (20)$$

$$r_{k,\sigma(k)} = C^*_{\sigma(k)} x_{o,k} + D^*_{\sigma(k)} d_{u,d} + H^*_{\sigma(k)} f_k \quad (21)$$

The following Theorem presents the sufficient conditions for design RFDF of switched systems (1)-(2) with average dwell-time and satisfying $\mathscr{H}_-/\mathscr{H}_\infty$ performance index given in (6) and (7).

*Theorem 1:*
*The system (20)-(21) is asymptotically stable and satisfies*

$$||r_{f,\sigma(k)}||_2 \quad > \quad \alpha_{\sigma(k)}||f||_2$$
$$||r_{d,\sigma(k)}||_2 \quad < \quad \beta_{\sigma(k)}||d_{u,d}||_2$$

*for any switching signal satisfying (11), if there exist matrices* $P_i > 0$ *and* $Q_i > 0$, *and scalars* $\alpha_{\sigma(k)} > 0$, $\beta_{\sigma(k)} > 0$, *so that*

$$\begin{bmatrix} -P_j & P_jA^* & P_jE^* \\ \ast & -(1-\zeta)P_i + C^{*T}C^* & C^{*T}D^* \\ \ast & \ast & D^{*T}D^* - \beta^2_{\sigma(k)}I \end{bmatrix} < 0 \qquad (22)$$

$$P_j - \mu P_i < 0 \qquad (23)$$

$$\begin{bmatrix} Q_j & Q_jA^* & Q_jF^* \\ \ast & (1-\zeta)Q_i + C^{*T}C^* & C^{*T}F_f^* \\ \ast & \ast & F_f^{*T}F_f^* - \alpha^2_{\sigma(k)}I \end{bmatrix} > 0 \qquad (24)$$

$$Q_j - \mu Q_i < 0 \qquad (25)$$

*for all* $i, j \in \{1,2,...,N\}$, $l \in \{1,2,...,m\}$ *and* $i \neq j$ *where* $\mu > 1$ *and* $0 < \zeta < 1$ *are given constants and* $\sigma(k) \in i$.
*Proof:*
Define the following MLF

$$V(x_{o,k}) = x^T_{o,k}P_{\sigma(k)}x_{o,k} \qquad \sigma(k) \in i, \quad i = \{1,2,...,N\} \qquad (26)$$

Formulate $\Delta V$ for (20)-(21) in a quadratic form gives:

$$\Delta V = \begin{bmatrix} x^T_o & d^T_{u,d} & f^T \end{bmatrix} [M] \begin{bmatrix} x_o \\ d_{u,d} \\ f \end{bmatrix} < 0 \qquad (27)$$

$$M = \begin{bmatrix} A^{*T}P_jA^* - (1-\zeta)P_i & A^{*T}P_jE^* & A^{*T}P_jF^* \\ \ast & E^{*T}P_jE^* & E^{*T}P_jF^* \\ \ast & \ast & F^{*T}P_jF^* \end{bmatrix} < 0 \quad (28)$$

The robustness index:

$$\mathscr{H}_\infty : ||r_d||_2 < \beta_{\sigma(k)}||d_{u,d}||_2, \qquad r_d = r|_{f=0}$$
$$\Rightarrow r^T_d r_d - \beta^2_{\sigma(k)} d^T_{u,d}d_{u,d} < 0$$
$$\Rightarrow r^T_d r_d - \beta^2_{\sigma(k)} d^T_{u,d}d_{u,d} =$$
$$\begin{bmatrix} x^T_o & d^T_{u,d} \end{bmatrix} \begin{bmatrix} C^{*T}C^* & C^{*T}D^* \\ \ast & D^{*T}D^* - \beta^2_{\sigma(k)}I \end{bmatrix} \begin{bmatrix} x_o \\ d_{u,d} \end{bmatrix} < 0$$

It can be concluded that:

$$\Delta V|_{f=0} + r^T_d r_d - \beta^2_i d^T_{u,d}d_{u,d} < 0 \qquad (29)$$

After some mathematical manipulations and using the Schur complement lemma, it leads to:

$$\begin{bmatrix} -P_j & P_jA^* & P_jE^* \\ \ast & -(1-\zeta)P_i + C^{*T}C^* & C^{*T}D^* \\ \ast & \ast & D^{*T}D^* - \beta^2_{\sigma(k)}I \end{bmatrix} < 0 \qquad (30)$$

$$P_j - \mu P_i < 0 \qquad (31)$$

where $P_j = \begin{bmatrix} P_{j1} & 0 \\ 0 & P_{j2} \end{bmatrix} > 0$, $P_i = \begin{bmatrix} P_{i1} & 0 \\ 0 & P_{i2} \end{bmatrix} > 0$, $\beta_{\sigma(k)} > 0$

And for the $\mathscr{H}_-$index:

$$\mathscr{H}_- : ||r_f||_2 > \alpha_{\sigma(k)}||f||_2 \qquad r_f = r|_{d_{u,d}=0}$$
$$\Rightarrow r^T_f r_f - \alpha^2_{\sigma(k)} f^T f > 0$$
$$\Rightarrow r^T_f r_f - \alpha^2_{\sigma(k)} f^T f =$$
$$\begin{bmatrix} x^T_o & f^T \end{bmatrix} \begin{bmatrix} C^{*T}C^* & C^{*T}H^* \\ \ast & H^{*T}H^* - \alpha^2_{\sigma(k)}I \end{bmatrix} \begin{bmatrix} x_o \\ f \end{bmatrix} > 0$$

Set the following MLF:

$$V(x_{o,k}) = x^T_{o,k}Q_\sigma x_{o,k} \qquad \sigma(k) \in i, \quad i = \{1,2,...,N\} \qquad (32)$$

it turns out:

$$-\Delta V|_{d_{u,d}=0} + r^T_f r_f - \alpha^2_{\sigma(k)} f^T f > 0 \qquad (33)$$

Consequently, using the Schur complement lemma will lead to the following matrix inequality:

$$\begin{bmatrix} Q_j & Q_jA^* & Q_jF^* \\ \ast & (1-\zeta)Q_i + C^{*T}C^* & C^{*T}F_f^* \\ \ast & \ast & F_f^{*T}F_f^* - \alpha^2_{\sigma(k)}I \end{bmatrix} > 0 \qquad (34)$$

$$Q_j - \mu Q_i < 0 \qquad (35)$$

where $Q_j = \begin{bmatrix} Q_{j1} & 0 \\ 0 & Q_{j2} \end{bmatrix} > 0$, $Q_i = \begin{bmatrix} Q_{i1} & 0 \\ 0 & Q_{i2} \end{bmatrix} > 0$, $\alpha_{\sigma(k)} > 0$.
This completes the proof. □
The following figure shows the scheme of FDF for switched systems.

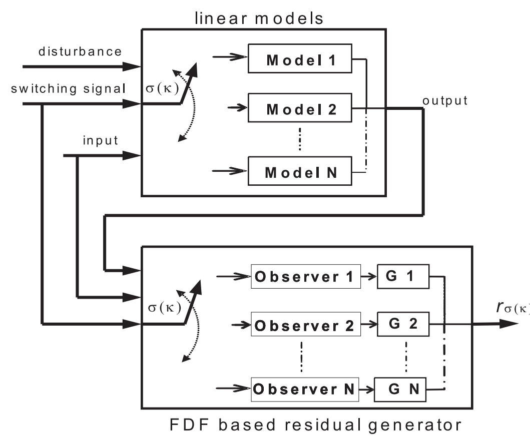


Fig. 1. Scheme of FDF for Hybrid Switched Systems

*Remark 2:*

- Matrix inequalities (22) and (24) are nonlinear matrix inequalities (NMI)s. It can be transformed to LMIs by using following definitions:

$$P_{j2}L_j = X_j \quad \Rightarrow \quad L_j = (P_{j2})^{-1}X_j \qquad (36)$$
$$Q_{j2}L_j = Y_j \quad \Rightarrow \quad L_j = (Q_{j2})^{-1}Y_j \qquad (37)$$

- In addition, $G_j$ is a square matrix, so it can be formulated as follows:

$$G^T_j G_j = M_j \Rightarrow G_j = (M_j)^{1/2}, \quad M_j > 0 \qquad (38)$$

- Equations (36) and (37) give two values for observer gain $L_j$, one solution to this problem is to set $P_{j2} = Q_{j2}$, so that the observer gain matrix will be $L_j = (P_{j2})^{-1}X_j$.

The previous results can be summarized by the following algorithm.

*Algorithm 1: Computation of the mixed* $\mathscr{H}_-/\mathscr{H}_\infty$ *FDF for switched systems under average dwell-time switching*

- **Step 0**: Rewrite the residual dynamics to the form (20)-(21), give the scalar values of $\mu$ and $\zeta$, and set the disturbance attenuation level $\beta$ to some value.
- **Step 1**: Set $P_{j2} = Q_{j2}$. Solve the matrix inequalities (22)-(25) based on the definitions (36)-(38), so that the sensitivity level $\alpha$ is maximized.
- **Step 2**: Set the final value of the observer gain as $L_j = (P_{j2})^{-1}X_j$, and the post filter matrix as $G_j = \sqrt{M_j}$.
- **Step 3**: Construct the residual generator (3)-(5).

*Remark 3:* This algorithm is for the off-line calculations, where the design parameters of the RFDF are found. Therefore, this algorithm will be repeated $N(N-1)l$ times to find these parameters, which will be saved and used in the online running of the system.
In order to achieve a successful fault detection, it is required to distinguish faults from unknown inputs and model uncertainties. This purpose can be achieved by residual evaluation and threshold setting, which will be given in the following section.

## V. Residual Evaluation and Threshold Setting

Depending on the system under consideration, different strategies for residual evaluation and threshold setting can be used, see e.g. [11]. In this paper, the *Root Mean Square* (RMS) function as a norm based evaluation function will be used to evaluate the switched residual signal, and an adaptive switched threshold will be set to detect the faults.

### A. RMS evaluation function for switched systems

RMS value measures the average energy of a signal over a time interval $(k, k+T)$. The RMS evaluation function of the residual signal for switched systems is set for each subsystem as follows:

$$J_{\sigma(k),RMS} = ||r_{\sigma(k)}(k)||_{RMS} = \sqrt{\frac{1}{T}\sum_{t=1}^{T}||r_{\sigma(k)}(k+t)||^2} \quad (39)$$

*Remark 4:* RMS value of a signal is related to $\mathscr{L}_2$-norm of this signal, as follows:

$$||r_{\sigma(k)}(k)||_{RMS} \leq \frac{1}{\sqrt{T}}||r_{\sigma(k)}(k)||_2$$

This evaluation function is chosen for switched systems for the following reasons:

1) It is widely used in practice.
2) It reduces the effect of unknown inputs, because of the average of the instantaneous value of the residual signal over the time window T.
3) RMS evaluation produces smoothness in residual signal over the moving time window.

After the residual signal has been evaluated, a threshold value should be set. Threshold setting will be explained in the following subsection.

### B. Threshold Setting

Threshold is the tolerant limit for unknown inputs and uncertainties during the fault free operation. This limit value can be computed as a constant or adaptive or dynamic value, see e.g. [11], [12], [18], [19] and the references therein. The threshold computation will be based on the maximum change in the (average) energy level of $r$ in response to the disturbances and model uncertainties for each subsystem.

In constant threshold computation, the maximum effect of the unknown inputs and model uncertainties for all subsystems (worst case) will be set as threshold value, where the disturbances and the control input are considered as unknown bound values. Since the control input $u$ is available online during system operation, it is evident to consider the instantaneous values of $u$ in the calculation of the threshold. Furthermore, the adaptation in the threshold can be achieved by considering the different bound values of the disturbances and the local performance index for each subsystem of the switched systems. Thereby, the fault detection rate will be enhanced.
To this end, the threshold value for switched systems is set as follows:

$$J_{th,\sigma(k),RMS} = \sup_{\substack{\|d_{(u,d)}\|_{RMS} < \delta_{(d_{\sigma(k)},2)} + \|u\|_{2,T} \\ \eta^l, l=\{1,\dots,m\}, f=0}} J_{\sigma(k),RMS} \quad (40)$$

where $\sigma(k) \in i$. As a result, the switch decision logic for detecting a fault for each subsystem is given as follows:

$$J_{\sigma(k),RMS} > J_{th,\sigma(k),RMS} \Rightarrow \text{fault}-\text{alarm}$$
$$J_{\sigma(k),RMS} \leq J_{th,\sigma(k),RMS} \Rightarrow \text{fault}-\text{free}$$

Therefore, the adaptive threshold for switched systems is defined as follows:
*Definition 2: Adaptive Threshold*
*Assume that $d_{u,d}$ is bounded for each subsystem in the sense of*

$$\|d_{(u,d)}\|_2 \leq \|d_{\sigma(k)}\|_2 + \|u\|_2 \leq \delta_{(d_{\sigma(k)},2)} + \|u\|_{2,T} \quad (41)$$

*Then the threshold can be set for **each subsystem** by*

$$J_{th,\sigma(k),RMS,2} = \sup_{\substack{\|d_{(u,d)}\|_{RMS} < \delta_{(d_{\sigma(k)},2)} + \|u\|_{2,T} \\ \eta^l, l=\{1,\dots,m\}, f=0}} J_{\sigma(k),RMS} \quad (42)$$

where $\delta_{(d_{\sigma(k)},2)}$ is the $\mathscr{L}_2$-norm of the disturbance for each subsystem, i.e. $\{\delta_{d1}, \delta_{d2}, ..., \delta_{dN}\}$.

The threshold computation is formulated as an optimization problem of finding minimum $\beta_{\sigma(k)}$, as follows:
*Theorem 2: Computation of adaptive threshold*
Given the system (20)-(21) with polytopic uncertainties, and $\beta_{\sigma(k)} > 0$, $\sigma(k) \in i$, $i = \{1,2,...,N\}$, and the evaluation window $T$, then the adaptive threshold is set to

$$J_{adap.th,\sigma(k),RMS,2}(k) = \frac{\beta_{\sigma(k)}}{\sqrt{T}}(\delta_{(d_{\sigma(k)},2)} + ||u(k)||_{2,T}) \quad (43)$$

where $\beta_{\sigma(k)}$ should satisfies

$$||r(k)||_2 \leq \beta_{\sigma(k)}||d_{(u,d)}(k)||_2 \qquad (44)$$

The proof of this theorem is derived from definition 2 and theorem 1.

*Remark 5:*

• The optimization problem (44) is already solved in the designed of residual signal, i.e. in solving the matrix inequality (22). Then the threshold is set by simple computation of equation (43).

• The threshold given in (43) consists of two time-varying parts: the first part depends on the bounds of the disturbances and individual performance index for each subsystem. The second part depends on the instantaneous energy change of the input signal.

Equations (41)-(43) showed that, the proposed idea of using the bound values of disturbances and the local performance index for each subsystem will reduce the threshold size and thus enhance the fault detectability. This result will be explained as follows:

• due to the nature of switched systems, the disturbance bound on each subsystem can be less than or equal the worst case disturbance $\delta_{dmax}$, which affects the whole system, i.e. $\delta_{d,\sigma(k)} \leq \delta_{dmax}$.

• the local performance index satisfies $\beta_{\sigma(k)} \leq \beta_{max}$. Therefore, instead of using the worst performance index $\beta_{max}$ for all subsystems, the local performance index will be used for each subsystem.

The following algorithm summarizes the design procedure:

*Algorithm 2:* Computation of adaptive threshold for switched systems

- **Step 0:** *The unknown input for each subsystem is assumed to be bounded by* $(\delta_{d_{\sigma(k)}})$, $\sigma(k) \in i$, $i = \{1,2,...,N\}$.
- **Step 1:** *solve the optimization problem*

$$min \beta_{\sigma(k)}$$

*subject to matrix inequalities (22) where* $\beta_{\sigma(k)} > 0$, $P_j > 0$, $P_i > 0$, *and given constant* $0 < \zeta < 1$. ***This step can be done off-line.***

- **Step 2:** *compute*

$$J_{adap.th,\sigma(k),RMS,2}(k) = \frac{\beta_{\sigma(k)}}{\sqrt{T}}(\delta_{(d_{\sigma(k)},2)} + ||u(k)||_{2,T})$$

*where* $||u||_{2,T}$ *is the instantaneous energy change of the switching signal, and* $T$ *is the evaluation window.*

## VI. Conclusions and Future Work

This paper proposed the design of RFDF with $\mathcal{H}_-/\mathcal{H}_\infty$ index for switched systems by considering the individual performance index of each subsystem. MLF with average dwell-time is used to ensure the stability of the switched systems. On the other hand, using the bounds of the disturbances and local performance index for each subsystem give the opportunity to design an adaptive threshold. The advantages of the adaptive threshold proposed in this paper is summarized as: 1) reduce of the threshold size, which increase the fault detection capability. 2) it can be implemented on-line easily, and it does not require too many on-line computations. The proposed theory has been successfully applied to the lateral vehicle dynamics benchmark (given in [11]:Chapter 3), but due to the space limitation, this example cannot be presented in this paper. The LMI which is used in this work can be relaxed by using the methods in [13], [20], and [21], which can be considered as an extension to this work.

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