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Stabilization Over Markov Feedback Channels

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***Abstract*—The problem of mean square stabilization of a discrete-time linear dynamical system over a Markov time-varying digital feedback channel is studied. In the scalar case, it is shown that the system can be stabilized if and only if a Markov jump linear system describing the evolution of the estimation error at the decoder is stable —videlicet if and only if the product of the unstable mode of the system and the spectral radius of a matrix that depends only on the Markov feedback rate is less than one. This result generalizes several previous *data rate theorems* that appeared in the literature, quantifying the amount of instability that can be tolerated when the estimated state is received by the controller over a noise free digital channel. In the vector case, a necessary condition for stabilizability is derived and a corresponding scheme is presented, which is tight in some special cases and which improves upon previous results on stability over Markov erasure channels.**

I. Introduction

We consider the problem of stabilization of a dynamical system where the estimated signal is transmitted for control over a time-varying communication channel, as depicted in Figure 1. This arises, for example, in pursuit evasion games where the state of the evader is estimated by distributed sensors and is communicated over a wireless fading channel to automatically control the pursuer [23].

The mathematical abstraction is that of a linear, discrete-time, dynamical system whose state is observed, quantized, encoded, and sent to a decoder over a noiseless digital link that supports the transmission of R_k bits at any given time step k, where R_k evolves according to a Markov chain representing the current state of the channel. Based on the decoded message, the control signal is computed and applied to the system. Both the encoder and the decoder are assumed to have causal knowledge of the channel state information, a legitimate assumption for slow wireless fading channels in which the channel conditions can be learnt with a short training sequence.

Following this model, Tatikonda and Mitter [26] examined the special case where the rate process is constant in time and the system has bounded disturbances. Nair and Evans [18] studied the case where the disturbances have unbounded support, maintaining the rate constant. Martin *et al.* [16] analyzed the case with time-varying independent and identically distributed (i.i.d.) rate but bounded system disturbances. The work in [17] allowed both a time-varying i.i.d. rate process and unbounded disturbances. Finally, You and Xie [27] considered

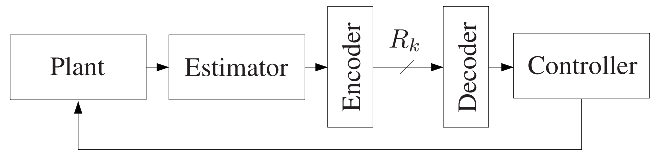

Fig. 1. Feedback loop model. The estimated state is quantized, encoded and sent to a decoder over a digital channel whose rate process R_k evolves as a Markov chain.

the case of unbounded disturbances and the time-varying channel rate taking values 0 or r according to the two-state Markov process depicted in Figure 2.

Results of these works analytically relate the speed of the dynamics of the plant to the information rate of the communication channel. They show that in order to guarantee stability, the rate must be large enough compared to the unstable modes of the system, so that it can compensate for the expansion of the state during the communication process. These kind of results are known in the literature as *data rate theorems* and there is interest in formulating them in the most general conditions.

In the present work, we consider a generalization that allows for unbounded system disturbances and models the time-varying rate of the channel as a homogeneous positive-recurrent Markov chain that takes values in a finite subset of the nonnegative integers. This takes into account arbitrary temporal correlations of the channel variations and includes all previous models mentioned above, so that our results recover the ones mentioned in the above papers as special cases. However, previous analysis methods cannot be applied directly and our analysis technique follows a different approach.

We derive the necessary and sufficient condition for mean square stabilization of a scalar system in terms of the stability of a Markov Jump Linear System (MJLS) [4] whose evolution depends on both the system's unstable mode and the Markov rate process. In the case of a plant with unstable mode $\lambda > 1$, we establish that stabilizing the system is equivalent to ensuring the stability of the MJLS having state dynamics $\lambda/2^{R_k}$, i.e., $z_{k+1} = \lambda/2^{R_k} z_k$, and transition jumps given by the Markov rate process. Intuitively, this equivalent MJLS describes the evolution of the estimation error at the decoder, which at every time step k increases by λ because of the system dynamics and is reduced by 2^{R_k} because of the information sent across the channel. A tight condition for second-moment stability is then expressed in terms of the spectral radius of the matrix governing the dynamics of the second moment of this MJLS.

A similar approach is followed to provide stability conditions for the case of vector systems. Necessary conditions are derived by proceeding in two steps. First, we assume that a "genie" helps the controller by stabilizing a subset of the unstable states. Then, we relate the stability of the reduced vector system to the one of a scalar MJLS whose evolution depends on the remaining unstable modes. By considering all possible subsets of unstable modes, we obtain a family of conditions that relate the degree of instability of the system to the parameters governing the rate process. On the other hand, a sufficient condition for mean-square stability is given using the control scheme described in [17]. This scheme yields to optimal performance only in some special cases, as already remarked in [17], but it can be easily analyzed using our result for scalar systems. Specifically, the sufficient condition is given as the intersection of the stability conditions for the scalar jump linear systems that describe the evolution of the estimation error for each unstable mode.

We now wish to spend some additional words on the related literature. Recent surveys on the theory of control with communication constraints appear in [10] and [19]. Broadly speaking, authors have followed two distinct approaches. The information-theoretic approach followed by the present paper and by [1], [5], [16], [18], [20], [26], [27], [28], aims to derive a data-rate theorem quantifying how much rate is needed to construct a stabilizing quantizer/controller pair. The network-theoretic approach followed by the works [8], [9], [21], [24] aims to determine the critical packet dropout probability above which the system cannot be stabilized by any control scheme. In this context, a packet models a real number, carrying an unbounded amount of information in its binary expansion of infinite precision. The work [17] created a bridge between the two approaches, as it recovers results of the packet loss model by assuming the rate R_k takes values 0 or r, and by letting $r \to \infty$. The same bridge also holds for our work that generalizes [17]. Indeed, when R_k is a Markov chain with state space $\{0, r\}$, we recover the results in [9] and [12] in the limit as $r \to \infty$.

The theory of MJLS has been previously used to investigate control problems over communication channels. Seiler and Sengupta [22] studied the mean-square stabilization of a dynamical system with random delays and used tools from MJLS to derive a linear matrix inequality condition for the existence of a stabilizing controller. A similar approach is taken in [13] to identify the most efficient estimation strategy to compensate for losses when controlling a system over a Markov erasure channel. Liu *et al.* [15] used control theoretic techniques to design capacity achieving codes for the communication problem over a finite-state Markov channel with feedback, by studying the second moment stability of a MJLS with output feedback and perfect knowledge of the Markov state.

The rest of the paper is organized as follows. Section II provides a description of the problem and introduces the necessary and sufficient condition for the stability of a MJLS. In Section III a tight condition for second-moment stability of scalar systems is stated. Section IV is devoted to the multi-dimensional case, for which necessary and sufficient conditions are provided. Section V concludes the paper.

II. Problem formulation

Consider the linear dynamical system

$$x_{k+1} = Ax_k + Bu_k + v_k, y_k = Cx_k + w_k, \quad k \in \mathbb{N} \quad (1)$$

where $x_k \in \mathbb{R}^d$ represents the state variable of the system, $u_k \in \mathbb{R}^m$ the control input, $v_k \in \mathbb{R}^d$ an additive disturbance independent of the initial condition x_0, $y_k \in \mathbb{R}^p$ the sensor measurement and $w_k \in \mathbb{R}^p$ the measurement noise. It is assumed that A is uniquely composed by unstable modes, so the open loop system is unstable. We assume the following:

A0. (A, B) is reachable and (A, C) is observable.
A1. x_0, v_j and w_k are mutually independent for all $j, k \in \mathbb{N}$.
A2. $\exists \epsilon > 0$ such that x_0, v_k and w_k have uniformly bounded $(2+\epsilon)$th absolute moments for all $k \in N$.
A3. The distribution of the noise is such that $e^{2h(v_k)/d} > 0$ for all $k \in \mathbb{N}$, where $h(v_k)$ denotes the differential entropy of v_k.

The state observer is connected to the controller through a noiseless digital communication link that at each time k allows transmission without errors of R_k bits, see Fig. 1. The rate process $\{R_k\}_{k\geq 0}$ is a homogeneous positive-recurrent Markov chain that takes values in a finite subset of the nonnegative integers

$$\mathcal{R} = \{r_1, \ldots, r_n\},$$

whose evolution through one time step is described by the transition probabilities

$$p_{ij} = \mathsf{P}\{R_{k+1} = r_j | R_k = r_i\}$$

for all $k \in \mathbb{N}$ and $i, j \in \{1, \ldots, n\}$. The rate process is independent of the other quantities describing the system and is causally known at the observer and the controller. For every $k \geq 0$, the coder is a function $s_k = s_k(y_0, \ldots, y_k)$ that maps all past and present measurements into the set $\{1, \ldots, 2^{R_k}\}$, the digital link is the identity function on the set $\{1, \ldots, 2^{R_k}\}$, and the control u_k is a function $\hat{x}_k(s_0, \ldots, s_k)$ that maps all past and present symbols sent over the digital link into $\mathbb{R}^d$.

Given this system, the objective is to find conditions on A and the rate process $\{R_k\}$ under which it is possible to design a control scheme, i.e., a sequence of coder/control functions, such that the closed loop system is mean-square stable

$$\sup_k \mathsf{E}\left[\|x_k\|^2\right] < \infty, \quad (2)$$

where the expectation is taken with respect to the random initial conditions, the additive disturbances, and the rate process R_k, and $\|\cdot\|^2$ denotes the L^2-norm.

Markov Jump Linear Systems.

Consider the scalar MJLS defined by

$$z_{k+1} = \frac{|\lambda|}{2^{R_k}} z_k + c, \tag{3}$$

where $z_k \in \mathbb{R}$ with $z_0 < \infty$, $c > 0$ is a constant, $\{R_k\}_{k\geq 0}$ is the Markov rate process described above. Let H be the $n \times n$ matrix with nonnegative real elements

$$h_{ij} = \frac{1}{2^{2r_j}} p_{ji}, \tag{4}$$

for all $i, j \in \{1, \dots, n\}$. The following lemma states the necessary and sufficient condition for the mean square stability of the system (3) in terms of the unstable mode $|\lambda|$ and the spectral radius of H. The spectral radius $\rho(\cdot)$ of a matrix is the maximum among the absolute values of its eigenvalues.

Lemma 1: Necessary and sufficient condition for the mean square stability of the system (3) is that

$$|\lambda|^2 < \frac{1}{\rho(H)}.$$

The proof of the above lemma is omitted as the claim is a special case of [4, Theorem 3.9] and [4, Theorem 3.33].

III. Scalar System

Consider the special case of a scalar system

$$x_{k+1} = \lambda x_k + u_k + v_k, \quad y_k = x_k + w_k, \quad \forall k \in \mathbb{N}. \tag{5}$$

where $|\lambda| \geq 1$.

Theorem 1: Under assumptions A0-A3, there exists a control that stabilizes the scalar system (5) in mean square sense if and only if the MJLS (3) is mean square stable, that is, if and only if

$$|\lambda|^2 < \frac{1}{\rho(H)}. \tag{6}$$

Remark 1: In the above condition the unstable mode of the system and the channel properties are decoupled. This means that for a given Markov rate process there exists a threshold above which the system cannot be stabilized by any control.

Application of Theorem 1 yields the following results as special cases.

a) Constant rate. When the channel supports a constant rate r, we have that $H = 1/2^{2r}$ and thus (6) reduces to the well known data rate theorem condition

$$r > \log|\lambda|$$

derived in [18], [25].

b) Independent rate process. Consider the special case of an i.i.d. rate process R_k where $R_k \sim R$ has probability mass function $p_i = \mathsf{P}\{R = r_i\}$, $r_i \in \mathcal{R}$. In this case, letting $p = (p_1, \dots, p_n)^{\mathrm{T}}$ and $h = (2^{-2r_1}, \dots, 2^{-2r_n})^{\mathrm{T}}$,

$$H = \big(p_1, \dots, p_n\big)^{\mathrm{T}} \big(2^{-2r_1}, \dots, 2^{-2r_n}\big) = p h^{\mathrm{T}}$$

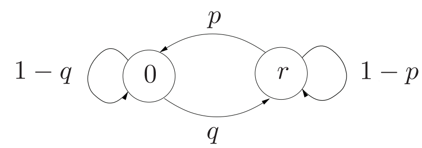

Fig. 2. A two-state Markov chain modeling a bursty packet erasure channel.

is a rank-one matrix whose only nonzero eigenvalue is $h^T p$. Therefore, Theorem 1 yields the result in [16], [17],

$$\begin{aligned} |\lambda|^2 \rho(H) &= |\lambda|^2 \big(2^{-2r_1}, \dots, 2^{-2r_n}\big)\big(p_1, \dots, p_n\big)^{\mathrm{T}} \\ &= \mathsf{E}\left[\frac{|\lambda|^2}{2^{2R}}\right] < 1. \end{aligned}$$

If we further specialize to the case $n = 2$, $r_1 = 0$, $r_2 = r$, and we let $r \to \infty$, then the stability condition $p_1 > 1/|\lambda|^2$ depends only on the erasure rate of the channel, i.e. we recover the packet loss model result in [8].

c) Two-state Markov process. In the special case illustrated in Fig. 2 in which $n = 2$, $p_{12} = q$, and $p_{21} = p$ for some $0 < p, q < 1$, we have

$$H = \begin{pmatrix} \frac{1}{2^{2r_1}}(1-q) & \frac{1}{2^{2r_2}} p \\ \frac{1}{2^{2r_1}} q & \frac{1}{2^{2r_2}}(1-p) \end{pmatrix} \tag{7}$$

and the condition in Theorem 1 reduces to

$$|\lambda|^2 \rho(H) = \tfrac{|\lambda|^2}{2} \mathrm{tr}(H) + \tfrac{|\lambda|^2}{2} \sqrt{\mathrm{tr}(H)^2 - 4\mathrm{det}(H)} < 1, \tag{8}$$

where $\mathrm{tr}(H)$ and $\mathrm{det}(H)$ denote the trace and determinant of H, respectively. The special case where $r_1 = 0$ and $r_2 = r$ was previously studied in [27] where, by following a different approach, the authors proved that necessary and sufficient condition for stabilization is that

$$\mathsf{E}\left[\frac{|\lambda|^{2\tau}}{2^{2r}}\right] < 1. \tag{9}$$

Here τ denotes the "hitting time" of state r, i.e., the time between two consecutive visits of that state, and the expectation is taken with respect to τ. Intuitively, this condition says that r should be large enough to compensate for the expansion of the state during the time in which packets are erased and thus no information can be sent from the observer to the controller. Condition (9) simplifies to

$$|\lambda|^2 < \begin{cases} \frac{1}{\mathrm{tr}(H)} & \text{if } \mathrm{det}(H) = 0, \\ \frac{\mathrm{tr}(H) - \sqrt{\mathrm{tr}(H)^2 - 4\mathrm{det}(H)}}{2\mathrm{det}(H)} & \text{otherwise,} \end{cases}$$

where H is given by (7) with $r_1 = 0$ and $r_2 = r$. Observe that

$$\frac{\mathrm{tr}(H) - \sqrt{\mathrm{tr}(H)^2 - 4\mathrm{det}(H)}}{2\mathrm{det}(H)} = \frac{2}{\mathrm{tr}(H) + \sqrt{\mathrm{tr}(H)^2 - 4\mathrm{det}(H)}},$$

so (8) is equivalent to (9) and our result recovers the one in [27] in the special case where $r_1 = 0$ and $r_2 = r$.

Taking the limit $r \to \infty$, we have that $\mathrm{det}(H) \to 0$ and $\mathrm{tr}(H) \to 1 - q$, and so the above inequalities simplify to the condition $(1-q)|\lambda|^2 < 1$, which means that the stability condition is only determined by the recovery rate q, i.e. we recover the packet loss model result in [9].

A. Necessity

The following lemma shows that the second moment of the state in (5) is lower bounded by the second moment of a MJLS that evolves as (3).

Lemma 2: For every $k \geq 0$,

$$\mathsf{E}[|x_k|^2] \geq \tfrac{1}{2\pi e}\,\mathsf{E}[|z_k|^2], \qquad (10)$$

where $\{z_k\}$ is a MJLS with dynamics $z_{k+1} = |\lambda|/2^{R_k} z_k$ and initial condition $z_0 = e^{h(x_0)}$.
The proof is given in the full version of the paper [3] and relies on standard information theoretic techniques. It follows that the state cannot be stabilized if this MJLS is second moment unstable. Hence, the "only if" condition in Theorem 1 follows directly from Lemma 1.

B. Sufficiency

1) Noiseless systems with bounded initial condition: We first consider the special case of a fully observed, discrete time, unstable, scalar system defined by

$$x_{k+1} = \lambda x_k + u_k, \quad k \geq 1, \qquad (11)$$

with $|x_0| \leq M_0$ for some bounded $M_0 > 0$. In this case, we have that

$$|x_k| \leq z_k, \quad k \geq 0, \qquad (12)$$

where z_k is a MJLS with dynamics

$$z_{k+1} = \frac{|\lambda|}{2^{R_k}} z_k, \quad k \geq 1$$

and $z_0 = M_0$. To see this, consider the following inductive proof. By construction, $|x_0| \leq M_0 = z_0$. Assume that the claim holds for all times up to k, so $|x_k| \leq z_k$. Suppose that the uncertainty set $[-z_k, z_k]$ is quantized using a R_k-bit uniform quantizer, and that the encoder communicates to the decoder the interval in which the state lies. Then, the decoder approximates the state by the centroid $\hat{x}_k$ of this interval and sends to the plant the control input $u_k = -\lambda \hat{x}_k$. By construction $|x_k - \hat{x}_k| \leq z_k/2^{R_k}$, thus

$$|x_{k+1}| = |\lambda||x_k - \hat{x}_k| \leq \frac{|\lambda|}{2^{R_k}} z_k = z_{k+1},$$

i.e., the claim holds at time $k+1$ as well. It follows that x_k is stable if the MJLS z_k is second moment stable. Hence, the "if" condition in Theorem 1 follows from Lemma 1.

By combining (10) and (12) we can see that the second moment of the state can be upper and lower bounded by two MJLSs that differ only in their initial conditions, and therefore share the same stability condition. What emerges from the analysis in this simple setting is that the MJLS $\{z_k\}_{k\geq 0}$ that determines the stability of the system describes the evolution of the estimation error at the decoder, which at every time step k increases by λ, because of the system dynamics, and is reduced by 2^{R_k}, because this is the best attainable accuracy of representation sending R_k bits across the channel.

2) Unbounded noise and initial condition: Consider now the system (5) assuming that conditions A0-A3 hold. The main challenge in this setting is that the disturbance has unbounded support, so it is not possible to confine the state dynamics within a finite interval as in the case of noiseless systems. Consequently, the uniform quantizer used for stabilizing noiseless systems has to replaced by a dynamic quantizer that follows a zoom-in zoom-out strategy [2], [14], where the range of the quantizer is increased (zoom-out phase) when atypically large disturbances affect the system, and decreased as the state reduces is size (zoom-in phase). We build a stabilizing control scheme based on the adaptive quantizer defined in [18].

In the remaining of this section we describe the main elements of our construction, while we refer the reader to [3] for a detailed description of the control scheme and its analysis. We assume that coder and decoder share at each time k an estimate of the state $\hat{x}_k$ that is recursively updated using the information sent through the channel. Time is divided into cycles of fixed duration of τ time steps. In each cycle the coder sends information about the state of the system at the beginning of the cycle. At the end of each cycle the decoder updates the estimate of the state and sends a control signal to the plant. The key step of the analysis is to show that the dynamics of the mean square estimation error (and thus of the state) can be bounded by a MJLS, so the stability of this MJLS implies the stability of the state. Specifically, we prove that at times $\tau, 2\tau, 3\tau, \ldots$, i.e., at the beginning of each cycle, the second moment of the estimation error $x_{j\tau} - \hat{x}_{j\tau}$ satisfies

$$\mathsf{E}[|x_{j\tau} - \hat{x}_{j\tau}|^2] \leq \mathsf{E}[z_{j\tau}^2], \quad j \geq 0,$$

where $z_{j\tau}$ is a process formed recursively as

$$z_{j\tau} = \phi \frac{|\lambda|^\tau}{2^{R_{(j-1)\tau} + \cdots + R_{j\tau-1}}} z_{(j-1)\tau} + \varsigma, \quad j \geq 1, \qquad (13)$$

for some bounded initial condition z_0, and constants $\phi > 1$ and $\varsigma > 0$. Notice that equation (13) represents the evolution at times $\tau, 2\tau, \cdots$ of the MJLS

$$z_k = \phi^{1/\tau} |\lambda| z_{k-1} + c, \quad k \geq 1.$$

for some appropriately chosen constant c. Thus, by Lemma 1 a sufficient condition for second moment stability of $\{z_k\}$ is that

$$\phi^{2/\tau} \rho\left(|\lambda|^2 H\right) < 1. \qquad (14)$$

On the other hand, if the condition of Theorem 1 is satisfied, that is, if $|\lambda|^2 \rho(H) < 1$, then we can choose the duration of a cycle τ large enough to ensure that (14) holds and, as a consequence, (13) is stable, the second moment of the estimation error at the beginning of each cycle is bounded, and the state remains second-moment bounded.

IV. Vector system

Consider the system (1), where $A \in \mathbb{R}^{d\times d}$ is uniquely composed by unstable modes. Let $\lambda_1, \ldots, \lambda_u \in \mathbb{C}$ be the distinct, non-conjugate eigenvalues of A (if λ_i, λ_i^* are complex conjugate eigenvalues, only one of them is considered). Let

m_i be the algebraic multiplicity of λ_i, $a_i = 1$ if $\lambda_i \in \mathbb{R}$ and $a_i = 2$ otherwise. We have $\sum_{i=1}^{u} a_i m_i = d$.
In order to decompose its dynamical modes, we express A into real Jordan canonical form J [11]. The matrix $J \in \mathbb{R}^{d \times d}$ is such that $A = T^{-1} J T$ for some similarity matrix T, and has the block diagonal structure $J = \text{diag}(J_1, \dots, J_u)$ where $J_i \in \mathbb{R}^{a_i m_i \times a_i m_i}$ and $\det J_i = \lambda_i^{a_i m_i}$. Let $\mathcal{U} = \{1, \dots, u\}$ denote the index set of the subsystems corresponding to the modes $\lambda_1, \dots, \lambda_u$. Consider the recursion

$$x_{k+1} = J x_k + T B u_k + T v_k, y_k = C T^{-1} x_k + w_k. \quad (15)$$

As (15) and (1) are related only by the matrix T, we assume that the system evolves according to (15).

A. Necessity

To find necessary conditions for stability, we make the optimistic assumption that a "genie" helps the controller by stabilizing part of the system, so that the information sent across the channel is only used to stabilize a subset of the unstable states. Formally, let $\nu_i \in \{0, \cdots, m_i\}$ denote the algebraic dimensionality of the ith unstable mode after the genie's intervention, and let $\mathcal{V} = \{\nu : \nu_i \in \{0, \cdots, m_i\}\} \subset \mathbb{N}^u$ denote the space of all possible algebraic dimensionalities. For each $\nu \in \mathcal{V}$, let $x_k(\nu)$ denote the unstable system obtained by removing from (15) the components that have been stabilized by the genie. Observe that $x_k(\nu)$ is real valued and has total dimension

$$d(\nu) = \sum_{i \in \mathcal{U}} a_i \nu_i. \quad (16)$$

Next, we find necessary conditions for the second-moment stability of the reduced system $x_k(\nu)$. The following lemma shows that the second moment $x_k(\nu)$ is lower bounded by the second moment of a scalar MJLS whose evolution depends on the unstable modes of $x_k(\nu)$ and their dimensionalities. The differential entropy of $x_0(\nu)$ is denoted by $h(x_0(\nu))$.

Lemma 3: For each $\nu \in \mathcal{V}$, let $z_k(\nu)$ be the scalar MJLS defined by

$$z_{k+1}(\nu) = \left(\frac{\prod_{i \in \mathcal{U}} |\lambda_i|^{a_i \nu_i}}{2^{R_k}} \right)^{1/d(\nu)} z_k(\nu),$$

with $z_0(\nu) = 2^{\frac{1}{d(\nu)} h(x_0(\nu))}$. Then, for every $k \geq 0$,

$$\mathsf{E}[|x_k(\nu)|^2] \geq \mathsf{E}[|z_k(\nu)|^2].$$

The proof of the above lemma is omitted since it closely follows the proof of Lemma 2 (cf. [3]). It follows that $x_k(\nu)$ cannot be stable if the MJLS $z_k(\nu)$ is second moment unstable. Finally, we can derive a set of necessary conditions in terms of the spectral radius of the matrix describing the dynamics of the second moment of $z_k(\nu)$. Let $H(\nu)$ be the $n \times n$ matrix with nonnegative real elements

$$h_{ij}(\nu) = \frac{1}{2^{\frac{2}{d(\nu)} r_j}} p_{ji} \quad (17)$$

for all $i, j \in \{1, \dots, n\}$. Combining Lemma 1 and Lemma 3 it then follows that

Theorem 2: Under assumptions A0-A3, necessary condition for the stabilization of the system (15) in the mean square sense (2) is that, for every $\nu \in \mathcal{V}$,

$$\lambda(\nu)^{2/d(\nu)} < \frac{1}{\rho(H(\nu))} \quad (18)$$

where $\lambda(\nu) = \prod_{i \in \mathcal{U}} |\lambda_i|^{a_i \nu_i}$, $d(\nu)$ is as in (16), and $H(\nu)$ is given by (17).

Remark 2: In the special case of a scalar system, we have that $\lambda(\nu) = \lambda$, $d(\nu) = 1$, and $H(\nu)$ is equal to (4), thus the above condition reduces to the necessary condition in Theorem 1.

Remark 3: Rewriting (18) as

$$\sum_{i \in \mathcal{U}} a_i \nu_i \log |\lambda_i| < -\tfrac{d(\nu)}{2} \log \rho(H(\nu))$$

it is clear that the subsystem modes and the channel properties are decoupled and that the Markov rate process poses an upper bound on the degree of instability of the system.

B. Sufficiency

A sufficient condition for mean-square stability is given extending the control scheme described in [17] and later in [27]. However, our scheme is more general than the previous ones, and it allows to *strictly* improve upon the sufficient condition obtained in [27] for the special case of a Markov erasure channel.

The main challenge in studying vector systems is how to design an optimal vector quantizer that dynamically adapts to the time-varying communication rate. The solution adopted in [17] for the case of an i.i.d. rate process is to quantize each component of the state using separate scalar quantizers, so the performance can be easily analyzed using the result for scalar systems. In this paper, we follow a similar method of proof and assume that at each time step the bits available for transmission are distributed among the various unstable modes of the system. More precisely, we define a bit allocation function $\alpha : \mathcal{R} \to [0,1]^u$ as a function having the properties that

$$\tfrac{1}{a_i m_i} \alpha_i(r) r \in \mathbb{N}, \quad i \in \mathcal{U}, \quad (19)$$

and

$$\alpha_1(r) + \cdots + \alpha_u(r) \leq 1 \quad (20)$$

for all $r \in \mathcal{R}$. The operational meaning of α is as follows. At every time k, $\frac{1}{a_i m_i} \alpha_i(R_k) R_k$ bits are used to quantize each scalar component of the ith unstable state of the system. Condition (19) enforces that the each subsystem is quantized using an integer number of bits, while (20) ensures that we do not use more bits than what is available for transmission. Let $\mathcal{A}$ be the set of all bit allocation functions. From the result on scalar systems stated in Section III, the ith subsystem is stable if the MJLS with dynamics

$$z_{k+1} = \frac{|\lambda_i|}{2^{\frac{1}{a_i m_i}\alpha_i(R_k)R_k}} z_k,$$

is stable. Paralleling the analysis for scalar system, define $H_l(\alpha)$ as the $n \times n$ real matrix with (i,j)th entry equal to

$$2^{-2\frac{\alpha_l(r_j)}{a_l m_l} r_j} p_{ji}. \tag{21}$$

Then, given a bit allocation function α, a sufficient condition for second moment stability of the system is that

$$|\lambda_i|^2 < \frac{1}{\rho(H_i(\alpha))} \tag{22}$$

for every $i \in \mathcal{U}$. A more general sufficient condition can be given by allowing the bit allocation function to change over time.

Theorem 3: Under the assumptions A0-A3 above, a sufficient condition for the stabilization of the system (15) in the mean-square sense is that

$$|\lambda_i|^2 < \frac{1}{\rho\big(H_i(\alpha_1)H_i(\alpha_2)\cdots H_i(\alpha_m)\big)^{\frac{1}{m}}} \quad \text{for all } i \in \mathcal{U}, \tag{23}$$

for some $m \geq 1$ and $\alpha_1, \cdots, \alpha_m \in \mathcal{A}$.

The main idea of the proof, given in [3], is to use different bit allocations functions in a cyclic repeated order via a time-sharing protocol.

By taking the $\log$ on both sides in (23), we can see that (23) defines an open hypercube in the domain of the unstable rates for each choice of m and of $\alpha_1, \cdots, \alpha_m \in \mathcal{A}$. Then, Theorem 2 states that it is possible to stabilize all those systems whose unstable rates lie inside the union of all such open hypercubes. Observe that the resulting region is not computable because there is no upper bound on the value of m. However, the next proposition, which is proved in [3], shows that the region is *convex*.

Proposition 1: For every $m \geq 1$ and $\alpha_1, \ldots, \alpha_m \in \mathcal{A}$, let $\mathcal{C}(m, \alpha_1, \cdots, \alpha_m)$ denote the set of rates satisfying (23). Then, the closure of

$$\bigcup_{m \geq 1} \bigcup_{\alpha_1, \ldots, \alpha_m \in \mathcal{A}} \mathcal{C}(m, \alpha_1, \cdots, \alpha_m)$$

is a convex set.

It follows from the above proposition that the set of rates satisfying the condition in Theorem 2 can be approximated by fixing a bound M for m and taking the *convex hull* of the union of the hypercubes over all $\alpha_1, \cdots, \alpha_m \in \mathcal{A}$ and $m \leq M$. In particular, by setting $M = 1$ we obtain that the inner bound contains the set of rates inside

$$\text{conv} \bigcup_{\alpha \in \mathcal{A}} \mathcal{C}(1, \alpha), \tag{24}$$

i.e., the convex hull of the union of the rates that can be stabilized using the same bit allocation function $\alpha \in \mathcal{A}$ at every instant of time. In general the inclusion is strict, as choosing $M > 1$ might yield a strictly larger region, but there are some special cases where the two sets are equal.

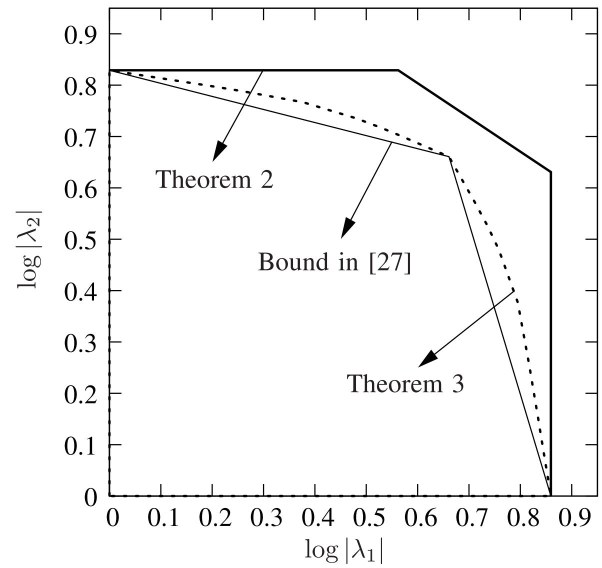

Fig. 3. Inner and outer bounds on the set of stabilizable rates over a Markov erasure channel.

As an application of Theorem 3 and Proposition 1, consider the following special cases:

a) Independent rate process. In the special case of an i.i.d rate process, for every $m \geq 1$ and bit allocation functions $\alpha_1, \ldots, \alpha_m \in \mathcal{A}$, (23) simplifies to

$$\log|\lambda_i| < -\sum_{j=1}^{m} \frac{1}{2m} \log \mathsf{E}\left[2^{-2\alpha_{j,i}(R)R}\right], \text{ for all } i \in \mathcal{U},$$

and therefore our condition recovers the sufficient condition given in [17]. It can be shown that in this case the set of rates satisfying Theorem 2 is exactly equal to (24).

b) Two-state Markov erasure process. Consider the Markov rate process illustrated in Fig. 2 in the special case where $r_1 = 0$ and $r_2 = r$. For this problem, a sufficient condition for mean-square stabilization was previously given in [27, Theorem 3]. Restating their result in our notation, You and Xie proved that it is possible to stabilize all rates inside (24). We claim that Theorem 2 above yields a strictly larger inner bound to the set of stabilizable rates. To see this, consider the special case where $r = 6$ and the transition probabilities are $p_{12} = 0.1$ and $p_{21} = 0.7$. Let the system matrix J have two distinct real eigenvalues λ_1 and λ_2 of multiplicity $\mu_1 = 2$ and $\mu_2 = 3$, respectively. Figure IV-B plots inner and outer bounds to the set of stabilizable rates $(\log|\lambda_1|, \log|\lambda_2|)$. The outer bound is delimited by the solid thick black line and is obtained by evaluating the bounds in Theorem 2. For the inner bounds, observe that in this setting there are 12 possible bit allocation functions, i.e., $|\mathcal{A}| = 12$, and therefore the inner bound in [27, Theorem 3] (delimited by the solid thin black line) is obtained taking the convex hull of 12 hypercubes. On the other hand, by evaluating Theorem 3 for $m \leq 8$ and every choice of $\alpha_1, \ldots, \alpha_m \in \mathcal{A}$ and by taking the convex hull of the union of the resulting hypercubes, we obtain the region enclosed by the dashed black line, which strictly includes the inner bound in [27, Theorem 3].

V. Concluding remarks

We studied the problem of mean-square stabilization of a linear, discrete-time, dynamical system over a time-varying, Markov digital feedback channel. We formulated a general version of the data rate theorem that extends previous formulations and provides a deeper understanding of the system's dynamics over correlated channels. The result has been obtained exploiting a reduction to MJLS and using the theory developed in this context. A missing point, that is still open for future research, is how results are modified in the presence of decoding errors, beside erasures.

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