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Residential Demand Response with Interruptible Tasks: Duality and Algorithms

Nikolaos Gatsis and Georgios B. Giannakis

Abstract— **This paper deals with optimal scheduling of demand response in a residential setup when the electricity prices are known ahead of time. Each end-user has a "must-run" load, and two types of adjustable loads. The first type must consume a specified total amount of energy over the scheduling horizon, but its consumption can be adjusted across the horizon. The second type of load has adjustable power consumption without a total energy requirement, but operation of the load at reduced power results in dissatisfaction of the end-user. Each adjustable load is interruptible in the sense that the load can be either operated (resulting in nonzero power consumption), or not operated (resulting in zero power consumption). Examples of such adjustable interruptible loads are charging a plug-in hybrid electric vehicle or operating a pool pump. The problem amounts to minimizing the cost of electricity plus user dissatisfaction, subject to individual load consumption constraints. The problem is nonconvex, but surprisingly it is shown to have zero duality gap if a continuous-time horizon is considered. This opens up the possibility of using Lagrangian dual algorithms without loss of optimality in order to come up with efficient demand response scheduling schemes.**

I. Introduction

The smart grid vision is to modernize the aging power grid infrastructure by capitalizing on the proven sensor, communication, and control technologies of today to address the pressing issues of environment, consumer demand, security and stability of energy generation, transportation, and consumption. One of the directions that are expected to be advanced toward this vision is enabling interaction of end-users with the grid [1].

A resource management task enabling interaction of end-users with the grid of the future is *demand response* (DR), also known as demand-side management [2]. DR aims to adapt the end-user power consumption in response to time-varying (or time-based) energy pricing, which is judiciously controlled by the utility companies to elicit desirable energy usage. On the one hand, end-users are well-motivated to shift nonurgent power consumption to periods of lower electricity price in order to reduce their utility bills. The utility company, on the other hand, benefits from smoothing out the peak demand, which has major impact on system reliability, generation cost, and meeting the pollution mandates.

DR is facilitated by deployment of *advanced metering infrastructure* (AMI), which comprises a two-way communication network between utility companies and the end-users [3]. Smart meters installed at end-users' premises are the AMI terminals at the end-users' side. These measure not just the total power consumption, but also the power consumption profile throughout the day, and report it to the utility company at regular time intervals. The utility company sends pricing signals to the smart meters through the AMI (real-time pricing), for the smart meters to adjust power consumption profiles of the various residential electric devices, in order to minimize the electricity bill and maximize end-user satisfaction.

This paper deals with optimal energy scheduling of interruptible devices in a residential setup. The interruptibility alludes to the fact that the device can be turned off and then on depending on the electricity cost, until it provides its service to the end-user.

Energy scheduling of noninterruptible devices, which can lead to convex problems, has been dealt with in [4]–[7]. Interruptible devices with discrete power consumption levels—leading to nonconvex problems—have also been considered using different optimization approaches [5], [7]–[10]. Stochastic counterparts where the task requests or the electricity prices are modeled as random processes have also been considered [7], [11].

The present work considers scheduling of two types of adjustable loads. The first must consume a specified total amount of energy over the scheduling horizon, but the consumption can be adjusted across the horizon. The second type of load has adjustable power consumption without a total energy requirement, but operation of the load at reduced power results in dissatisfaction of the end-user. Each adjustable load is interruptible in the sense that the load can be either operated (resulting in a nonzero power consumption in a continuous interval), or not operated (resulting in zero power consumption). An example of the first type is charging a plug-in hybrid electric vehicle; while an air conditioning unit (A/C) provides an example of the second type. The resulting formulation is nonconvex, and distinct from the problems with discrete energy levels. Different from other works in the literature, the approach taken here relies on Lagrangian duality. The main findings are that: (a) formulating the problem over a continuous time horizon has zero duality gap; and (b) if the time is discretized, the problem has vanishing duality gap as the discretization becomes finer.

The rest of this paper is organized as follows. Section II formulates the continuous-time and discrete-time optimization problems, and provides the duality gap results. The subgradient method is used to solve the dual problem in Section III. Section IV presents numerical tests, and Section V concludes this paper, and gives pointers to future directions.

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N. Gatsis and G. B. Giannakis are with the Department of Electrical and Computer Engineering at the University of Minnesota, Minneapolis, MN (emails: {gatsisn, georgios}@umn.edu).

II. Problem Formulation and Duality Properties

A. Continuous-Time Formulation

Consider a residence with a smart meter communicating with various devices at the residence. The smart meter also communicates with the utility company through the AMI. The utility company has announced the prices for the scheduling horizon (e.g., next day) ahead of time. Let $[0, T]$ denote the scheduling horizon, and $C(s(t), t)$ the cost charged to the end-user at time t, where $s(t)$ is the total residential power consumption. Note that the cost is time-varying, and hence explicitly incorporates the real-time pricing paradigm of the future grid.

In the DR formulation, it is imperative to capture diverse end-user requirements, as well as different classes of devices and tasks involved in the scheduling. Here, we consider a base residential load that is not to be shifted, and two classes of interruptible devices, denoted respectively as $\mathcal{K}_1$ and $\mathcal{K}_2$. The base load is denoted by $p_0(t)$, and can be lights or computers. Moreover, let $p_k(t)$ be the power consumption of device k, where $k \in \mathcal{K}_1$ or $k \in \mathcal{K}_2$ (by convention, $0 \notin \mathcal{K}_1$ and $0 \notin \mathcal{K}_2$). The particular characteristics of those classes are as follows:

1) Class $\mathcal{K}_1$ contains devices with a prescribed energy requirement $\check{E}_k$ that has to be completed over a duration $\check{T}_k$, starting from time $\check{\alpha}_k$. The power consumed is constrained to $\{0\} \cup [\check{p}_{k\min}, \check{p}_{k\max}]$ with $\check{p}_{k\min} > 0$ capturing the situation that a device cannot operate at arbitrarily low power. Examples are operating a pool pump or charging a PHEV. The energy constraint can be written as
$$\frac{1}{T}\int_{\check{\alpha}_k}^{\check{\alpha}_k+\check{T}_k} p_k(t)dt = \check{E}_k,\ k \in \mathcal{K}_1 \tag{1}$$
where due to the normalization factor $1/T$, the quantities $p_k(t)$ and $\check{E}_k$ have units of power.
2) Class $\mathcal{K}_2$ includes devices operating with power in $\{0\} \cup [\check{p}_{k\min}, \check{p}_{k\max}]$. There is no total energy requirement, but instead, a disutility function $V_k(p_k(t), t)$ that is introduced to capture dissatisfaction of the end-user for operating away from a nominal point, as in Fig. 1 for example. The premise is that the end-user may choose to operate away from a nominal point, if this can reduce the electricity bill, as will be seen shortly in the optimization formulation. An example in this class is an A/C unit. As with class $\mathcal{K}_1$, it is possible to define an interval $[\check{\alpha}_k, \check{\alpha}_k + \check{T}_k]$ over which the device is to be operated. In this case, we have that $V_k(p_k(t), t) = 0$ for $t < \check{\alpha}_k$ or $t \geq \check{\alpha}_k + \check{T}_k$.

The interruptible nature is apparent due to the constraint set $\{0\} \cup [\check{p}_{k\min}, \check{p}_{k\max}]$ for the instantaneous power consumption. The case of noninterruptible tasks can be accommodated by considering a constraint set $[\check{p}_{k\min}, \check{p}_{k\max}]$ with $\check{p}_{k\min} > 0$ over the desired hours of operation.

Let $\mathcal{K}$ denote the set of all adjustable devices ($\mathcal{K}_1 \cup \mathcal{K}_2$) and the base load. The *continuous-time* residential DR scheduling amounts to solving the following optimization problem:

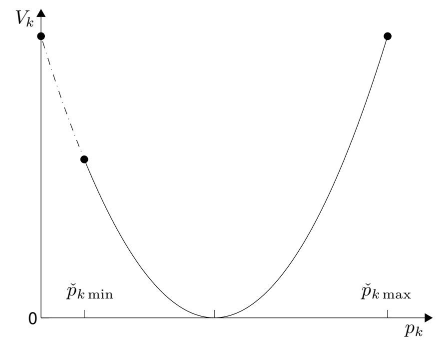

Fig. 1. Example of a convex disutility function. The point where the function achieves is minimum can be a desirable set-point by the end-user.

$$\begin{aligned}
\mathsf{P} = \min \quad & \frac{1}{T}\int_0^T C\left(\sum_{k\in\mathcal{K}} p_k(t), t\right) dt \\
& + \frac{1}{T}\int_0^T \sum_{k\in\mathcal{K}_2} V_k(p_k(t), t) dt \quad (2a)\\
\text{subj. to} \quad & \frac{1}{T}\int_0^T p_k(t)dt \geq \check{E}_k,\ k \in \mathcal{K}_1 \quad (2b)\\
& p_k(t) \in \{0\} \cup [\check{p}_{k\min}, \check{p}_{k\max}],\\
& \quad t \in [\check{\alpha}_k, \check{\alpha}_k + \check{T}_k],\ k \in \mathcal{K}_1 \cup \mathcal{K}_2 \quad (2c)\\
& p_k(t) = 0,\ t \notin [\check{\alpha}_k, \check{\alpha}_k + \check{T}_k],\ k \in \mathcal{K}_1 \cup \mathcal{K}_2 \quad (2d)\\
\text{variables} \quad & p_k(t) \quad (0 \leq t \leq T,\ k \in \mathcal{K}_1 \cup \mathcal{K}_2).
\end{aligned}$$

The optimization variables in (2) are the power consumptions of the adjustable devices ($k \in \mathcal{K}_1 \cup \mathcal{K}_2$), and are functions of time over the interval $[0, T]$, as indicated in the last line of formulation (2). As such, they are infinite-dimensional. Note that (1) has been relaxed to inequality in (2b)—this is valid in practice under condition C1, which will be given shortly. The factor $(1/T)$ multiplying all integrals in (2) is only for symmetry with the discrete-time case, as will be seen later. Regarding units, $p_k(t)$ and $\check{E}_k$ have units of power.

Problem (2) is nonconvex due to constraint (2c). The following assumptions are made:

C1. Functions $C\left(\sum_k p_k, t\right)$ and $V_k(p_k, t)$ with respect to their first argument are defined, continuous, and convex, over $\times_{k\in\mathcal{K}_1\cup\mathcal{K}_2}[0, p_{k\max}]$ for all $t \in [0, T]$. Function $C\left(\sum_k p_k, t\right)$ is also strictly increasing in its first argument. Moreover, functions $C\left(\sum_k p_k(t), t\right)$ and $V_k(p_k(t), t)$ are integrable whenever the functions $p_k(t)$ are (Borel) measurable. Finally, for constraint (2b) to be practically meaningful, the condition $\check{p}_{k\min}\check{T}_k < \check{E}_k T$ is supposed to hold.

C2. There exist $\{p_k(t)\}$, $k \in \mathcal{K}_1$ satisfying (2c) and (2d) so that (2b) holds as strict inequality.

Both conditions are mild and are expected to hold in DR setups. The fact that $C(s, t)$ is strictly increasing in s together with the condition $\check{p}_{k\min}\check{T}_k < \check{E}_k T$ imply that (2b) holds as *equality* at the optimal point. The inequality $\check{p}_{k\min}\check{T}_k <$

$\check{E}_k T$ essentially means that operating the device $k \in \mathcal{K}_1$ at its lowest power level over the specified interval is not enough in order to provide the required energy. Condition C2 corresponds to the standard Slater constraint qualification.

Taking a Lagrangian dual approach, let λ_k denote the Lagrange multiplier corresponding to (2b); $\boldsymbol{\lambda}$ be a vector collecting all Lagrange multipliers; and $\boldsymbol{p}(t)$ be a vector collecting all power consumptions $p_k(t)$, $k \in \mathcal{K}_1 \cup \mathcal{K}_2$. Let $\Pi(t)$ denote the $(|\mathcal{K}_1| + |\mathcal{K}_2|)$-dimensional region representing the instantaneous constraints (2c) and (2d). This region is a function of t because for each k, the constraint set may change depending on whether $t \in [\check{\alpha}_k, \check{\alpha}_k + \check{T}_k]$. Keeping those constraints implicit, the Lagrangian function corresponding to (2) takes the following form, after straightforward arrangements:

$$L(\boldsymbol{\lambda}, \boldsymbol{p}(t)) = \frac{1}{T}\int_0^T \left(C\left(\sum_{k\in\mathcal{K}} p_k(t), t\right) + \sum_{k\in\mathcal{K}_2} V_k(p_k(t), t) - \sum_{k\in\mathcal{K}_1} \lambda_k p_k(t) \right) dt + \sum_{k\in\mathcal{K}_1} \lambda_k \check{E}_k. \quad (3)$$

The dual function and the dual problem are, respectively,

$$q(\boldsymbol{\lambda}) = \min_{\boldsymbol{p}(t)\in\Pi(t)} L(\boldsymbol{\lambda}, \boldsymbol{p}(t)) \quad (4)$$

$$\mathsf{D} = \max_{\boldsymbol{\lambda}\geq \mathbf{0}} q(\boldsymbol{\lambda}). \quad (5)$$

Weak duality implies that $\mathsf{D} \leq \mathsf{P}$. Despite nonconvexity, the following result asserts that there is no duality gap.

Proposition 1. *Strong duality holds under conditions C1 and C2, i.e.,*

$$\mathsf{D} = \mathsf{P}. \quad (6)$$

Intuitively, continuous-time averaging smooths out the non-convexity, and thus eliminates the duality gap induced by (2c). The proof is tailored after a related result in wireless networking [12, Lemma 1], and is presented in [13]. Proposition 1 implies that solving the dual problem is optimal. Two questions arise naturally: (a) how the dual problem can be solved efficiently, and (b) how the solution of the dual problem can be leveraged in order to give a solution of the primal problem. These questions are addressed in Section III. To this end, it will be helpful to examine the duality properties of a discretized version of (2), and this is the subject of the ensuing subsection.

B. Discrete-Time Formulation

Consider a partition of $[0, T]$ into N intervals, $\{0, \frac{T}{N}, \frac{2T}{N}, \ldots, T\}$. Let $p_k^n := p_k[(n-1)T/N]$, $C^n(.) := C[., (n-1)T/N]$, and $V_k^n(.) := V_k[., (n-1)T/N]$, for $n = 1, \ldots, N$. It is supposed that the partition is fine enough so that the time instants $\check{\alpha}_k$ and $\check{\alpha}_k + \check{T}_k$ are members of the partition. The *discrete-time* version of (2) is formulated by replacing the integrals with left Riemann sums, and takes the following form

$$\mathsf{P}_N = \min \quad \frac{1}{N}\sum_{n=1}^{N} C^n\left(\sum_k p_k^n\right) + \frac{1}{N}\sum_{n=1}^{N}\sum_{k\in\mathcal{K}_2} V_k^n(p_k^n) \quad (7a)$$

$$\text{subj. to} \quad \frac{1}{N}\sum_{n=1}^{N} p_k^n \geq \check{E}_k,\ k \in \mathcal{K}_1 \quad (7b)$$

$$p_k^n \in \{0\} \cup [\check{p}_{k\,\min}, \check{p}_{k\,\max}],\ k \in \mathcal{K}_1 \cup \mathcal{K}_2,\quad n \in \frac{N}{T}\cdot\{\check{\alpha}_k, \ldots, \check{\alpha}_k + \check{T}_k - 1\} \quad (7c)$$

$$p_k^n = 0,\ k \in \mathcal{K}_1 \cup \mathcal{K}_2,\quad n \notin \frac{N}{T}\cdot\{\check{\alpha}_k, \ldots, \check{\alpha}_k + \check{T}_k - 1\} \quad (7d)$$

$$\text{variables} \quad p_k^n \quad (n \in \{1, \ldots, N\},\ k \in \mathcal{K}_1 \cup \mathcal{K}_2).$$

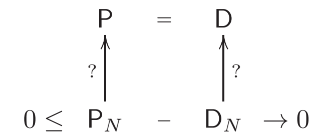

Fig. 2. Primal and dual values of continuous- and discrete-time problems

Problem (7) is nonconvex in general, due to constraint (7c). Nevertheless, it is a mixed integer program with special separable structure [14, Sec. 5.6.1]. The dual problem can be formulated analogously to the continuous-time case by introducing a Lagrange multiplier for constraint (7b), and keeping the rest of the constraints implicit. Let D_N denote the optimal value of this dual problem. The following duality gap estimate asserts that the duality gap vanishes as the partition size increases.

Proposition 2. *For the duality gap of* (7)*, under conditions C1 and C2, it holds that*

$$0 \leq \mathsf{P}_N - \mathsf{D}_N = O\left(\frac{1}{N}\right). \quad (8)$$

Proposition 2 asserts that the duality gap of (7) vanishes as $N \to \infty$. The proof amounts to showing that problem (7) satisfies certain conditions which guarantee vanishing duality gap [14, Sec. 5.6.1], [13]. It is worth noting that Proposition 2 does not imply that P_N converges to the optimal value P of (2) as $N \to \infty$; see also Fig. 2. This result would be useful in approximating the continuous-time problem with the discrete-time one. Nevertheless, the results of Propositions 1 and 2 illustrated in Fig. 2 point to this direction. Motivated by these duality gap results, the next section develops an algorithm to solve (5), which relies on the discretized problem (7).

III. Subgradient Method

The subgradient method will be employed to solve the dual problem (5); see e.g., [15, Sec. 8.2]. The motivation is that solving the dual problem is optimal due to Proposition 1. The method iterates between two steps, namely (a) Lagrangian minimization, in the same fashion as (4); and (ii) Lagrange multiplier update.

Let ℓ be the iteration index, and $\boldsymbol{\lambda}(0) \geq \mathbf{0}$ be the initial Lagrange multiplier vector. The Lagrangian minimization step takes the form

$$\boldsymbol{p}^\dagger(t;\ell) \in \arg\min_{\boldsymbol{p}(t)\in\Pi(t)} L(\boldsymbol{\lambda}(\ell),\boldsymbol{p}(t)) \tag{9}$$

The Lagrange multiplier $\boldsymbol{\lambda}(\ell)$ is constant for the minimization in (9). Note also that a function of time is sought at each iteration.

The following lemma asserts that the minimization decomposes into minimizations per time instant t.

Lemma 1. *The value $\boldsymbol{p}^\dagger(t;\ell)$ can be obtained as the solution of the following optimization problem at $t\in[0,T]$*

$$\min \quad C\left(\sum_{k\in\mathcal{K}} p_k, t\right) + \sum_{k\in\mathcal{K}_2} V_k(p_k,t) - \sum_{k\in\mathcal{K}_1} \lambda_k(\ell)p_k \tag{10a}$$

$$\text{subj. to} \quad p_k \in \{0\}\cup[\check{p}_{k\,\min},\check{p}_{k\,\max}], \quad t\in[\check{\alpha}_k,\check{\alpha}_k+\check{T}_k],\ k\in\mathcal{K}_1\cup\mathcal{K}_2 \tag{10b}$$

$$p_k = 0,\ t\notin[\check{\alpha}_k,\check{\alpha}_k+\check{T}_k],\ k\in\mathcal{K}_1\cup\mathcal{K}_2 \tag{10c}$$

$$\text{variables} \quad p_k \quad (k\in\mathcal{K}_1\cup\mathcal{K}_2).$$

The key idea to note is that the optimization variable in (10) is a *vector*, as opposed to a function in (9). Lemma 1 relies on minimizing the integrand in the Lagrangian function per t, which is reminiscent of the procedure to obtain the well-known waterfilling in wireless communications [16, Ch. 4]. The proof of Lemma 1 follows next.

Proof of Lemma 1. Let $t\in[0,T]$ be arbitrary, and let $\boldsymbol{p}^\dagger(t;\ell)$ be the solution of (10) given $\boldsymbol{\lambda}(\ell)$ for the particular t. Moreover, let $\boldsymbol{p}(t)\in\Pi(t)$ be arbitrary. It holds by definition that

$$C\left(\sum_{k\in\mathcal{K}} p_k^\dagger(t;\ell),t\right) + \sum_{k\in\mathcal{K}_2} V_k(p_k^\dagger(t;\ell),t) - \sum_{k\in\mathcal{K}_1}\lambda_k(\ell)p_k^\dagger(t;\ell) \leq C\left(\sum_{k\in\mathcal{K}} p_k(t),t\right) + \sum_{k\in\mathcal{K}_2} V_k(p_k(t),t) - \sum_{k\in\mathcal{K}_1}\lambda_k(\ell)p_k(t) \tag{11}$$

Integrating the latter over $[0,T]$ yields

$$L(\boldsymbol{\lambda}(\ell),\boldsymbol{p}^\dagger(t;\ell)) \leq L(\boldsymbol{\lambda}(\ell),\boldsymbol{p}(t;\ell)) \tag{12}$$

Therefore, $\boldsymbol{p}^\dagger(t;\ell)$ is the function minimizing the Lagrangian in (9). □

It is difficult to perform the minimization for all $t\in[0,T]$ in practice, and we will shortly attempt to bypass this issue by performing the minimization over a partition of $[0,T]$.

Regarding solvers of (10), note that if the function $C(.,t)$ is linear, then the minimization also decomposes into *per device* problems, which are straightforward to solve in closed form.

For more general costs, problem (10) becomes a convex optimization problem if the on/off status of devices ($p_k=0$ or $p_k\in[\check{p}_{k\,\min},\check{p}_{k\,\max}]$) per device is fixed. The number of on/off combinations is in general $2^{|\mathcal{K}_1|+|\mathcal{K}_2|}$, which may not be prohibitively large. The convex problem that has to be solved given the on/off status has special structure which can further be leveraged to devise an efficient solution. Specifically, it involves box constraints and typically a combination of linear, piecewise linear, or quadratic functions as objective.

TABLE I

ALGORITHM FOR LAGRANGIAN MINIMIZATION FOR 2 DEVICES GIVEN LAGRANGE MULTIPLIERS

if 1 is on and 2 is off **then**
  $p_1^\dagger = \left[\frac{\lambda-b}{2a} - p_0(t)\right]_{\check{p}_{1\,\min}}^{\check{p}_{1\,\max}}$
end if
if 2 is on and 1 is off **then**
  $p_2^\dagger = \left[\frac{2w\check{p}_{2\,\max}-(b+2ap_0(t))}{2a+2w}\right]_{\check{p}_{2\,\min}}^{\check{p}_{2\,\max}}$
end if
if 1 and 2 are on **then**

  //Case 1: Check if $\check{p}_{1\,\min} < p_1^\dagger < \check{p}_{1\,\max}$
  $p_2 = \left[\check{p}_{2\,\max} - \frac{\lambda}{2w}\right]_{\check{p}_{2\,\min}}^{\check{p}_{2\,\max}}$
  $p_1 = \frac{\lambda-b}{2a} - (p_0(t)+p_2)$
  if $\check{p}_{1\,\min} < p_1 < \check{p}_{1\,\max}$ **then**
    $p_1^\dagger = p_1$; $p_2^\dagger = p_2$
  end if

  //Case 2: Check if $p_1^\dagger = \check{p}_{1\,\max}$
  $p_2 = \left[\frac{2w\check{p}_{2\,\max}-[b+2a(p_0(t)+\check{p}_{1\,\max})]}{2a+2w}\right]_{\check{p}_{2\,\min}}^{\check{p}_{2\,\max}}$
  if $2a(p_0(t)+\check{p}_{1\,\max}+p_2)+b-\lambda \leq 0$ **then**
    $p_1^\dagger = \check{p}_{1\,\max}$; $p_2^\dagger = p_2$
  end if

  //Case 3: Check if $p_1^\dagger = \check{p}_{1\,\min}$
  $p_2 = \left[\frac{2w\check{p}_{2\,\max}-[b+2a(p_0(t)+\check{p}_{1\,\min})]}{2a+2w}\right]_{\check{p}_{2\,\min}}^{\check{p}_{2\,\max}}$
  if $2a(p_0(t)+\check{p}_{1\,\min}+p_2)+b-\lambda \leq 0$ **then**
    $p_1^\dagger = \check{p}_{1\,\min}$; $p_2^\dagger = p_2$
  end if
end if

It is also possible to obtain a (nearly) closed-form solution if the cost is quadratic, $C(s,t)=as^2+bs$ for all t, the disutility function is quadratic, $V_k(p_k,t)=w(\check{p}_{k\,\max}-p_k)^2$, and there are two interruptible devices, one per class with consumptions $p_1(t)$ and $p_2(t)$, respectively. Table I lists the related expressions, which are not difficult to derive from the optimality conditions of convex problems with box constraints [17, Example 2.1].

Having determined the primal minimizers $\boldsymbol{p}^\dagger(t;\ell)$, the Lagrange multipliers are updated as follows (β_ℓ is the iteration-dependent stepsize):

$$\lambda_k(\ell+1) = \max\left\{\lambda_k(\ell) - \beta_\ell\left(\frac{1}{T}\int_0^T p_k^\dagger(t;\ell)dt - \check{E}_k\right)\right\}. \tag{13}$$

In order to evaluate the integral in (13), the functions $p_k^\dagger(t;\ell)$ are needed. The integral can be approximated from values of $p_k^\dagger(t;\ell)$ at the partition $\{0,\frac{T}{N},\frac{2T}{N},\ldots,T\}$ as follows:

$$\int_0^T p_k^\dagger(t;\ell)dt \approx \frac{1}{N}\sum_{n=1}^N p_k^\dagger[(n-1)T/N;\ell]. \tag{14}$$

The approach of partitioning $[0,T]$ in order to facilitate (9) and (13) can be interpreted as application of the subgradient method to the dual of the discrete-time problem. If a non-summable but square-summable stepsize is used, the method will converge to the optimal Lagrange multipliers for (7) [15, Sec. 8.2]. The final step is to recover a primal solution from the dual solution.

This is an issue that requires further research. A possible approach is to substitute the optimal Lagrange multipliers into (10) and minimize the Lagrangian function at a chosen partition. The power consumption schedule obtained this way will be optimal if it satisfies any of the following two conditions: (a) it is primal feasible, and satisfies the complementary slackness condition [17, Prop. 5.1.4]; or (b) it is the unique minimizer of the Lagrangian [18, p. 248]. This schedule can be considered to be piecewise constant over the continuous time-interval $[0, T]$. The challenge facing this method is that it is not straightforward to show satisfaction of either condition in the general case. Numerical tests illustrating the merits of this method are presented in the ensuing section.

Remark. There are types of residential loads, most notably loads with intertemporal constraints, that cannot be classified under the interruptible loads analyzed in this paper. Naturally, results like Proposition 1 or Lemma 1 may not hold in such cases. See also [19] for a recent development based on Lagrangian duality which focuses on deferrable loads, i.e., loads whose starting time can be variable, but they are not interruptible.

IV. Numerical Tests

Scheduling of 2 devices (one from each class) over $T = 5$ hours is considered here. The power consumptions are denoted as $p_1(t)$ and $p_2(t)$. There is also a base load, $p_0(t) = 0.2\,\text{kW}$ for $0 \leq t < 2$, and $p_0(t) = 0$ otherwise. The device parameters are $\check{p}_{1\min} = \check{p}_{2\min} = 0.1\,\text{kW}$, $\check{p}_{1\max} = \check{p}_{2\max} = 1\,\text{kW}$, and $\check{E}_1 = 0.3\,\text{kW}$. The cost is $C(s,t) = 0.01s^2 + 0.8s$ for all t (where s is in kW). Device 2 is constrained to be off during the first and the last hour, i.e., $\check{\alpha}_2 = 1$ and $\check{\alpha}_2 + \check{T}_2 = 4$. The disutility function is $V_2(p_2, t) = (\check{p}_{2\max} - p_2)^2$ for $1 \leq t < 4$, and 0 otherwise. The stepsize is selected to be $\beta_\ell = 10/(\ell + 10)$, $\ell \geq 1$, and the partition size $N = 10$. The algorithm of Table I was used for minimizing the Lagrangian. There is a single Lagrange multiplier in this problem. Fig. 3 depicts the convergence of the subgradient method.

The final schedule is obtained by minimizing the Lagrangian function given the optimal Lagrange multipliers, and is depicted in Fig. 4. It can be observed that for both devices the instantaneous power consumption constraints are satisfied. Device 1 is off between hours 1 and 2, because the total power consumption from the base load and device 2 is already high. It is interesting to note that the area under the grey curve normalized by T has the value of $\check{E}_1 = 0.3$. Therefore, the solution is feasible, and also satisfies (7b) and (2b). The power consumption schedule in this case is optimal for (7), since it satisfies the necessary and sufficient optimality conditions [17, Prop. 5.1.4].

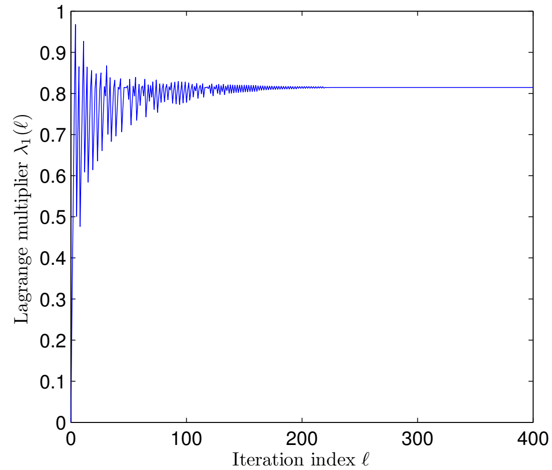

Fig. 3. Convergence of Lagrange multiplier

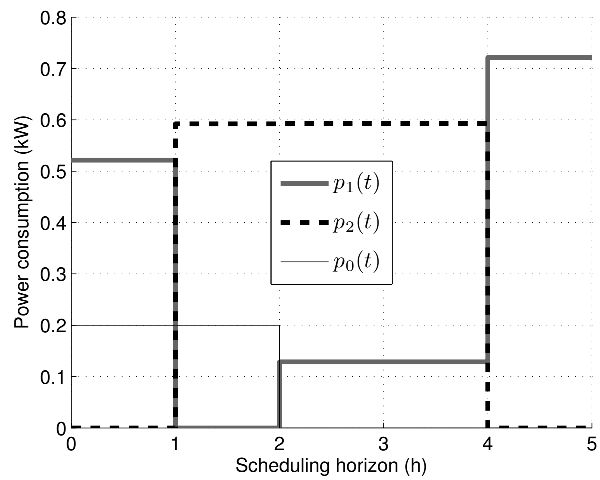

Fig. 4. Schedule of 3 devices.

Increasing the partition size is also investigated. The results are shown in Fig. 5. For all partition sizes, the resulting power consumption schedule is feasible, and in fact optimal (as in the previous discussion related to Fig. 4). The dual value is obtained in a straightforward manner from the subgradient method. It is observed that the two values are identical even for finite N. If these are the same as P, then the behavior depicted in Fig. 5 indicates that there is a piecewise constant solution to (2) in the present case. The behavior in Fig. 5 is consistent with Proposition 2.

V. Conclusions and Future Directions

This paper deals with optimal residential energy scheduling. The end-user has a "must-run" load, and two types of adjustable loads. The first type must consume a specified total amount of energy over the scheduling horizon, but the consumption can be adjusted across the horizon. The second type of load has adjustable power consumption without a total energy requirement, but operation of the load at

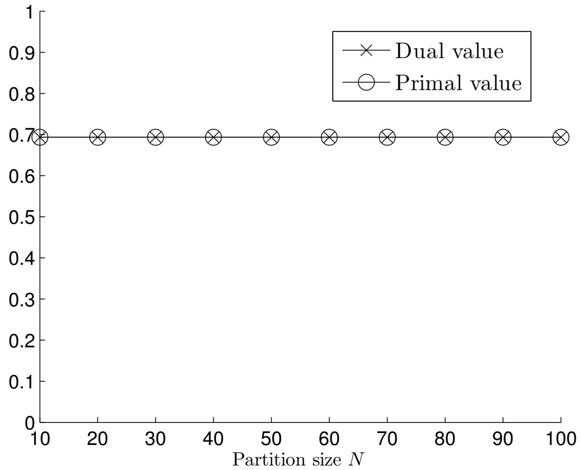

Fig. 5. Duality gap between primal value P_N and dual value D_N.

reduced power results in dissatisfaction of the end-user. Each adjustable load is interruptible in the sense that the load can be either operated (resulting in nonzero power consumption), or not (resulting in zero power consumption). The on/off feature of the problem results in nonconvexity. Two issues regarding duality of this problem are positively resolved. First, the problem is shown to have zero duality gap, if it is formulated over a continuous time interval. Second, a regular discretized version of the problem is shown to have vanishing duality gap as the partition size of the continuous time-interval grows. Numerical experiments illustrate the usefulness of the subgradient method as a solver leveraging these results.

Two immediate research goals are identified. The first is to further investigate the relationship between formulations (2) and (7). Specifically, for the purpose of approximating the continuous-time problem with its discretized version, it is useful to establish that P_N approaches P. Propositions 1 and 2 illustrated in Fig. 2 and the numerical results are positive indicators of this fact. Related results developed for difficult nonconvex programs or other continuous-time formulations may be useful to this end; see e.g., [20], [21], and references therein. As a refinement of this issue, it is useful to examine when the solutions of (2) can be piecewise constant, in which case (7) can yield a schedule which is optimal for (2) as well.

The duality gap results motivate the use of dual methods for efficient solutions to energy scheduling. Therefore, the second research issue amounts to recovering primal solutions (energy schedules) from dual solutions. This is a general issue with Lagrangian duality in integer programming, but the important feature here is to leverage the vanishing duality gap. Tricks for related separable mixed integer programs may be useful [14, Sec. 5.6].

The aforementioned research goals rely on the duality gap results established here, and will be undertaken in future submissions. The underpinning idea is to reveal "hidden convexity" structures, which can be instrumental in developing efficient solutions to practical home automation systems.

REFERENCES

[1] (2010, July 14) Smart grid basics. Federal Smart Grid Task Force. [Online]. Available: http://www.smartgrid.gov/basics

[2] Demand response. U.S. Department of Energy, Office of Electricity Delivery and Energy Reliability. [Online]. Available: http://www.oe.energy.gov/demand.htm

[3] A. M. Giacomoni, S. M. Amin, and B. F. Wollenberg, "Reconfigurable interdependent infrastructure systems: Advances in distributed sensing, modeling, and control," in *Proc. American Control Conf.*, San Francisco, CA, June–July 2011.

[4] F. C. Schweppe, B. Daryanian, and R. D. Tabors, "Algorithms for a spot price responding algorithm," *IEEE Trans. Power Systems*, vol. 4, no. 2, pp. 507–516, May 1989.

[5] A.-H. Mohsenian-Rad and A. Leon-Garcia, "Optimal residential load control with price prediction in real-time electricity pricing environments," *IEEE Trans. Smart Grid*, vol. 1, no. 2, pp. 120–133, Sept. 2010.

[6] A. J. Conejo, J. M. Morales, and L. Baringo, "Real-time demand response model," *IEEE Trans. Smart Grid*, vol. 1, no. 3, pp. 236–242, Dec. 2010.

[7] I. Koutsopoulos and L. Tassiulas, "Control and optimization meet the smart power grid: Scheduling of power demands for optimal energy management," 2010. [Online]. Available: http://arxiv.org/abs/1008.3614

[8] S. Hatami and M. Pedram, "Minimizing the electricity bill of cooperative users under a quasi-dynamic pricing model," in *Proc. 1st IEEE Int. Conf. Smart Grid Communications*, Gaithersburg, MD, Oct. 2010, pp. 421–426.

[9] L. D. Ha, S. Ploix, E. Zamai, and M. Jacomino, "Tabu search for the optimization of household energy consumption," in *Proc. IEEE Int. Conf. Information Reuse and Integration*, Waikoloa Village, HI, Sept. 2006, pp. 86–92.

[10] M. A. A. Pedrasa, T. D. Spooner, and I. F. MacGill, "Coordinated scheduling of residential distributed energy resources to optimize smart home energy services," *IEEE Trans. Smart Grid*, vol. 1, no. 2, pp. 134–143, Sept. 2010.

[11] D. O'Neill, M. Levorato, A. Goldsmith, and U. Mitra, "Residential demand response using reinforcement learning," in *Proc. 1st IEEE Int. Conf. Smart Grid Communications*, Gaithersburg, MD, Oct. 2010, pp. 409–414.

[12] K. Rajawat, N. Gatsis, and G. B. Giannakis, "Cross-layer designs in wireless fading networks with multicast," *IEEE/ACM Trans. Netw.*, 2011, to appear. [Online]. Available: http://arxiv.org/abs/1003.5239

[13] N. Gatsis and G. B. Giannakis, "A Lagrangian duality approach to demand-side management," in preparation.

[14] D. P. Bertsekas, *Constrained Optimization and Lagrange Multiplier Methods*. Belmont, MA: Athena Scientific, 1996.

[15] D. P. Bertsekas, A. Nedić, and A. Ozdaglar, *Convex Analysis and Optimization*. Belmont, MA: Athena Scientific, 2003.

[16] A. Goldsmith, *Wireless Communications*. New York, NY: Cambridge University Press, 2005.

[17] D. P. Bertsekas, *Nonlinear Programming*. Belmont, MA: Athena Scientific, 1999.

[18] S. Boyd and L. Vandenberghe, *Convex Optimization*. Cambridge, UK: Cambridge University Press, 2004.

[19] S.-J. Kim and G. B. Giannakis, "Efficient and scalable demand response for the smart power grid," in *Proc. 4th Int. Workshop Computational Advances in Multi-Sensor Adaptive Processing*, San Juan, Puerto Rico, Dec. 2011.

[20] Z.-Q. Luo and S. Zhang, "Duality gap estimation and polynomial time approximation for optimal spectrum management," *IEEE Trans. Signal Process.*, vol. 57, no. 7, pp. 2675–2689, July 2009.

[21] X. Wang, S. Zhang, and D. Y. Yao, "Separated continuous conic programming: Strong duality and an approximation algorithm," *SIAM J. Control Optim.*, vol. 48, no. 4, pp. 2118–2138, 2009.