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Input-To-State Stabilization of Low-Complexity Model Predictive Controllers for Linear Systems

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***Abstract*— Research on sub-optimal Model Predictive Control (MPC) has led to a variety of optimization methods based on explicit or online approaches, or combinations thereof. Its foremost aim is to guarantee essential controller properties, i.e. recursive feasibility, stability, and robustness, at reduced and predictable computational cost, i.e. computation time and storage space. This paper shows how the input sequence of any (not necessarily stabilizing) sub-optimal controller and the shifted input sequence from the previous time step can be used in an optimal combination, which is easy to determine online, in order to guarantee input-to-state stability (ISS) for the closed-loop system. The presented method is thus able to stabilize a wide range of existing sub-optimal MPC schemes that lack a formal stability guarantee, if they can be considered as a continuous map from the state space to the space of feasible input sequences.**

I. INTRODUCTION

The growing complexity of modern control systems and the increasing availability of powerful hardware has extended the scope of applications for Model Predictive Control (MPC). Unlike traditional control methods, it requires the solution of an optimization problem (MPC problem) over a receding finite prediction horizon at every time step.

For linear systems with fast dynamics and high sampling rates, which are the focus of this paper, restricted hardware capacities—both in terms of computational speed and storage capacity—remain a critical limiting factor. One approach to reduce the computation efforts is explicit MPC (see [2]), where the polyhedral piecewise-affine solution of the MPC problem is pre-computed and stored for every relevant initial condition. For systems of small dimensions, the storage requirements are typically small, and the online procedure reduces to a fast look-up operation. However, as its worst-case complexity grows exponentially with the problem size, explicit MPC loses much of its effectiveness for larger systems, where it is outperformed by appropriately tailored online algorithms.

The need for increased efficiency has led research to focus on sub-optimal MPC, for which essential properties such as recursive feasibility, stability, or robustness, however, are often difficult to establish. For online MPC, a common approach is to stop an iterative algorithm early, e.g. as in [15] for interior point methods, [3] for active set methods, or [11] for fast gradient methods. In explicit MPC, approximations to the exact solution, but having a lower complexity, are constructed.

Most recent explicit MPC approaches partition the state space into regions of a predefined shape, like hypercubes (see [6], [13]) or simplices (see [14]), and interpolate the stored solution at its extreme points (see [7]). While they permit to construct MPC controllers of very low complexity, they require a (often rather high) degree of complexity for stability guarantees. This paper presents a simple add-on scheme, to be used in conjunction with sub-optimal control schemes of arbitrarily low complexity, that provides robust stability. Its only key requirement is that the controller represents a continuous map from the state space into the space of feasible full-horizon input sequences.

The method is based on conventional Lyapunov stability theory for sub-optimal MPC, as described in [12]; however the decrease in the cost function is achieved not by close approximation of the optimal solution, but by a combination of (a) the current sub-optimal input sequence and (b) the shifted input sequence of the previous time step. The idea has originally been proposed by [14], yet only for the nominal case and without robustness properties. In this paper, it is modified so as to yield input-to-state stability (ISS) in the presence of state disturbances.

The method makes minor modifications to the MPC problem and introduces a simple and fast online procedure (Section III), which usually amounts to a few matrix-vector multiplications. It admits a rigorous proof of ISS in the presence of state disturbances (Section IV). Finally, practical application of the method is demonstrated for a single mass oscillator (Section V).

II. NOTATION & PRELIMINARIES

A. Notation

$\mathbb{N} = \{0, 1, 2, ...\}$ denotes the set of natural numbers and 0, $\mathbb{R}$ the set of real numbers, and $\mathbb{R}_+$ ($\mathbb{R}_{0+}$) the set of positive (non-negative) real numbers. In the product space $\mathbb{R}^n$, $\mathbb{B}^n$ is the closed unit ball in the Euclidean norm $\|\cdot\|$. The space of real sequences $\{t_n\}_{n\in\mathbb{N}}$ is denoted by $\mathbb{R}^{\mathbb{N}}$.

A *polyhedron* is the finite intersection of closed half-spaces in $\mathbb{R}^n$, and a *polytope* is a bounded polyhedron.

For some index $k \in \mathbb{N}$, non-bold letters indicate vectors $x_k \in \mathbb{R}^n$, and bold letters $\mathbf{x}_k := \{x_{k|k}, x_{k+1|k}, ..., x_{k+N|k}\}$ an ordered collection of vectors $x_{k+i|k} \in \mathbb{R}^n$ that can also be considered as a large stacked-up vector $\mathbf{x}_k \in \mathbb{R}^{(N+1)n}$.

A function $\alpha : \mathbb{R}_{0+} \to \mathbb{R}_{0+}$ is a K-function if it is continuous, strictly monotonically increasing, and $\alpha(0) = 0$; it is a K_∞-function if in addition $\alpha(r) \to \infty$ as $r \to \infty$.

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A function $\beta : \mathbb{R}_{0+} \times \mathbb{R}_{0+} \to \mathbb{R}_{0+}$ is a KL-function if for any fixed $t \in \mathbb{R}_{0+}$ $\beta(\cdot, t)$ is a K-function and for any fixed $r \in \mathbb{R}_{0+}$ $\beta(r, \cdot)$ is monotonically decreasing and $\beta(r,t) \to 0$ as $t \to \infty$.

Let (X, d_X) be a metric space and S be a subset, written $S \subset X$. The *point-to-set distance* of some $x \in X$ to S is

$$d_X(x, S) := \inf_{s \in S} d_X(x, s) \ ;$$

the distance to the empty set $d_X(x, \emptyset) := \infty$ by convention. For any $\varepsilon > 0$, the ε*-neighborhood* of S is denoted by

$$U_\varepsilon S := \{\, x \in X \mid d_X(x, S) < \varepsilon \,\} \ .$$

B. Control System with State Disturbance

Consider a linear time-invariant system in discrete-time

$$x_{k+1} = Ax_k + Bu_k + w_k \ , \qquad x_0 \in \mathbb{X} \ , \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$, whose trajectory of states $x_k \in \mathbb{R}^n$ must be kept in the state constraint set $\mathbb{X} \subset \mathbb{R}^n$ for all $k \in \mathbb{N}$. Here $u_k \in \mathbb{U} \subset \mathbb{R}^m$ is the external control input and $w_k \in \mathbb{W} \subset \mathbb{R}^n$ a random state disturbance at step k.

Assumption II.1 (System Dynamics). *(a) The pair of matrices A, B is stabilizable. (b) The state is measured at every step k. (c) The* state constraint set $\mathbb{X}$ *is convex and contains the origin in its interior. (d) The set of* admissible controls $\mathbb{U}$ *is compact, convex, and contains the origin in its interior.*

Assumption II.2 (Disturbances). *The* disturbance set $\mathbb{W}$ *is compact and contains the origin; neither $\mathbb{W}$ nor the probability distribution of w_k are known for controller design.*

For closed-loop controllers, *control sequences* $\{u_k\}_{k\in\mathbb{N}}$ are provided by a *state feedback law* $\kappa : \mathcal{X}_\kappa \to \mathbb{R}^m$ on some domain $\mathcal{X}_\kappa \subset \mathbb{X}$, assigning $u_k := \kappa(x_k)$.

Definition II.3 ((Robustly) Feasible Controls). *A control sequence (or state feedback law) is said to be* (recursively) feasible for x_0 *if (a) it is admissible and (b) the resulting state trajectory satisfies the state constraints at all times $k \in \mathbb{N}$. It is called* robustly (recursively) feasible for x_0 *if the above conditions are met for all possible outcomes of the disturbance sequence $\{w_k\}_{k\in\mathbb{N}} \in \mathbb{W}^{\mathbb{N}}$.*

Definition II.4 ((Robustly) Positively Invariant Set). *A set $\mathcal{X}_\kappa \subset \mathbb{X}$ is called* positively invariant (PI) for (1) under κ *if for all $\xi \in \mathcal{X}_\kappa$ it holds that: (a) $\kappa(\xi) \in \mathbb{U}$ and (b) $[A\xi + B\kappa(\xi)] \in \mathcal{X}_\kappa$. It is called* robustly positively invariant (RPI) for (1) under κ *if, moreover, $[A\xi + B\kappa(\xi)] \oplus \mathbb{W} \subset \mathcal{X}_\kappa$.*

In this paper, a more general *parameterized feedback law* $\kappa : \mathbb{P} \times \mathcal{X}_\kappa \to \mathbb{R}^m$ will be considered. Namely, it allows for $u_k := \kappa(p_k, x_k)$ to depend also on a specific sequence of parameters p_k in some compact set $\mathbb{P}$. The sequence is recursively defined for all $k \in \mathbb{N}$ by some non-linear transition map $f : \mathbb{P} \times \mathbb{X} \to \mathbb{P}$:

$$p_{k+1} = f(p_k, x_k) \ , \qquad p_0 \in \mathbb{P} \ .$$

Note that these 'dynamics' use the state trajectory as 'inputs'. Definitions II.3 and II.4 hold analogously.

C. Point-to-Set Mappings

Let X and Y be metric spaces and Λ be a point-to-set mapping from X into the power set 2^Y; in short $\Lambda : X \rightrightarrows Y$.

Definition II.5 (Continuity). [1, p. 25] $\Lambda : X \rightrightarrows Y$ *is (a)* closed at $\bar{x} \in X$ *if for any two sequences $\{x_t\}_{t\in\mathbb{N}}$ and $\{y_t\}_{t\in\mathbb{N}}$, where $y_t \in \Lambda(x_t)$ for all $t \in \mathbb{N}$ and*

$$x_t \to \bar{x} \qquad \text{and} \qquad y_t \to \bar{y}$$

as $t \to \infty$, it holds that $\bar{y} \in \Lambda(\bar{x})$;
(b) Hausdorff upper semicontinuous (H-u.s.c.) at $\bar{x} \in X$ *if for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\Lambda(x) \subset U_\varepsilon \Lambda(\bar{x}) \qquad \forall\ x \in U_\delta\{\bar{x}\} \ ;$$

(c) Hausdorff lower semicontinuous (H-l.s.c.) at $\bar{x} \in X$ *if for every $\varepsilon > 0$ there exists $\delta > 0$ such that*

$$\Lambda(\bar{x}) \subset U_\varepsilon \Lambda(x) \qquad \forall\ x \in U_\delta\{\bar{x}\} \ .$$

If Λ is both H-u.s.c. and H-l.s.c. at $\bar{x} \in X$, then it is H-continuous at $\bar{x} \in X$*. The qualifier 'at $\bar{x}$' is omitted if a property holds for all $\bar{x} \in X$.*

Proposition II.6. [1, p. 26] *If the mapping $\Lambda : X \rightrightarrows Y$ is H-u.s.c. at $\bar{x} \in X$ and the set $\Lambda(\bar{x})$ is closed, then Λ is closed at $\bar{x}$.*

D. Lyapunov Stability Theory

This section briefly introduces input-to-state stability (e.g. [10], [5]), in particular for constrained systems (e.g. [9]).

Definition II.7 (Input-to-State Stability). *Consider the dynamic system* (1) *under a feedback law κ that is robustly recursively feasible on some RPI set $\Gamma_\kappa \subset \mathcal{X}_\kappa$. The origin is* input-to-state stable (ISS) on Γ_κ *if there exist a KL-function β and a K-function τ such that*

$$\|x_k\| \le \beta\big(\|x_0\|, k\big) + \tau\big(\sup_{i \le k-1} \|w_i\|\big) \qquad \forall\ k \in \mathbb{N} \ ,$$

for $x_0 \in \Gamma_\kappa$ and any disturbance sequence $\{w_k\}_{k\in\mathbb{N}} \in \mathbb{W}^{\mathbb{N}}$.

Definition II.7 reduces to *asymptotic stability (AS) of the origin on Γ_κ* if $\{w_k\}_{k\in\mathbb{N}} = 0$. The stability analysis of a system under a parameterized feedback law $\kappa(p_k, x_k)$ entails the dependence of the Lyapunov function on the parameters $p_k \in \mathbb{P}$; it shall therefore be referred to as a *parameterized Lyapunov function* (a similar concept is used in [12]).

Definition II.8 (Parameterized ISS Lyapunov Function). $V : \mathbb{P} \times \mathcal{X}_\kappa \to \mathbb{R}_{0+}$ *is called a* parameterized ISS Lyapunov function for system (1) under κ on Γ_κ, *if $\Gamma_\kappa \subset \mathcal{X}_\kappa$ is a RPI set for* (1) *under κ, and V satisfies the following conditions:*
(a) There exist two K_∞-functions α_1 and α_2 such that

$$\alpha_1(\|\xi\|) \le V(p_k, \xi) \le \alpha_2(\|\xi\|) \ \ \forall\ p_k \in \mathbb{P}, \forall\ \xi \in \Gamma_\kappa. \tag{2}$$

(b) There exists a K_∞-function α_3 and K-function σ with

$$V(p_{k+1}, x_{k+1}) \le V(p_k, x_k) - \alpha_3(\|x_k\|) + \sigma(\|w_k\|) \tag{3}$$

for all $k \in \mathbb{N}$, any $x_0 \in \Gamma_\kappa$, and any $\{w_k\}_{k\in\mathbb{N}} \in \mathbb{W}^{\mathbb{N}}$.

Definition II.8 reduces to an *ISS Lyapunov function* if the dependence on the parameters $p_k \in \mathbb{P}$ is removed; it reduces to a *parameterized Lyapunov function* if the term $\sigma(\|w_k\|)$ is removed in equation (3); and it reduces to a classic *Lyapunov function* if both of these simplifications are made.

The following theorem is a straightforward extension of ISS Lyapunov theory for parameterized systems (see [8]).

Theorem II.9 (ISS Lyapunov Stability). [4, p. 2131] *Let κ be a parameterized feedback law and Γ_κ be a RPI set for* (1) *under κ. If there is a parameterized ISS Lyapunov function for* (1) *under κ on Γ_κ, then the origin is ISS on Γ_κ.*

Corollary II.10. [12, p. 649] *Let κ be a parameterized feedback law and Γ_κ be a PI set for* (1) *under κ. If there is a parameterized Lyapunov function for* (1) *under κ on Γ_κ, and if $\{w_k\}_{k\in\mathbb{N}} = 0$, then the origin is AS on Γ_κ.*

III. PROPOSED SUB-OPTIMAL MODEL PREDICTIVE CONTROLLER

A. MPC State Feedback

This section briefly introduces the MPC state feedback κ_N^o which, with some abuse of the term, is referred to as the *optimal state feedback law*. It is based on a finite *prediction horizon* $N > 0$ and a *stage cost function* $\ell : \mathbb{X} \times \mathbb{U} \to \mathbb{R}_{0+}$, penalizing the state and the control input at every predicted step over the horizon (for more details see e.g. [10]).

Assumption III.1 (Cost Function). *The stage cost function ℓ is continuous, $\ell(0,0) = 0$, and it has some lower-bounding K_∞-function α_l,*

$$\ell(\xi,\upsilon) \geq \alpha_l\left(\|\xi\|\right) \qquad \forall\ \xi \in \mathbb{X},\ \forall\ \upsilon \in \mathbb{U}\ .$$

Assumption III.2 (Terminal Set). *(a) There exists a terminal set $\mathcal{X}_f \subset \mathbb{X}$ which is compact, convex, and contains the origin in its interior.*
(b) On $\mathcal{X}_f$, there is a terminal state feedback law *κ_f such that $\mathcal{X}_f$ is a PI set for* (1) *under κ_f.*
(c) There exists a terminal cost function *$\ell_f : \mathcal{X}_f \to \mathbb{R}_{0+}$ which is continuous, $\ell_f(0) = 0$, and it has some upper-bounding K_∞-function α_u,*

$$\ell_f(\xi) \leq \alpha_u\left(\|\xi\|\right) \qquad \forall\ \xi \in \mathcal{X}_f\ .$$

Moreover, ℓ_f is a control Lyapunov function for system (1)*:*

$$\min_{\upsilon\in\mathbb{U}}\left\{\ell_f(A\xi + B\upsilon) - \ell_f(\xi) + \ell(\xi,\upsilon)\right\} \leq 0 \quad \forall\ \xi \in \mathcal{X}_f.$$

The *MPC cost function* $J_N : \mathbb{X} \times \mathbb{U}^N \to \mathbb{R}_{0+}$ is

$$J_N\left(x_{k|k}, \mathbf{u}_k\right) = \sum_{i=0}^{N-1} \ell\left(x_{k+i|k}, u_{k+i|k}\right) + \ell_f\left(x_{k+N|k}\right),$$

where the *predictive model dynamics*

$$x_{k+i+1|k} = Ax_{k+i|k} + Bu_{k+i|k} \quad \forall\ i = 0, ..., N-1 \quad (4)$$

are understood to be substituted recursively in order to remove the dependence on all states other than $x_{k|k}$. This 'sequential approach' is chosen to facilitate the notation.

Remark III.3. *From Assumptions III.2(c) and III.1, and the continuity of the predictive dynamics* (4)*, it follows immediately that the MPC cost function J_N is continuous.*

All of this is assembled into the *MPC Problem*:

$$\min_{\mathbf{u}_k} \quad J_N\left(x_k, \mathbf{u}_k\right) \quad (5a)$$
$$\text{s.t.} \quad x_{k+i+1|k} = Ax_{k+i|k} + Bu_{k+i|k}, \quad x_{k|k} = x_k, \quad (5b)$$
$$\mathbf{u}_k \in \mathbb{U}^N\ , \quad (5c)$$
$$\mathbf{x}_k \in \mathbb{X}^N \times \mathcal{X}_f\ , \quad (5d)$$

where $i \in \{0, ..., N-1\}$. Problem (5) represents an optimization problem parameterized by the initial state x_k; it is solved for a *(full-horizon) input vector* $\mathbf{u}_k$ of the lowest possible cost. Let $\mathcal{X}_N$ denote the set of all initial states for which there exists a solution; $\Pi : \mathcal{X}_N \rightrightarrows \mathbb{U}^N$ be the *feasible set map* and $\Phi : \mathcal{X}_N \rightrightarrows \mathbb{U}^N$ be the *solution map*, i.e. $\Pi(x_k)$ and $\Phi(x_k)$ are the sets of feasible and cost-minimal input vectors, respectively. Moreover, define $\phi : \mathcal{X}_N \to \mathbb{R}_{0+}$ as the *extreme value map*, i.e. $\phi(x_k)$ is the minimal cost at x_k.

The MPC state feedback law $\kappa_N^o(x_k)$ returns the first element (a vector of dimension m) of some input vector from $\Phi(x_k)$ (a vector of dimension Nm).

B. Suboptimal State Feedback

Consider a feasible solution map $s : \mathcal{S}_N \to \mathbb{U}^N$ for the MPC Problem, defined on $\mathcal{S}_N \subset \mathcal{X}_N$, i.e. for $\xi \in \mathcal{S}_N$ $s(\xi)$ returns a feasible (yet not necessarily optimal) point of (5). It is assumed that the evaluation of s is much cheaper than solving the MPC Problem, in terms of computation time and/or storage space.

Assumption III.4 (Sub-Optimal Solution Map). *(a) The map $s : \mathcal{S}_N \to \mathbb{U}^N$ is defined on a compact set $\mathcal{S}_N \subset \mathcal{X}_N$ with the origin in its interior, (b) $s(\xi)$ is feasible for all $\xi \in \mathcal{S}_N$, and (c) s is a continuous function with $s(0) = 0$.*

Assumption III.4 does not suffice to guarantee stability of the closed-loop system if (in analogy to optimal MPC) the sub-optimal controller were to use the first element of $s(x_k)$, as it does not ensure a cost decrease. Feasibility implies stability, in the sense of [12], only if a cost decrease can be ensured—e.g. by further iterations of some descent algorithm. As discussed in [14], in the nominal case a convex combination of the sub-optimal input trajectory with the shifted input sequence from the previous step always achieves a cost decrease. However, this only works if the shifted input sequence remains feasible, which does not hold, in general, in the presence of state disturbances. Therefore some alterations to this approach are introduced in the next section that allow to establish input-to-state stability.

C. Affine Combination Feedback

Definition III.5 (Shift Operator). *For a given $x_{k|k} \in \mathcal{X}_N$, the shift operator $\boldsymbol{\sigma}_{x_{k|k}} : \Pi(x_{k|k}) \to \mathbb{U}^N$ removes the first input element from an input vector and adds a terminal feedback input at its tail:*

$$\boldsymbol{\sigma}_{x_{k|k}}\mathbf{u}_k = \left\{u_{k+1|k},\ ...,\ u_{N-1|k},\ \kappa_f(x_{k+N|k})\right\}\ .$$

Note $\boldsymbol{\sigma}_{x_{k|k}}\mathbf{u}_k$ is feasible for $x_{k+1|k}$, as $\mathbf{u}_k$ is feasible for $x_{k|k}$ and $\kappa_f(x_{k+N|k}) \in \mathcal{X}_f$. For clarity of notation, the index of $\boldsymbol{\sigma}_{x_{k|k}}$ will be omitted, as it is understood from the context.

The modifications proposed to the approach of [14] are twofold. First, the MPC Problem for which the sub-optimal map s (satisfying Assumption III.4) provides a feasible solution is modified. Namely, the state and terminal constraints are tightened by some $\delta > 0$, i.e. $\mathbb{X}$ and $\mathcal{X}_f$ are replaced by $\mathbb{X} \ominus \delta\mathbb{B}^n$ and $\mathcal{X}_f \ominus \delta\mathbb{B}^n$, respectively.

A formulation of the resulting problem, referred to as the *Tightened MPC Problem*, is omitted. Note that the terms 'state constraint set', 'terminal constraint set', or 'feasible' shall remain with respect to the original MPC Problem.

Remark III.6. *δ can be regarded as a design parameter. Together with properties of the system and the sub-optimal solution, it determines the 'ISS gain' in a trade-off against the (maximum) size of the controller domain.*

The second modification is to solve the following *Affine Combination Problem* (similar, but not identical to [14]) online at every step to obtain the sub-optimal control input:

$$\min_{\alpha} \quad J_N(x_k, \mathbf{u}_k) \tag{6a}$$
$$\text{s.t.} \quad \mathbf{u}_k = \alpha\boldsymbol{\sigma}_{x_{k-1|k-1}}\mathbf{u}_{k-1} + (1-\alpha)s(x_k) \ , \tag{6b}$$
$$|\alpha| \leq 1 \ , \tag{6c}$$
$$x_{k+i+1|k} = Ax_{k+i|k} + Bu_{k+i|k}, \quad x_{k|k} = x_k, \tag{6d}$$
$$\mathbf{u}_k \in \mathbb{U}^N \ , \tag{6e}$$
$$\mathbf{x}_k \in \mathbb{X}^N \times \mathcal{X}_f \ , \tag{6f}$$

where $i \in \{0, ..., N-1\}$. Its decision variable is a scalar $\alpha \in [-1, 1]$ that determines an optimal combination of the sub-optimal input vector $s(x_k)$ with the shifted input vector $\boldsymbol{\sigma}\mathbf{u}_{k-1}$. Problem (6) can be initialized by setting $\boldsymbol{\sigma}\mathbf{u}_{-1} := 0$.

Problem (6) includes the parameter $\boldsymbol{\sigma}\mathbf{u}_{k-1} \in \mathbb{U}^N$ in addition to x_k. Indexed by this parameter, let $\Xi_{\boldsymbol{\sigma}\mathbf{u}_{k-1}} : \mathcal{S}_N \rightrightarrows \mathbb{B}^1$ and $\Psi_{\boldsymbol{\sigma}\mathbf{u}_{k-1}} : \mathcal{S}_N \rightrightarrows \mathbb{B}^1$ denote its *feasible set map* and its *solution map*, and $\psi_{\boldsymbol{\sigma}\mathbf{u}_{k-1}} : \mathcal{S}_N \to \mathbb{R}_{0+}$ be its *extreme value map*. The sub-optimal parameterized feedback law $\kappa_N^s(\boldsymbol{\sigma}\mathbf{u}_{k-1}, x_k)$ returns the first element of the input vector obtained from some element of $\Psi_{\boldsymbol{\sigma}\mathbf{u}_{k-1}}$.

Remark III.7. *(a) Unlike the state feedback law κ_N^o, κ_N^s is a parameterized feedback law with the shifted input vectors being the parameters with their own dynamics and contained in a compact set, namely $\mathbb{U}^N$ (compare Section II-B).*
(b) Depending on the disturbance w_{k-1}, $\boldsymbol{\sigma}\mathbf{u}_{k-1} \in \Pi(x_{k|k-1})$ may or may not be feasible for x_k, i.e. it is not necessarily an element of $\Pi(x_k)$.
(c) By virtue of the constraint system of (6)*, any feasible solution to* (6) *corresponds to a feasible input vector for* (5)*. Moreover, a feasible solution to* (6) *always exists, because $s(x_k)$ (corresponding to $\alpha = 0$) is feasible for* (5)*, and even for the Tightened MPC Problem, by Assumption III.4.*

Remark III.8. *In many practical cases* (6) *can be solved analytically. More details on this are found in Section V for the numerical example.*

AS for the proposed controller when $\{w_k\}_{k\in\mathbb{N}} = 0$ follows from the existing theory (e.g. in [14]); moreover, it is a special case of Theorem IV.7. The next section is concerned with proving ISS for the proposed controller.

IV. INPUT-TO-STATE STABILITY

A. Continuity of the Cost Function

The key to obtaining ISS is to prove continuity of the extreme value function, to which this section is dedicated. Note that for the existing approach of [14], the optimal cost may be discontinuous as a result of the shifted input sequence becoming infeasible.

Theorem IV.1. *Choose any $\mathbf{u}_\sigma \in \mathbb{U}^N$. The extreme value function of the Affine Combination Problem*

$$\psi_{\mathbf{u}_\sigma} : \mathcal{S}_N \to \mathbb{R}_{0+}$$

is continuous at any $x_0 \in \mathcal{S}_N$.

The proof is based on Theorem 4.2.1 (1,2) and Lemma 2.2.1 in [1]. Indeed, for any given $\mathbf{u}_\sigma \in \mathbb{U}^N$ it is sufficient that the cost function $J_N(\,\cdot\,, \mathbf{u}_\sigma)$ be continuous, and that the feasible set map $\Xi_{\mathbf{u}_\sigma} : \mathcal{S}_N \rightrightarrows \mathbb{R}$ and the optimal solution map $\Psi_{\mathbf{u}_\sigma} : \mathcal{S}_N \rightrightarrows \mathbb{R}$ be closed (and their images non-empty).

These results shall be proven in a sequence of lemmas, whose proofs are based on a certain geometric perspective on the problem which is now described. Let $x_{\sigma,0}(x_0), ..., x_{\sigma,N}(x_0)$ denote the predicted state trajectory starting at x_0 and driven by the input vector $u_{\sigma,0}, ..., u_{\sigma,N-1}$. Note that while the inputs are fixed (and admissible), each state of the trajectory is a continuous function of $x_0 \in \mathcal{S}_N$, and not necessarily inside $\mathbb{X}$. Similarly, let $u_{\pi,0}(x_0), ..., u_{\pi,N-1}(x_0)$ be the feasible input vector provided by $s(x_0)$, and $x_{\pi,0}(x_0), ..., x_{\pi,N-1}(x_0)$ be the corresponding state trajectory. Note that each input and state is a continuous function of x_0, and the trajectory is feasible (by Assumption III.4).

Consider the state space $\mathbb{R}^n$ at any step $k \in \{0, ..., N\}$. The state (or terminal) constraint set is convex and contains $x_{\pi,k}(x_0)$ (with a distance to the boundary of at least δ, due to the constraint tightening), while not necessarily containing $x_{\sigma,k}(x_0)$. By virtue of the linear dynamics, the affine combination parameter α defines a closed line segment

$$L_k^x := \{\, \alpha x_{\pi,k}(x_0) + (1-\alpha)x_{\sigma,k} \mid |\alpha| \leq 1\} \subset \mathbb{R}^n \ . \tag{7}$$

A similar view holds for the input space $\mathbb{R}^m$ at any step $k \in \{0, ..., N-1\}$. $\mathbb{U}$ is convex, containing both $u_{\pi,k}(x_0)$ and $u_{\sigma,k}(x_0)$ in its interior or on its boundary. Again, the affine combination parameter α defines a closed line segment

$$L_k^u := \{\, \alpha u_{\pi,k}(x_0) + (1-\alpha)u_{\sigma,k} \mid |\alpha| \leq 1\} \subset \mathbb{R}^m \ . \tag{8}$$

For the purpose of clarity, but without loss of generality, assume that $\mathbb{U}$ and $\mathcal{X}_f$ are polytopes and that $\mathbb{X}$ is a polyhedron, i.e. described by a finite number of linear inequalities.

Lemma IV.2. *The feasible set $\Xi_{\mathbf{u}_\sigma}(x_0)$ is a non-empty closed interval $[\underline{\alpha}(x_0), \overline{\alpha}(x_0)] \subset [-1, 1]$ containing $\{0\}$ for any $x_0 \in \mathcal{S}_N$.*

Proof: For each step $k \in \{0, ..., N\}$, the intersection of the closed line segment L_k^x (or L_k^u) with the closed state or terminal constraint set (or input constraint set) is a closed line segment of smaller or equal size. Hence the set of feasible α, with respect to step k, is some closed interval in $[-1, 1]$.

The result is immediate, since the set of α that are feasible with respect to all constraints is given as their (finite) intersection; moreover, $\alpha = 0$ is always feasible, as mentioned in Remark III.7(c). ∎

Next, it is shown that the interval's upper and lower bound vary continuously with x_0 (Lemma IV.3), which is used to establish H-continuity of the feasible set map (Lemma IV.4).

Lemma IV.3. *The limits $\underline{\alpha}(x_0)$ and $\overline{\alpha}(x_0)$ of the feasible interval in Lemma IV.2 are continuous functions of x_0.*

Note that the conditions that α be contained in a compact interval and is upper bounded by 1 are crucial for this proof. Despite all continuity assumptions stated above, there exist simple examples in which the limits $\underline{\alpha}(x_0)$ and $\overline{\alpha}(x_0)$ are discontinuous if this assumption were not satisfied.

Proof: Again, for clarity the limitations on $\underline{\alpha}(x_0)$ and $\overline{\alpha}(x_0)$ imposed at each step $k \in \{0, ..., N\}$ by the state $[\underline{\alpha}_k^x(x_0), \overline{\alpha}_k^x(x_0)]$ and by the input $[\underline{\alpha}_k^u(x_0), \overline{\alpha}_k^u(x_0)]$ are considered separately. If each of them can be shown to be a continuous function of x_0, then so are the maximum and minimum of a finite number of them:

$$\underline{\alpha}(x_0) = \max_k \max\{\underline{\alpha}_k^x(x_0), \underline{\alpha}_k^u(x_0)\} \ ,$$
$$\overline{\alpha}(x_0) = \min_k \min\{\overline{\alpha}_k^x(x_0), \overline{\alpha}_k^u(x_0)\} \ .$$

The input line segment L_k^u is such that $u_{\sigma,k}$ (where $\alpha = 1$) is feasible and fixed with respect to x_0, and $u_{\pi,k}(x_0)$ is feasible and varies continuously with x_0. This allows to deduce the following: (i) Clearly, $\overline{\alpha}_k^u = 1$ for any x_0. (ii) By virtue of the lower bound at -1, $\underline{\alpha}_k^u(x_0)$ varies continuously with x_0, possibly as the intersection of L_k^u with the boundary of the convex set $\mathbb{U}$, even if the points $u_{\sigma,k}$ and $u_{\pi,k}$ coincide.

The state line segment L_k^x is such that $x_{\pi,k}(x_0)$ (where $\alpha = 0$) varies continuously with x_0 and always remains feasible with a distance of at least δ to the boundary of the (state or terminal) constraint set. On the other hand, $x_{\pi,k}(x_0)$ (where $\alpha = 1$) varies continuously with x_0, yet may become infeasible. By virtue of these continuities, both $\underline{\alpha}_k^x(x_0)$ and $\overline{\alpha}_k^x(x_0)$ vary continuously with x_0, possibly as the intersection of L_k^x with the boundary of the convex set $\mathbb{X}$ (or $\mathcal{X}_f$), even if the points $x_{\sigma,k}$ and $x_{\pi,k}$ coincide. ∎

Corollary IV.4. *The feasible set map $\Xi_{\mathbf{u}_\sigma} : \mathcal{S}_N \rightrightarrows \mathbb{B}^1$ is H-continuous at any $x_0 \in \mathcal{S}_N$.*

Proof: Straightforward extension of Lemma IV.3. ∎

Corollary IV.5. *The feasible set map $\Xi_{\mathbf{u}_\sigma} : \mathcal{S}_N \rightrightarrows \mathbb{B}^1$ is closed at any $x_0 \in \mathcal{S}_N$.*

Proof: Immediate consequence of Proposition II.6, given that $\Xi_{\mathbf{u}_\sigma}(x_0)$ is closed (Lemma IV.2) and H-u.s.c. (Corollary IV.4). ∎

Lemma IV.6. *The solution map $\Psi_{\mathbf{u}_\sigma} : \mathcal{S}_N \rightrightarrows \mathbb{B}^1$ is closed at any $x_0 \in \mathcal{S}_N$.*

Proof: In this case, the requirements for closedness of a set-valued map by Definition II.5(a) are verified directly.

Consider any two sequences $\{x_t\}_{t\in\mathbb{N}} \subset \mathcal{S}_N$ and $\{\alpha_t\}_{t\in\mathbb{N}} \subset \mathbb{B}^1$ such that $x_t \to x_0$ and $\alpha_t \to \alpha_0$ as $t \to \infty$ and $\alpha_t \in \Psi_{\mathbf{u}_\sigma}(x_t)$ for all $t \in \mathbb{N}$. It must be proven that $\alpha_0 \in \Psi_{\mathbf{u}_\sigma}(x_0)$.

Notice first that α_0 is feasible, because $\alpha_t \in \Xi_{\mathbf{u}_\sigma}(x_t)$ for all $t \in \mathbb{N}$ and therefore $\alpha_0 \in \Xi_{\mathbf{u}_\sigma}(x_0)$ because $\Xi_{\mathbf{u}_\sigma}$ is closed (Corollary IV.5). It remains to be shown that α_0 minimizes the cost (6a). Suppose there exists $\alpha^\star \neq \alpha_0$ inducing a lower value in the cost function J_N, i.e.

$$J_N\left(x_0, \alpha_0 s(x_0) + (1-\alpha_0)\mathbf{u}_\sigma\right) - J_N\left(x_0, \alpha^\star s(x_0) + (1-\alpha^\star)\mathbf{u}_\sigma\right) = \varepsilon > 0 \ .$$

As will be proven, this contradicts the optimality of some combination (x_t, α_t) that is sufficiently close to (x_0, α_0).

Since J_N and s are continuous, and $x_0 \in \mathcal{S}_N$ and $\alpha \in \mathbb{B}^1$ are contained in compact sets, there exists $\delta_\alpha > 0$ such that

$$|\alpha - \tilde{\alpha}| < \delta_\alpha \quad \Longrightarrow \quad \Big| J_N\left(x, \alpha s(x) + (1-\alpha)\mathbf{u}_\sigma\right) - J_N\left(x, \tilde{\alpha} s(x) + (1-\tilde{\alpha})\mathbf{u}_\sigma\right)\Big| \leq \frac{\varepsilon}{6}$$

for any $x \in \mathcal{S}_N$; and some $\delta_x > 0$ such that

$$\|x - x_0\| < \delta_x \quad \Longrightarrow \quad \Big| J_N\left(x, \alpha s(x) + (1-\alpha)\mathbf{u}_\sigma\right) - J_N\left(x_0, \alpha s(x_0) + (1-\alpha)\mathbf{u}_\sigma\right)\Big| \leq \frac{\varepsilon}{6}$$

for any $\alpha \in \mathbb{B}^1$. Note that there is no mention of feasibility here. The H-l.s.c. of Π (Corollary IV.4), however, guarantees existence of some $\delta_\pi > 0$ such that

$$\Xi_{\mathbf{u}_\sigma}(x_0) \subset U_{\delta_\alpha} \Xi_{\mathbf{u}_\sigma}(x) \qquad \forall \ \ x \in U_{\delta_\pi}\{x_0\} \ . \qquad (10)$$

Pick t large enough such that $\|x_t - x_0\| < \min\{\delta_x, \delta_\pi\}$ and also $\|\alpha_t - \alpha_0\| < \delta_\alpha$. Then clearly

$$\Big| J_N\left(x_t, \alpha_t s(x_t) + (1-\alpha_t)\mathbf{u}_\sigma\right) - J_N\left(x_0, \alpha_0 s(x_0) + (1-\alpha_0)\mathbf{u}_\sigma\right)\Big| \leq \frac{\varepsilon}{3} \ .$$

Moreover, equation (10) implies the existence of some feasible $\tilde{\alpha}^\star$ for x_t which is δ_α-close to $\alpha^\star \in \Xi_{\mathbf{u}_\sigma}(x_0)$. Therefore

$$\Big| J_N\left(x_t, \tilde{\alpha}^\star s(x_t) + (1-\tilde{\alpha}^\star)\mathbf{u}_\sigma\right) - J_N\left(x_0, \alpha^\star s(x_0) + (1-\alpha^\star)\mathbf{u}_\sigma\right)\Big| \leq \frac{\varepsilon}{3} \ ,$$

establishing the contradiction. ∎

This completes the proof of Theorem IV.1.

B. Input-to-State Stability

Let $\Gamma_s \subset \mathcal{S}_N$ be the set of all initial conditions for which the proposed parameterized controller κ_N^s is robustly recursively feasible, i.e. for which the closed-loop trajectory does not leave $\mathcal{S}_N$ for any disturbance sequence $\{w_k\}_{k\in\mathbb{N}} \in \mathbb{W}^\mathbb{N}$. The next theorem (in conjunction with Theorem II.9) proves ISS of the origin on Γ_s, for system (1) under κ_N^s.

Theorem IV.7. *Let $\{\mathbf{u}_k\}_{k\in\mathbb{N}}$ be any sequence of feasible input vectors for $\{x_k\}_{k\in\mathbb{N}}$ resulting from* (6)*. The optimal cost function $\psi_{\boldsymbol{\sigma}\mathbf{u}_{k-1}}$, parameterized by the sequence $\{\boldsymbol{\sigma}\mathbf{u}_{k-1}\}_{k\in\mathbb{N}}$, is a parameterized ISS Lyapunov function for system* (1) *under κ_N^s on Γ_s.*

Proof: The parameter sequence $\boldsymbol{\sigma}\mathbf{u}_{k-1}$ is contained in the compact set $\mathbb{U}^N$. With $\alpha_l(\cdot)$ from Assumption III.1 and $\alpha_u(r) := \sup_{\|\xi\|\le r} J(\xi, s(\xi))$, $\psi_{\boldsymbol{\sigma}\mathbf{u}_{k-1}}$ is lower and upper bounded by two K_∞-functions:

$$\alpha_l(\|\xi\|) \le \psi_{\boldsymbol{\sigma}\mathbf{u}_{k-1}}(\xi) \le \alpha_u(\|\xi\|) \ ,$$

for all $\xi \in \Gamma_s$ and $\boldsymbol{\sigma}\mathbf{u}_{k-1} \in \mathbb{U}^N$.

Because $x_0 \in \Gamma_s$, $x_k \in \mathcal{S}_N$ for all $k \in \mathbb{N}$. For any $x_k \in \mathcal{S}_N$, let $\boldsymbol{\sigma}\mathbf{u}_{k-1} \in \mathbb{U}^N$ be the shifted input vector resulting from (6). It must be shown that the nominal cost decrease is lower bounded by a K_∞-function α_l,

$$\psi_{\boldsymbol{\sigma}\mathbf{u}_k}(Ax_k + Bu_{k|k}) - \psi_{\boldsymbol{\sigma}\mathbf{u}_{k-1}}(x_k) \le -\alpha_l(\|x_k\|) \ ,$$

and an additional cost caused by w_k is upper bounded by some K-function σ,

$$\psi_{\boldsymbol{\sigma}\mathbf{u}_k}\left(Ax_k + Bu_{k|k} + w_k\right) - \psi_{\boldsymbol{\sigma}\mathbf{u}_{k-1}}(x_k) \le \\ -\alpha_l(\|x_k\|) + \sigma(\|w_k\|) \ .$$

The former statement is equivalent to AS in [14]: as $\boldsymbol{\sigma}\mathbf{u}_k$ is feasible for $[Ax_k + Bu_{k|k}]$ (Remark III.7(b)), the optimal input provided by (6) decreases by at least one stage cost.

The latter statement follows as $\psi_{\boldsymbol{\sigma}\mathbf{u}_k}$ is continuous (by Theorem IV.1) on the compact set $\mathcal{S}_N$ for any $\boldsymbol{\sigma}\mathbf{u}_k \in \mathbb{U}^N$, hence it is bounded. Thus the definition

$$\sigma(\omega) := \sup_{\boldsymbol{v}\in\mathbb{U}^N}\left(\sup_{\|w\|\le\omega}\left\{\psi_{\boldsymbol{v}}(\xi+w) - \psi_{\boldsymbol{v}}(\xi)\middle|\xi, (\xi+w) \in \mathcal{S}_N\right\}\right)$$

yields a desired upper-bounding K-function. ■

V. NUMERICAL EXAMPLE

Consider a single mass oscillator with mass $m = 1$, stiffness $k = 5$, damping $d = 0.01$, and controlled by a force $u(t)$; the sampling time is $\Delta t = 0.2$. Let $x_k := [v_k, p_k]^\top$ denote its velocity and position at step k; then the dynamics are described by

$$x_{k+1} = \begin{bmatrix} 0.900 & -0.966 \\ 0.193 & 0.902 \end{bmatrix} x_k + \begin{bmatrix} 0.193 \\ 0.020 \end{bmatrix} u_k + w_k \ . \quad (11)$$

The disturbance support set is chosen as

$$[-0.400, -0.004]^\top \le w_k \le [0.400, 0.004]^\top \ ,$$

and the input and the state constraint sets as

$$-5.000 \le u_k \le 5.000 \ , \quad (12a)$$

$$[-10, -5]^\top \le [v_k, p_k]^\top \le [10, 5]^\top \ . \quad (12b)$$

For a quadratic cost function with $N = 20$, $Q = \mathrm{diag}[1, 10]$, and $R = 2$, cost function and constraints can be written as

$$J_N(x_k, \mathbf{u}_k) = \frac{1}{2}\mathbf{u}_k^\top H \mathbf{u}_k + x_k^\top F^\top \mathbf{u}_k \ , \quad (13)$$

$$G\mathbf{u}_k \le e + Ex_k \ , \quad (14)$$

for appropriate $H \succ 0$, F, G, e, and E (e.g. [2]).

In this particular case the Affine Combination Problem has an analytic solution, obtained by substituting $\alpha\mathbf{u}_\sigma + (1-\alpha)\mathbf{u}_\pi$ into (13). This yields a scalar quadratic equation in α with

$$\alpha_k^\star = \frac{-\left[\mathbf{u}_\pi^\top H + x_k^\top F\right][\mathbf{u}_\sigma - \mathbf{u}_\pi]}{[\mathbf{u}_\sigma - \mathbf{u}_\pi]^\top H [\mathbf{u}_\sigma - \mathbf{u}_\pi]}$$

as its unconstrained minimizer. Substitution of $\alpha\mathbf{u}_\sigma + (1-\alpha)\mathbf{u}_\pi$ into the constraints (14) gives a vector of scalar inequalities for α (in addition to $|\alpha| \le 1$)

$$[G(\mathbf{u}_\sigma - \mathbf{u}_\pi)]\,\alpha \le e + Ex_k - G\mathbf{u}_\pi \ ,$$

defining a (non-empty) closed interval for $\alpha_k \in [\underline{\alpha}, \overline{\alpha}]$. Depending on $\alpha_k^\star$, the optimal solution to (6) is $\underline{\alpha}$, $\alpha_k^\star$, or $\overline{\alpha}$.

In this example, a terminal set with the state feedback of a linear quadratic regulator is employed (e.g. [2]). The constraints are tightened by $\delta = 0.1$. The sub-optimal controller $s(x)$ is provided by interpolation of the optimal solution stored at triangulated sampling points, which are selected as the union of the extreme points of $\mathcal{S}_{20}$ together with 5 randomly placed points in $\mathcal{S}_{20} \setminus \mathcal{X}_f$.

By the results of this paper, system (11) is input-to-state stable under κ_{20}^s on some $\Gamma_s \subset \mathcal{S}_{20}$. Whereas for most practical systems it is prohibitively expensive to compute RPI sets, the set Γ_s can be approximated by simple forward simulation, as illustrated in Figure 1.

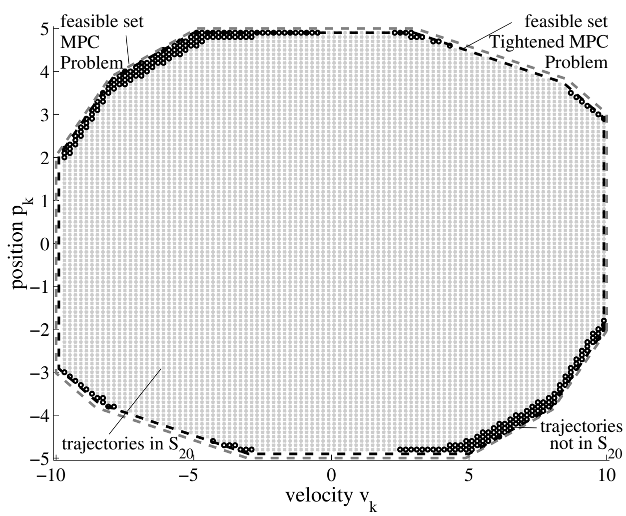

Fig. 1. Initial Conditions Remaining in $\mathcal{S}_N$

Figure 1 displays initial conditions on a grid in $\mathcal{S}_{20}$, marked by a grey cross if all trajectories remained inside $\mathcal{S}_{20}$ within 5 simulated steps, as tested for *all* possible disturbance combinations from the extreme points of $\mathbb{W}$; and by a black circle otherwise. This approximation proved to be highly reliable in all further simulations.

Figure 2 compares the closed-loop trajectories of the optimal MPC controller (grey lines) with those of the sub-optimal controller (black lines) for some initial conditions in $\mathcal{S}_{20}$. Both trajectories are subjected to the same disturbance sequence, selected according to a uniform distribution on $\mathbb{W}$.

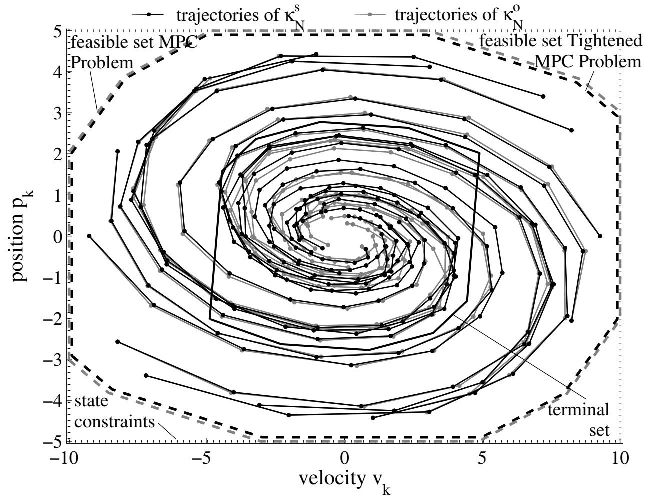

Fig. 2. Sample Trajectories in Phase Plane

The results of a cost analysis are illustrated in Figure 3. It compares the closed-loop costs J_{20}^s incurred by κ_{20}^s as excess percentage over the closed-loop costs J_{20}^o of κ_{20}^o. The first case (light grey bars, referred to as the 'nominal case') is for $\{w_k\}_{k\in\mathbb{N}} = 0$, while in the second case (dark grey bars, referred to as the 'robust case') the trajectories are subjected to a uniformly distributed disturbance sequence in $\mathbb{W}$. In total, 100 random initial conditions are selected for each of 50 different placements of the 5 sampling points, and the simulation was performed for 50 steps.

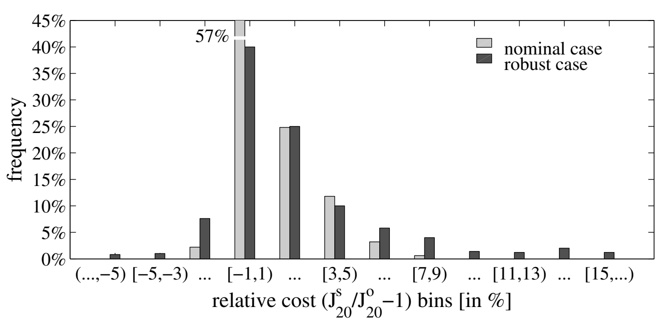

Fig. 3. Sub-optimal vs. Optimal Closed-Loop Cost

In *some instances*, the sub-optimal controller produced lower costs than the optimal controller, while *on average* the former was outperformed by the latter. The deviations in the robust case were generally larger, reaching up to 25%, than in the nominal case, for which 10% was never exceeded.

Figure 4 depicts the corresponding frequency distribution of the optimal parameter value α. In the nominal case the shifted input vector (corresponding to $\alpha = 1$) was used extensively, and only little of the stored input vector (corresponding to $\alpha = 0$) got mixed into the actually applied input. The opposite holds for the robust case, where the controller had to rely more on the stored input vector as a result of the shifted input vector becoming less advantageous or infeasible.

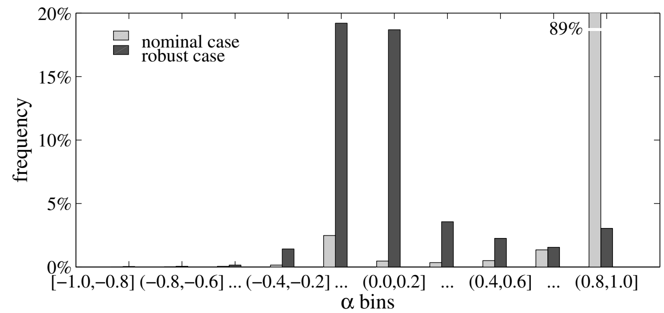

Fig. 4. Frequency Distribution of $\boldsymbol{\alpha}$

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