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Fractional Sliding Mode Observer Design for a Class of Uncertain Fractional Order Nonlinear Systems

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Abstract—**This paper investigates the problem of state estimation for a class of fractional order nonlinear systems with uncertainty, using sliding mode technique. In other words, the purpose of the problem is to develop a fractional order sliding mode observer. Through the fractional order extension of the Lyapunov stability criterion, the stability analysis of the error system is completed and it is showed that the observer design guarantees the convergence of the estimation error. Two illustrative examples are provided to approve the theoretical results.**

I. Introduction

THE state estimation of nonlinear systems based on the output measurements is a very important issue in many engineering applications, since the information about the system states are necessary in the design of any controller but, in real systems, all the states are seldom available. Therefore, the problem of state observers that predicts the present system state, in the case of integer order systems, is an issue that have been addressed in many papers and many important results are available in the recent literature [1-3]. High gain observers [4], sliding mode observers [5] and Kalman-like observers are some observers that have been proposed for integer order nonlinear systems. Among all the proposed observers in the literature, sliding mode techniques have good robustness against system uncertainties [6].

On the other hand, fractional calculus has attracted many interests in recent years and numerous physical real world phenomena have been modeled effectively with fractional order dynamics [7, 8]. Besides, there are a growing number of fractional calculus applications in different areas [9, 10]. One of these areas is control theory [11, 12]. In recent years, many basic issues of control theory such as stability [13, 14], controllability [15], observability [16] are extended in order to deal with more general systems containing non-integer derivatives. Very recently, control of FO systems has become a very important and interesting topic for the system control community. In literature, different controllers have been proposed and extensions of many different control approaches have been used to accommodate fractional order systems [17-20].

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However, the lack of the extension of the existing observers for fractional order systems is sensible. In literature, there are very limited reports on the estimation and compensation of disturbance [21, 22]. Besides, to the authors' best knowledge, there are few works dealing with the problem of designing a fractional order observer for fractional order systems [23, 24]. In [23], the problem of creating the state observer for fractional order linear systems is investigated and a design scheme for initialized fractional order state estimator is introduced. For fractional order nonlinear systems, a simple fractional order observer design is proposed in [24]. But, the model uncertainties have not been taken into consideration in the proposed approaches. So, it is still of considerable importance to seek direct systematic approaches for designing observers for fractional order systems.

In this paper, a novel robust fractional order sliding mode observer is presented to solve the problem of state reconstruction for fractional order nonlinear systems with uncertain nonlinearities. It is shown that the proposed observer guarantees that the state estimation errors are convergent to zero. The fractional order Lyapunov approach is exploited to analyze the stability of the estimation error system. It ought to be mentioned that the proposed observer is very simple and constructive for practical applications. Moreover, utilizing fractional calculus, a new stability analysis method is given for the error dynamics when the fractional order observer is applied. As both the system model and the observer have fractional order dynamics, using the integer order Lyapunov stability theory leads to a more complex observer design and more restrictive assumptions, which may not be applicable to real world applications.

The rest of the paper is organized as follows: In Section 2, some basic concepts of fractional calculus is described. In Section 3, a class of uncertain fractional-order nonlinear systems is introduced and its properties are discussed. In Section 4, a novel fractional order observer is presented and stability analysis of the fractional-order error system when the proposed observer is applied is given. In Section 5, the observer scheme has been tested via numerical simulations and the corresponding results are presented to confirm the usefulness and effectiveness of the proposed observer for state estimation of fractional-order nonlinear systems with model uncertainties and external disturbances. Finally, some concluding remarks are drawn in Section 6.

II. BASIC DEFINITION AND PRELIMINARIES OF FRACTIONAL ORDER CALCULUS

Fractional order integration and differentiation are the generalization of the integer-order ones. Efforts to extend the specific definitions of the traditional integer order to the more general arbitrary order context led to different definitions for fractional derivatives [25]. Two of the most commonly used definitions are Riemann-Liouville, and Caputo definitions.

Definition 1. [11] The αth-order Riemann-Liouville fractional derivative of function $f(t)$ with respect to t and the terminal value t_0 is given by

$$D^{\alpha} f(t) = \frac{d^{\alpha} f(t)}{dt^{\alpha}} = \frac{1}{\Gamma(m-\alpha)} \frac{d^m}{dt^m} \int_{t_0}^{t} \frac{f(\tau)}{(t-\tau)^{\alpha-m+1}} d\tau, \quad (1)$$

and, the Riemann-Liouville definition of the αth-order fractional integration is given by

$${}_{t_0}I_t^{\alpha} f(t) = I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_{t_0}^{t} \frac{f(\tau) d\tau}{(t-\tau)^{1-\alpha}} \quad (2)$$

where m is the first integer larger than α, i.e. $m-1 \le \alpha < m$ and Γ is the Gamma function.

Definition 2. [11] The Caputo fractional derivative of order α of a continuous function $f : R^{+} \to R$ is defined as follows

$$D^{\alpha} f(t) = \begin{cases} \dfrac{1}{\Gamma(m-\alpha)} \displaystyle\int_{0}^{t} \frac{f^{(m)}(\tau)}{(t-\tau)^{\alpha-m+1}} d\tau, & m-1 < \alpha < m \\ \dfrac{d^m}{dt^m} f(t), & \alpha = m, \end{cases} \quad (3)$$

where m is the first integer larger than α.

Definition 3. [26] The solution of

$$D^{q} x(t) = f(x,t) \quad (4)$$

is said to be Mittag-Leffler stable if

$$\|x(t)\| \le \left\{ m\left[x(t_0)\right] E_q(-\lambda (t-t_0)^q) \right\}^{b} \quad (5)$$

where t_0 is the initial time, $q \in (0,1)$, $\lambda > 0$, $b > 0$, $m(0) = 0$, $m(x) \ge 0$, and $m(x)$ is locally Lipschitz on with Lipschitz constant m_0.

In the proof of main results, we need the following theorem.

Theorem 1. [26] Let x=0 be an equilibrium point for the non-autonomous fractional-order system (4), where $f(t,x)$ satisfies the Lipschitz condition with Lipschitz constant l>0.

Assume that there exist a Lyapunov candidate $V(t,x)$ satisfying

$$\begin{aligned} &\alpha_1 \|x\|^{a} \le V(t,x) \le \alpha_2 \|x\|^{ab} \\ &D^{\beta} V(t,x) \le -\alpha_3 \|x\|^{ab} \end{aligned} \quad (6)$$

where α_1, α_2, α_3, a, b are positive constants and $\beta \in (0,1)$. Then the equilibrium point of system (4) is Mittag-Leffler stable.

Remark 1. [27] Mittag-Leffler Stability implies asymptotic stability.

III. SYSTEM DESCRIPTION

Consider a fractional order nonlinear system

$$\begin{aligned} &D^{\alpha} x = Ax + H(x,u) + Bu + \Psi(y,u,t) \\ &y = Cx \end{aligned} \quad (7)$$

where $\alpha \in (0,1)$ and x, u, y represents the state variables, input and output, respectively. A is a constant matrix and B is a constant input weighting vector.

Assumption 1. The known nonlinear term $H(x,u)$ is a Lipschitz function with respect to x, i.e.

$$\|H(x,u) - H(\hat{x},u)\| \le k \|x - \hat{x}\| \quad (8)$$

where k is a positive constant.

Assumption 2. The unknown nonlinear term $\Psi(y,u,t)$ represents all modeling uncertainties and disturbances experienced by the system which satisfies

$$\|\Psi(y,u,t)\| \le \rho(y,u,t) \quad (9)$$

where the function $\rho(y,u,t)$ is known.

Assumption 3. The matrix pair (A,C) is observable.

It follows from Assumption 3 that there exist a matrix L such that A-LC is stable, and thus for any Q>0, the Lyapunov equation

$$(A-LC)^{T} P + P(A-LC) = -Q \quad (10)$$

has a unique solution P>0.

Remark 2. [15] Consider a fractional order system given by the following linear state space form with finite dimension n:

$$\begin{cases} D^{\alpha}x = Ax + Bu \\ y = Cx \end{cases}, \qquad x(0) = x_0 \tag{11}$$

System (11) is observable on $[t_0, t_1]$ if and only if

$$O \triangleq \begin{bmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{bmatrix} \tag{12}$$

has rank n.

IV. Sliding Mode Observer Design

In this section, a fractional order sliding mode observer is proposed for the uncertain fractional order nonlinear system described in the previous part and the stability of the estimation error dynamics is discussed utilizing the tool of fractional calculus.

Theorem 2. Consider the uncertain fractional-order nonlinear system (7) satisfying Assumption 1-3 and

$$\Upsilon \equiv \sum_{k=1}^{\infty} \frac{\Gamma(1+\alpha)}{\Gamma(1+k)\Gamma(1-k+\alpha)} (D^{k}e)(D^{\alpha-k}e), \qquad \longrightarrow \ \|\Upsilon\| \le \gamma\|e\| \tag{13}$$

where γ is a positive constant [28]. Then, the sliding mode observer with the following design

$$\begin{aligned} &D^{\alpha}\hat{x} = A\hat{x} + H(\hat{x},u) + Bu + L(y - C\hat{x}) + \upsilon \\ &y = C\hat{x} \end{aligned} \tag{14}$$

where the function υ is defined by

$$\begin{aligned} &\upsilon = \Lambda \frac{x-\hat{x}}{\|x-\hat{x}\|} \\ &\Lambda = \gamma + \rho(y,u,t) \end{aligned} \tag{15}$$

and with the observer design parameter L such that

$$\begin{aligned} &Q = (A - LC) + (A + LC)^{T} \\ &\lambda_{\min}(Q) > 2k \end{aligned} \tag{16}$$

guarantees that the error equation

$$e = x - \hat{x} \tag{17}$$

is asymptotically stable, regardless of whichever admissible uncertainty affects the system's model.

Proof. Using Eq. (7) and (14), we have

$$\begin{aligned} D^{\alpha}e &= D^{\alpha}x - D^{\alpha}\hat{x} \\ &= A(x-\hat{x}) + H(x,u) - H(\hat{x},u) \\ &\quad + \Psi(y,u,t) - LC(x-\hat{x}) - \upsilon \\ &= (A-LC)e + H(x,u) - H(\hat{x},u) \\ &\quad + \Psi(y,u,t) - \upsilon \end{aligned} \tag{18}$$

Now, consider a Lyapunov candidate function $V = 2e^{T}e$. By using (18) and the Leibniz's rule for fractional differentiating [29], the fractional derivative of V is given by

$$\begin{aligned} D^{\alpha}V &= (D^{\alpha}e)^{T}e + e^{T}(D^{\alpha}e) \\ &\quad + 2\left(\sum_{k=1}^{\infty} \frac{\Gamma(1+\alpha)}{\Gamma(1+k)\Gamma(1-k+\alpha)} (D^{k}e)(D^{\alpha-k}e)\right) \\ &= (D^{\alpha}e)^{T}e + e^{T}(D^{\alpha}e) + 2\Upsilon \\ &= e^{T}\left((A-LC)^{T} + (A-LC)\right)e \\ &\quad + 2e^{T}(H(x,u) - H(\hat{x},u) + \Psi(y,u,t) - \upsilon) + 2\Upsilon \end{aligned} \tag{19}$$

From Asuumption 3, it follows that

$$\begin{aligned} D^{\alpha}V &\le -e^{T}Qe + 2\Upsilon \\ &\quad + 2e^{T}(H(x,u) - H(\hat{x},u) + \Psi(y,u,t) - \upsilon) \end{aligned} \tag{20}$$

From the know lemma $\lambda_{\min}\|e\|^{2} \le e^{T}Qe \le \lambda_{\max}\|e\|^{2}$, Eq. (20) can be rewritten as

$$\begin{aligned} D^{\alpha}V &\le -\lambda_{\min}\|e\|^{2} - 2e^{T}\upsilon + 2\|\Upsilon\| \\ &\quad + 2\|e\|(\|H(x,u) - H(\hat{x},u)\| + \|\Psi(y,u,t)\|) \end{aligned} \tag{21}$$

Using Eq. (8), (9) and (13), it yields

$$\begin{aligned} D^{\alpha}V &\le -\lambda_{\min}\|e\|^{2} - 2e^{T}\upsilon + 2\gamma\|e\| \\ &\quad + 2\|e\|(k\|e\| + \rho(y,u,t)) \\ &= -(\lambda_{\min} - 2k)\|e\|^{2} + 2(\gamma + \rho(y,u,t))\|e\| - 2e^{T}\upsilon \end{aligned} \tag{22}$$

Substituting (15) in (22), it can be easily concluded

$$\begin{aligned} D^{\alpha}V &\le -(\lambda_{\min} - 2k)\|e\|^{2} \\ &\quad + 2(\gamma + \rho(y,u,t))\|e\| - 2(\gamma + \rho(y,u,t))\frac{e^{T}e}{\|e\|} \end{aligned} \tag{23}$$

Consequently, we have

$$D^{\alpha}V \leq -(\lambda_{\min} - 2k)\|e\|^2 \tag{24}$$

which denotes that the error dynamics asymptotically converges to zero according to Theorem 1, if the observer parameter L is chosen appropriately. Therefore, it can be concluded that the estimated trajectories attain to the original system trajectories. This implies that using the sliding mode observer (14), one can estimate the original system internal variables with a good accuracy. The proof is complete. □

V. Simulation Results

In this section, we will give two illustrative examples to show the applicability and efficiency of the proposed observer. The systems are selected such that they belong to the class of chaotic systems. High sensitivity of the chaotic systems to the initial conditions makes a challenge in handling them and estimating the state trajectories.

A. Fractional Order Financial System

The first model introduced in this part describes a fractional-order financial system [30] of three nonlinear differential equations that exhibits chaotic flow. The system has three state variables x, y and z which stands for the interest rate, the investment demand, and the price index, respectively. The fractional-order model of the system is described by

$$D^{\alpha}x = \begin{bmatrix} -a & 0 & 1 \\ 0 & -b & 0 \\ -1 & 0 & -c \end{bmatrix} x + \begin{bmatrix} xy \\ 1 - x^2 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u + 2\cos(t)\left(y(t) + 1\right) \tag{25}$$

$$y = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} x$$

in which a is the saving amount, b is the cost per investment, and c is the elasticity of demand of commercial markets and all three system parameters, a, b, and c, are positive real constants. In this paper, it is assumed that α=0.9, a=1, b=0.1, and c=1.

Besides, the parameters' values used to simulate the observer (14) are as follows

$$\Psi(y,u,t) = 2\left(\cos(t)y(t) + \cos(t)\right) \xrightarrow{\|\Psi(y,u,t)\| \leq \rho(t,y)} \rho(t,y) = 2\left(|y| + 1\right) \tag{26}$$

Using the proposed observer (14), we obtained the simulation results given in figures 1 and 2.

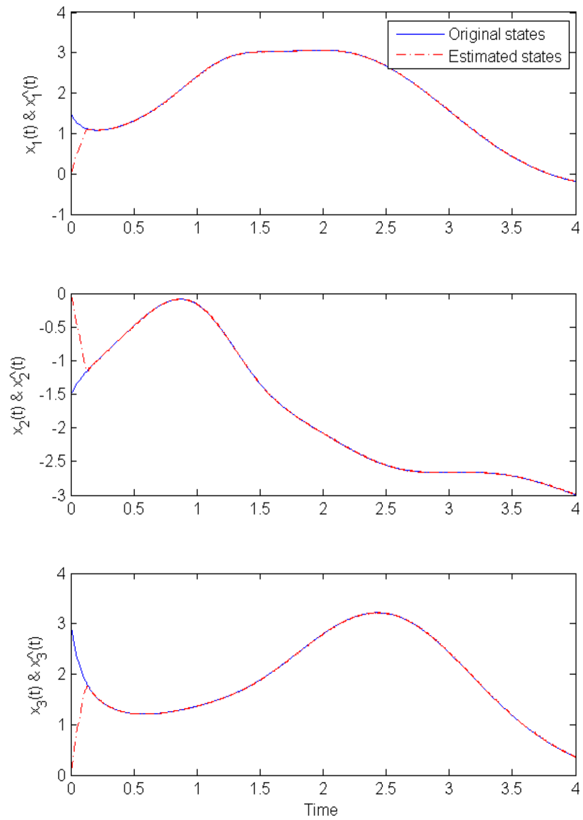

Fig. 1. Estimate of fractional order financial system's state evolution with fractional order sliding mode observer (14).

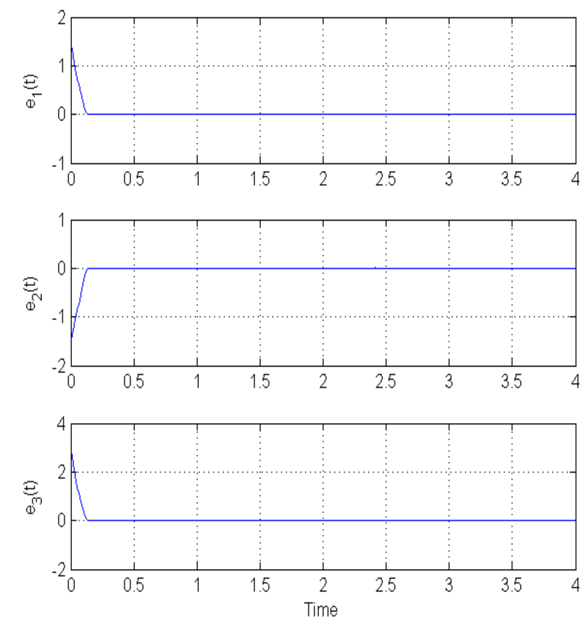

Fig. 2. Estimate error evolution with fractional order sliding mode observer (14) when applied to the fractional order financial system.

Figure 1 shows the time evolution of three system internal variables and their estimation with the fractional order sliding mode observer (14). Figure 2 illustrates the time history of the estimation error. It can be seen from the simulations that the observer (14) make the state estimations approach the actual states precisely. In other words, even though the nonlinear uncertainty term is included in the system's model, the error converges exactly to zero.

B. *Fractional Order Lu System*

As the second example for the effectiveness of state estimation via fractional order observer (14), the fractional order Lu system [31] with the following representation is considered:

$$ {}_0D_t^{\alpha}x = \begin{bmatrix} -\rho & \rho & 0 \\ 0 & \nu & 0 \\ 0 & 0 & -\mu \end{bmatrix} x + \begin{bmatrix} 0 \\ -x_1x_3 \\ x_1x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} u + \sin(t)\left(y(t)+1\right) \tag{27} $$

$$ y = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} x $$

where $(\rho,\mu,\upsilon) = (35,3,28)$ .We also assume that the order of derivatives in Eq. (27) is α=0.9. The so called Lu system is known as a bridge between Lorenz system and Chen system.

From the above data, the following information can be easily inferred

$$ \Psi(y,u,t) = \left(\sin(t)y(t) + \sin(t)\right) \xrightarrow{\|\Psi(y,u,t)\| \le \rho(t,y)} \rho(t,y) = 1 + |y| \tag{28} $$

Choosing suitable matrix *L,* the fractional order sliding mode observer (14) can be designed and the state estimation can be obtained.

The plots in figure 3 compare the evolution of the three components of the fractional order Lu system state with their respective estimates provided by the observer (14) with the above settings. Figure 4 reports the evolution of the estimation error. The plot clearly shows that the state estimation error converges to zero with a fast rate and an acceptable performance is achieved using the proposed fractional order sliding mode observer. The simulation results also show that the observer scheme is robust with respect to modeling uncertainties and environmental disturbances experienced by the system, i.e. the proposed observer clearly cancels the uncertainty and disturbance effect in the state estimation.

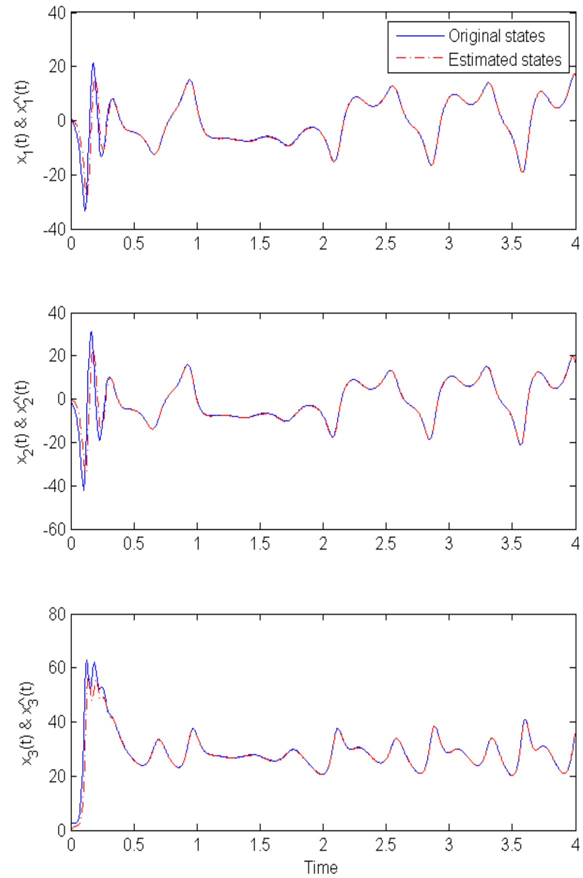

Fig. 3. Estimate of fractional order Lu system's state evolution with fractional order sliding mode observer (14).

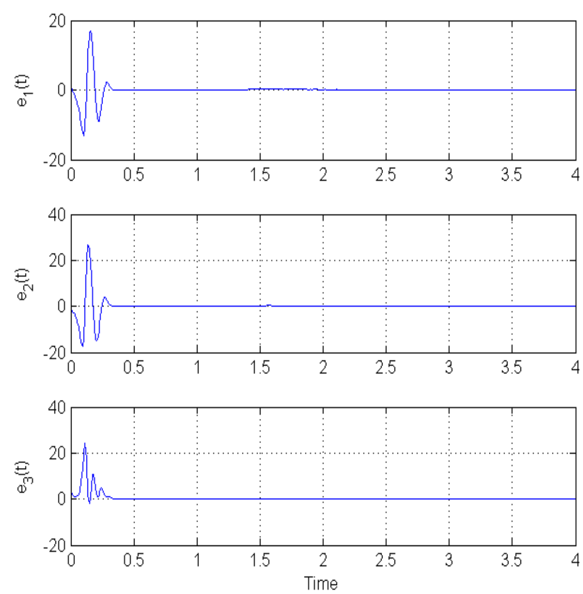

Fig. 4. Estimate error evolution with fractional order sliding mode observer (14) when applied to the fractional order Lu system.

VI. Conclusion

In this paper, a novel fractional order sliding mode observer is developed to estimate the fractional order system state variables. The proposed observer can be applied to a class of uncertain fractional order nonlinear systems. Analysis of the resulting error system is given by the fractional Lyapunov stability theory. It should be noticed that the new stability analysis is given using fractional calculus, which results in a simple fractional order sliding mode observer. It is shown that the observation errors obtained from the observer are asymptotically convergent to zero. The effectiveness of our proposed estimation technique is demonstrated by the numerical simulations.

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