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Convenient Model for Systems with Hystereses-Control

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Abstract— **We establish a model of a system with hystereses, which allows for standard stability analysis of fixed points and closed orbits. To this end, we represent a system with hystereses as a piecewise-affine switched system that consists of a family of dynamical systems defined on disjoint polyhedral sets. The discrete transitions are realized by reset maps defined on the facets of these polyhedral sets. We have shown that the state space of a resulting switched system is a smooth manifold, the Cartesian product of a torus with an Euclidean space. Additionally, we construct the charts explicitly. Thereby, the analysis of a system with hystereses can be seen as the analysis of a dynamical system on a manifold. This dynamical system in a chart corresponds to a differential equation with discontinuous right hand side, which solution is shown to exist and to be unique.**

I. Introduction

Modeling of systems with hystereses were studied in [1], [2], [3], [4]. In these works, scalar hysteresis operators, including Preisach and Duhem operators, were examined. Furthermore, existence and uniqueness of solutions of ordinary and partial differential equations coupled with hysteresis operators were investigated.

We study in this work systems with multiple hystereses, seen as switched system. By a hysteresis, we understand a binary mechanism that switches the dynamics whenever a state reaches either its upper or lower limits.

As for the study of any phenomena in dynamical systems, the very first challenge is to establish a convenient definition of a state space and a notion of a trajectory such that existing theories may be applied. For this purpose, we model a system with hystereses as a switched system. A switched system is a hybrid system which consists of several subsystems and a rule that orchestrates the switching among them. In this paper, the state space of a switched system is a disjointed union of polyhedral sets. The discrete transitions are realized by reset maps defined on the facets of the polyhedral sets. The reset maps are regarded as generators of an equivalence relation allowed by gluing the polyhedral sets together. The resulting quotient state space is a quotient space. This idea has been used before in [5], [6] and [7]; whereas, the original contribution of this work is to show that the quotient state space of a system with hystereses can be described as a smooth manifold with system dynamics given by piecewise smooth trajectories. If n is the number of states and $m \leq n$ is the number of hystereses, then this manifold is the Cartesian product of an m-torus and an $(n-m)$-dimensional Euclidean space. An advantage of the current approach is that this construction allows for the application of standard methods from analysis of differential equations with discontinuous right hand side [8], [9] in the study of systems with multiple hystereses.

Furthermore, we construct explicitly coordinate charts on the studied manifold. In specific applications, this is a valuable tool, as it allows for concrete analytical studies and numerical computations. For example, stability analysis of a critical point can be carried out by means of Lyapunov stability [9], [10], [11]. Moreover, stability analysis and control synthesis of periodic orbits can be conducted by means of a Poincaré map [12], [13], [14], [15] and [16], [17].

The article is organized as follows. Section II sets up the notation and terminology. Section III is divided into four parts. In the first two subsections, we establish the foundations: a system with hystereses is modeled as a switched system, and system trajectories are defined. The next subsection contains the first main result stating that the quotient state space of the resulting switched system is a smooth manifold. The final subsection contains the second main result, which shows that the dynamics of a system with hystereses can be equivalently expressed as a dynamical system on this manifold.

II. Preliminaries

We write $F \preceq P$ to indicate that F is a face of a polyhedral set P, and $F \prec P$ to indicate that F is a proper face of P ($F \preceq P$ and $F \neq P$). A map $f : P \to P'$ is said to be polyhedral if it is a continuous injection, and if for any $F \preceq P$ there is $F' \preceq P'$ with $\dim(F) = \dim(F')$ such that $f(F) \subseteq F'$.

For a given subset U of a topological space X, by $\mathrm{cl}(U)$, $\mathrm{int}(U)$ and $\mathrm{bd}(U)$ we denote the closure, the interior and the boundary of U in X, respectively.

We denote by $\mathbb{R}_+$ the set of non-negative reals $[0,\infty[$, by $\mathbb{Z}_+$ the set of non-negative integers $\{0, 1, 2, \ldots\}$, and by $\mathbb{N}$ the set of natural numbers $\{1, 2, \ldots\}$.

III. Hystereses as Switched Systems

We consider an n dimensional system with m hystereses

$$\dot{x} = \xi(x; \delta_1, \ldots, \delta_m) = \xi_\delta(x). \tag{1}$$

The value of each δ_i is either 0 or 1 and is determined by m hystereses

$$\delta_i = \begin{cases} 1 & \text{if } x_i \geq X_i^u \\ 0 & \text{if } x_i \leq X_i^l \\ \delta_i & \text{if } X_i^l < x_i < X_i^u, \end{cases} \tag{2}$$

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where X_i^u and X_i^l are the predefined, respectively, upper and lower limit for x_i. By convention, $\delta_i = 0$ for any initial condition $x_i(t_0) \in]X_i^l, X_i^u[$ for (1).

A. System with Hystereses as a Switched System

To begin with, we consider the following scenario. Let

$$x(t_0) \in \;]X_1^l, X_1^u[\times \ldots \times]X_m^l, X_m^u[\times \mathbb{R}^{n-m};$$

hence, by convention $\delta = 0$. Suppose now that at time t, x_i reaches the upper limit X_i^u, then $\delta_i = 1$. This scenario indicates that the system with m hystereses comprises 2^m dynamical systems

$$\dot{x} = \xi_\delta(x), \; \delta \in \mathbf{2}^m, \tag{3}$$

with $\mathbf{2} = \{0, 1\}$ and $\xi_\delta(x) = \xi(x; \delta)$, defined on the polyhedral set

$$Q = [X_1^l, X_1^u] \times \ldots \times [X_m^l, X_m^u] \times \mathbb{R}^{n-m}. \tag{4}$$

A discrete transition between these systems takes place whenever a trajectory reaches the boundary of Q. This description goes with the concept of a switched system.

Definition 1 (Switched System): A switched system (of dimension n) is a triple $(\mathcal{P}, \mathcal{S}, \mathcal{R})$ where

- $\mathcal{P} = \mathcal{P}_D$ is a finite family of polyhedral sets $\mathcal{P} = \{P_\delta \subset \mathbb{R}^n \mid P_\delta \text{ a polyhedral set}, \dim(P_\delta) = n, \; \delta \in D\}$, and D is a finite index set.
- $\mathcal{S}$ is a finite family of smooth vector fields $\mathcal{S} = \{\xi_\delta : P_\delta \to \mathbb{R}^n \mid P_\delta \in \mathcal{P}, \; \delta \in D\}$.
- $\mathcal{R} = \mathcal{R}_J$ is a finite family of polyhedral maps, called reset maps, $\mathcal{R} = \{R_j : F \to F' \mid F \prec P \in \mathcal{P}, \; F' \prec P' \in \mathcal{P}, \; \dim(F) = \dim(F') = n-1, \; j \in J\}$, and J is a finite index set.

Next, we demonstrate that a system with m hystereses is a switched system $(\mathcal{P}, \mathcal{S}, \mathcal{R})$. The set $\mathcal{P}$ consists of 2^m copies of the polyhedral set Q in (4)

$$\mathcal{P} = \{P_\delta = Q \times \{\delta\} | \; \delta \in \mathbf{2}^m\}. \tag{5}$$

Formally, in (5), we have separated (made disjoint) each of the copies of Q.

Let $\delta^0[a, b] = \{a\}$, and $\delta^1[a, b] = \{b\}$. For a system with m hystereses, the facets on the polyhedral set

$$Q = [X_1^l, X_1^u] \times \ldots \times [X_i^l, X_i^u] \times \ldots \times [X_m^l, X_m^u] \times \mathbb{R}^{n-m} \tag{6}$$

are

$$\begin{aligned} F_i^\alpha = F_i^\alpha(Q) = [X_1^l, X_1^u] \times \ldots \times \delta^\alpha[X_i^l, X_i^u] \times \ldots \\ \ldots \times [X_m^l, X_m^u] \times \mathbb{R}^{n-m} \end{aligned} \tag{7}$$

for $i \in \{1, \ldots, m\}$ and $\alpha \in \mathbf{2}$. The facet operators commute in the following sense

$$F_i^\alpha \circ F_j^\beta(Q) = F_{j-1}^\beta \circ F_i^\alpha(Q), \quad i < j, \; \alpha, \beta \in \mathbf{2}.$$

Example 1: For a three dimensional system with two hystereses, the polyhedral set Q has four facets ($\alpha \in \mathbf{2}$)

$$\begin{aligned} F_1^\alpha = F_1^\alpha(Q) &= \delta^\alpha[X_1^l, X_1^u] \times [X_2^l, X_2^u] \times \mathbb{R} \\ F_2^\alpha = F_2^\alpha(Q) &= [X_1^l, X_1^u] \times \delta^\alpha[X_2^l, X_2^u] \times \mathbb{R} \end{aligned}$$

This particular case will be used to exemplify the concepts introduced throughout the paper.

The set $\mathcal{S}$ consists of 2^m systems given by (3). Whereas, the set $\mathcal{R}$ of reset maps is

$$\mathcal{R} = \{R_i(\delta) | \; (i, \delta) \in \{1, \ldots, m\} \times \mathbf{2}^m\}.$$

To define the maps $R_i(\delta)$, let a map $l : \{1, \ldots, m\} \times \mathbf{2}^m \to \mathbf{2}^m$ be given by

$$l(i, \delta) = (\delta_1, \ldots, \delta_{i-1}, \delta_i + 1, \delta_{i+1}, \ldots, \delta_m),$$

where the results of the summation are computed modulo 2. Intuitively, the map l takes a polyhedral set enumerated by δ to the future polyhedral set. Notice that only ith coordinate of δ is changed, which indicates that the switching takes place as a result of x_i having reached its upper or lower boundary.

The reset map

$$R_i(\delta) : F_i^{l_i(i,\delta)} \times \{\delta\} \to F_i^{l_i(i,\delta)} \times \{l(i, \delta)\}$$

is subsequently defined by

$$R_i(\delta)\,(x, \delta) = (x, l(i, \delta)).$$

Example 2: To characterize the set $\mathcal{R}$ of a three dimensional system with two hystereses, the map l is given by

$$l(i, \delta) = (l_1(i, \delta), l_2(i, \delta)) = \begin{cases} (\delta_1 + 1, \delta_2) & \text{if} \quad i = 1 \\ (\delta_1, \delta_2 + 1) & \text{if} \quad i = 2. \end{cases}$$

The set $\mathcal{R} = \{R_i(\delta) | \; (i, \delta) \in \{1, 2\} \times \mathbf{2}^2\}$ consists of eight reset maps. The switched system is illustrated in Fig. 1. Here, each element of Q has been (orthogonally) projected onto the (x_1, x_2)-space. Hence, the polyhedral sets P_δ are represented by squares. The three squares $P_{(0,1)}$, $P_{(1,0)}$, $P_{(1,1)}$ have been vertically and/or horizontally reflected. The stippled lines in the drawing indicate the reset maps in $\mathcal{R}$.

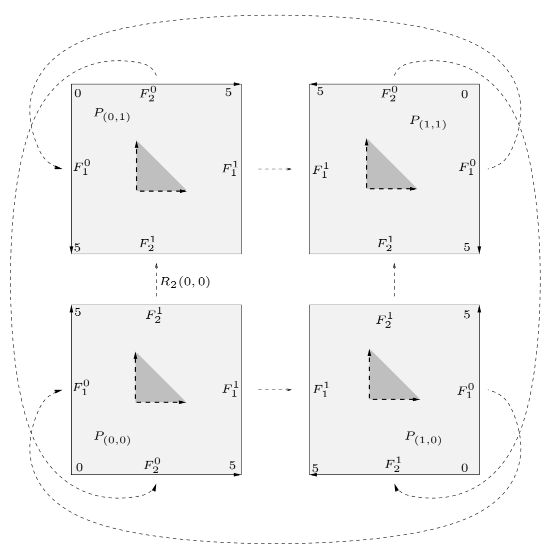

Fig. 1. The x_1x_2-state space of a system with two hystereses, where for clarity of illustration we assume $(X_i^l, X_i^u) = (0, 5)$. The reset maps are indicated by the stippled lines. The last $(n - m)$ coordinates have been suppressed; thus, each $P_\delta = Q \times \{\delta\}$ is illustrated by a square. By abuse of notation, the facets of P_δ are denoted by F_i^α (instead of $F_i^\alpha \times \{\delta\}$).

B. Trajectories of a System with Hystereses

A vital object for studying the behavior of any dynamical system is its trajectory. In order to introduce the notion of a trajectory of a switched system, we bring in a concept of a time domain.

In the following, we denote sets of the form $\{a, \dots\}$ with $a \in \mathbb{Z}_+$ as $\{a, \dots, \infty\}$. Let $k \in \mathbb{N} \cup \{\infty\}$; a subset $\mathcal{T}_k \subset \mathbb{R}_+ \times \mathbb{Z}_+$ will be called a time domain if there exists an increasing sequence $\{t_i\}_{i\in\{0,\dots,k\}}$ in $\mathbb{R}_+ \cup \{\infty\}$ such that

$$\mathcal{T}_k = \bigcup_{i\in\{1,\dots,k\}} (T_i \times \{i\})$$

where $T_i = [t_{i-1}, t_i]$ if $i \in \{1, \dots, k-1\}$, and

$$T_k = \begin{cases} [t_{k-1}, t_k] & \text{if } t_k < \infty \\ [t_{k-1}, \infty[& \text{if } t_k = \infty. \end{cases}$$

Note that $T_i = [t_{i-1}, t_i]$ for all i if $k = \infty$. We say that the time domain is infinite if $k = \infty$ or $t_k = \infty$. The sequence $\{t_i\}_{i\in\{0,\dots,k\}}$ corresponding to a time domain will be called a switching sequence.

Definition 2 (Trajectory): A trajectory of the switched system $(\mathcal{P}_D, \mathcal{S}, \mathcal{R}_J)$ is a pair $(\mathcal{T}_k, \gamma)$, where $k \in \mathbb{N} \cup \{\infty\}$ is fixed, and

- $\mathcal{T}_k \subset \mathbb{R}_+ \times \mathbb{Z}_+$ is a time domain with corresponding switching sequence $\{t_i\}_{i\in\{0,\dots,k\}}$,
- $\gamma : \mathcal{T}_k \to X = \bigcup_{\delta\in D} P_\delta$ is continuous (X has the disjoint union topology) and satisfies:
  1) For each $i \in \{1, \dots, k-1\}$, there exist $\delta \neq \delta' \in D$ such that $\gamma(t_i; i) \in \mathrm{bd}(P_\delta)$, and $\gamma(t_i; i+1) \in \mathrm{bd}(P_{\delta'})$.
  2) For each $i \in \{1, \dots, k\}$, there exists $\delta \in D$ such that the Cauchy problem
  $$\frac{\partial}{\partial t}\gamma(t; i) = \dot{\gamma}(t; i) = \xi_\delta(\gamma(t; i)), \qquad (8)$$
  $$\gamma(t_{i-1}; i) \in P_\delta$$
  has a solution on $T_i \subset \mathcal{T}_k$.
  3) For each $i \in \{1, \dots, k-1\}$, there exists $j \in J$ such that $R_j(\gamma(t_i; i)) = \gamma(t_i; i+1)$.

A trajectory at x is a trajectory $(\mathcal{T}_k, \gamma)$ with $\gamma(t_0; 1) = x$.

C. State Space as a Manifold

To study any dynamical system, the starting point is a convenient definition of the state space. It was suggested in [6] and [7] to glue the state spaces of respective subsystems of a switched system together on the subsets identified by the reset maps. We adapt this concept in this article, and additionally, we impose a differentiable structure on the resulting space. This is essential for any analysis of dynamical systems.

For a switched system $(\mathcal{P}, \mathcal{S}, \mathcal{R})$, we define the state space which is the union of polyhedral sets

$$X = \bigcup_{\delta\in\mathbf{2}^2} P_\delta.$$

with disjoint union topology. Let $\sim \subset X \times X$ be the equivalence relation generated by the relations $x \sim R(x)$ for all reset maps $R \in \mathcal{R}$ and all points x in the domain of R. The quotient state space of the switched system is the quotient space

$$X^* = X/\sim,$$

which we will refer to in the sequel simply as the state space. The equivalence class of $x \in X$ is denoted by

$$[x] = \{y \in X |\ \exists R_1, \dots R_l \in \mathcal{R} \cup \mathcal{R}^{-1} \text{ such that } y = R_l \cdot \dots \cdot R_1(x)\}$$

with $\mathcal{R}^{-1} = \{R^{-1} |\ R \in \mathcal{R}\}$.

Example 3: In particular, for a three dimensional system with two hystereses, let $x \in X$. If $x \in \mathrm{int}(P_\delta)$ for some $\delta \in \mathbf{2}^2$, the equivalence class $[x] = \{(x; \delta)\}$. If $x \in \mathrm{int}(F_1^1 \times \{(0,0)\})$, then $[x] = \{(x; 0, 0), (x; 1, 0)\}$, and if $x \in (F_1^1 \cap F_2^1) \times \{(0,0)\}$, we have $[x] = \{(x; 0, 0), (x; 1, 0), (x; 0, 1), (x; 1, 1)\}$. Furthermore, X^* is the product of a 2-torus with the reals, $X^* = \mathbf{T}^2 \times \mathbb{R}$; Fig. 1 illustrates this situation.

Theorem 1: The state space X^* of an n-dimension system with m hystereses is the Cartesian product of an m-torus with $(n-m)$-dimensional Euclidean space, $X^* = \mathbf{T}^m \times \mathbb{R}^{n-m}$.

Proof: We only give a sketch of proof since details involve lengthy but straight forward combinatorics.

Let $I_i(\delta)$ denote the interval $[X_i^l, X_i^u]$ in the definition of P_δ in (6) and (5). Note that identifications take place between elements of $I_i(\delta)$ and $I_j(\delta')$ iff $i = j$ and $\delta \neq \delta'$. Therefore, without loss of generality, we can restrict our attention to $I_i(\delta),\ \delta \in \mathbf{2}^m$ for fixed $i \in \{1, 2, \dots, m\}$.

Consider an $I_i(\delta)$ with $\delta_i = 0$. By means of the reset maps $R_j(\delta),\ j \neq i$, the whole interval $I_i(\delta)$ is identified with $m-1$ intervals $I_i(\delta')$ with $\delta'_i = 0$ and $\delta'_j = \delta_j$ except for precisely one j. This procedure also applies for each of the intervals $I_i(\delta')$. Continuing this way, we conclude that the 2^{m-1} intervals corresponding to the deltas having values 0 at the i'th entry are all identified. The same conclusion also holds for the 2^{m-1} intervals corresponding to deltas having 1 at the i'th entry. Therefore, we are left with only two intervals corresponding to $\delta_i = 0$ and $\delta_i = 1$. These are identified at their endpoints, by reset maps $R_i(\delta),\ \delta \in \mathbf{2}^m$, which gives rise to a one-sphere. This completes the proof. ■

In the following, we will explicitly construct a differentiable structure on the state space X^* of a system with hystereses. At the outset, we define a set

$$K^l = \{F \preceq P_\delta |\ P_\delta \in \mathcal{P},\ \dim(F) = l\},$$

and for $S \subset Q$ with Q given in (4). For an arbitrary but fixed δ', let

$$K^l(S) = \{F \in K^l |\ \pi^{-1} \circ \pi(S \times \delta') \cap F = \emptyset\}$$

with the map $\pi : X \to X^*$ denoting the canonical projection $\pi(x) = [x]$. Hence, the set $K^l(S)$ consists of all l-dimensional faces in $\mathcal{P}$ that do not contain points identified, via $\sim$, with $S \times \{\delta'\}$, i.e., all l-dimensional faces $F \in \mathcal{P}$ such that $[F] \cap [S \times \{\delta'\}] = \emptyset$.

Example 4: In particular, for $n = 3$ and $m = 2$,

$$K^3 = \mathcal{P}, \{F_i^\alpha \times \{\delta\} | \ (\alpha, i, \delta) \in \mathbf{2} \times \{1, 2\} \times \mathbf{2}^2\} \subset K^2,$$

and

$$K^2 \left(F_1^{\sigma_1} \circ F_2^{\sigma_2}(Q)\right), \quad \text{for } \sigma \in \mathbf{2}^2$$

contains all the facets of polyhedral sets in $\mathcal{P}$ that do not contain the line

$$F_1^{\sigma_1} \circ F_2^{\sigma_2}(Q) = \delta^{\sigma_1}[X_1^l, X_1^u] \times \delta^{\sigma_2}[X_2^l, X_2^u] \times \mathbb{R}$$

For $\sigma \in \mathbf{2}^m$, let

$$U_\sigma = X - K^{n-1}\left(F_1^{\sigma_1} \circ \ldots \circ F_m^{\sigma_m}(Q)\right),$$

and define $\psi_\sigma : U_\sigma \to \mathbb{R}^n$ by $\psi_\sigma(x; \delta) = J_\delta(x - r_\sigma)$, where J_δ is n by n diagonal matrix

$$(J_\delta)_{ij} = \begin{cases} 0 & \text{for } i \neq j \\ (-1)^{\delta_i} & \text{for } i = j, \ i \leq m \\ 1 & \text{for } i = j, \ i > m \end{cases}$$

and

$$(r_\sigma)_i = \begin{cases} X_i^u \sigma_i + X_i^l(1 - \sigma_i) & \text{for } i \leq m \\ 0 & \text{for } i > m. \end{cases}$$

Specifically, the new coordinates $y = \psi_\sigma(x)$ for $i \in \{1, \ldots, m\}$ are

$$y_i = (-1)^{\delta_i}(x_i - X_i^u \sigma_i - X_i^l(1 - \sigma_i)),$$

and thus $y_i \in]X_i^l - X_i^u, X_i^u - X_i^l[$. We refer to the pair (U_σ, ψ_σ) defined above as a chart on X. Figure 2 illustrates the image of ψ_σ for a system with two hystereses.

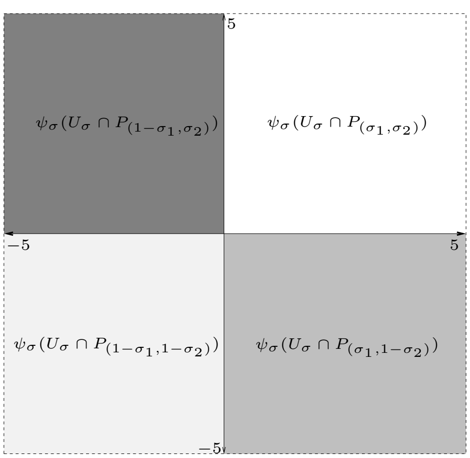

Fig. 2. For $(X_i^l, X_i^u) = (0, 5)$, the set $\psi_\sigma(U_\sigma) =]-5, 5[\times]-5, 5[$ is projected to the y_1y_2-space. Each of the four the quadrants indicates where ψ_σ maps the polyhedral set $U_\sigma \cap P_\delta$ to.

Proposition 1: Let $\sigma \in \mathbf{2}^m$.

1) The sets U_σ and $U_\sigma^* = \pi(U_\sigma)$ are open in X (with disjoint union topology) and X^* (with quotient topology), respectively. Moreover, $X^* = \bigcup_{\sigma \in \mathbf{2}^m} U_\sigma^*$ and $X = \bigcup_{\sigma \in \mathbf{2}^m} U_\sigma$.
2) For any x and $x' \in U_\sigma$, $\psi_\sigma(x) = \psi_\sigma(x')$ if and only if $x \sim x'$. Moreover, ψ_σ is a continuous.
3) For any $P \in \mathcal{P}$, the restriction $\psi_\sigma|_{U_\sigma \cap P}$ is a restriction of an affine isomorphism $\mathbb{R}^n \to \mathbb{R}^n$.
4) Let $V = \psi_\sigma(U_\sigma)$. There is a homeomorphism ψ_σ^* completing the diagram ($\psi_\sigma = \psi_\sigma^* \circ \pi$)

$$\begin{tikzcd} U_\sigma \arrow[r, "\pi_\sigma"] \arrow[d, "\psi_\sigma"'] & U_\sigma^* \arrow[dl, dotted, "\psi_\sigma^*"] \\ V & \end{tikzcd} \tag{9}$$

Proof: Properties 1), 2) and 3) follow immediately from the definitions. Whereas, Property 4) follows from Corollary 22.3 in [18]. ∎

We make the following two observations based on Proposition 1. For any $\sigma, \sigma' \in \mathbf{2}^m$ and any $\delta \in \mathbf{2}^m$, the composition $\psi_{\sigma'}^* \circ \psi_\sigma^{*-1}\big|_{\psi_\sigma(U_\sigma \cap U_{\sigma'} \cap P_\delta)}$ is an affine isomorphism. Thus, $\psi_{\sigma'}^* \circ \psi_\sigma^{*-1}$ is piecewise affine on $\psi_\sigma(U_\sigma \cap U_{\sigma'})$. Moreover, the family of (affine) charts $\{(U_\sigma^*, \psi_\sigma^*)\}_{\sigma \in \mathbf{2}^m}$ constitutes a differentiable structure on X^*.

Henceforth, we refer to a pair $(U_\sigma^*, \psi_\sigma^*)$ defined in diagram (9) as an affine chart on X^*.

So far, we have described the state space for a system with hystereses as a single smooth manifold with explicitly given charts. Now, we are ready for the next step, which is to formulate the dynamics of the system on the resulting manifold.

D. Dynamics on the State Space Manifold

In the present case, the resulting manifold has a partition induced by the polyhedral sets of the switched system. In each cell of the partition, the dynamics is smooth but discontinuous on the facets. This complicates the analysis via local charts beyond the theory of smooth dynamical systems. To resolve this problem, we employ the concept of a local switched system, which is defined on the image of the local chart. The next definition formalizes this notion.

Definition 3 (Local Switched System): A local switched system (of dimension n) is a triple $(W, \mathcal{C}, \mathcal{F})$ where

- W is a polyhedral set of dimension n in $\mathbb{R}^n$.
- $\mathcal{C} = \{Q_i | \ i \in I\}$ is a family of polyhedral sets which partition W.
- $\mathcal{F}$ is a family of smooth functions $\mathcal{F} = \{f_i : Q_i \to \mathbb{R}^n \mid i \in I\}$.

The dynamics of the local switched system $(W, \mathcal{C}, \mathcal{F})$ is governed by the following differential inclusions

$$\dot{y}(t) \in F(y(t)) \text{ (almost everywhere)} \tag{10}$$

where the set valued map F is defined by

$$F : \ W \to 2^{\mathbb{R}^n}; \ y \mapsto \{v \in E | \ v = f_i(y) \text{ if } y \in Q_i\}$$

with $2^{\mathbb{R}^n}$ the power set of $\mathbb{R}^n$. Basic properties and stability of local switched system were studied in the authors previous work [20]. In the next proposition, we show that the system with hystereses looks locally like a local switched system; recall that the dynamics of the system is given by ξ_δ in (3).

Proposition 2: Let (U_σ, ψ_σ) be a chart on X. For any n-dimensional polyhedral set $W \subset V = \psi_\sigma(U_\sigma)$, let

$$\begin{aligned}\mathcal{C} =& \{Q_\delta |\ Q_\delta = W \cap \psi_\sigma(P_\delta \cap U_\sigma),\ \delta \in \mathbf{2}^m\}\\ \mathcal{F} =& \{f_\delta : Q_\delta \to \mathbb{R}^n \mid f_\delta = D\psi^\delta \xi_\delta \circ (\psi^\delta)^{-1},\\ &\qquad \psi^\delta = \psi_\sigma|_{\psi_\sigma^{-1}(Q_\delta)},\ \delta \in \mathbf{2}^m\}\end{aligned}$$

then the triple $(W, \mathcal{C}, \mathcal{F})$ is a local switched system.

Figure 3 illustrates Proposition 2 for a system with two hystereses.

Proof: Since $W \subset V$ and $U_\sigma \subset \bigcup_{\delta \in \mathbf{2}^m} P_\delta$ we have $W = \bigcup_{\delta \in \mathbf{2}^m} Q_\delta$. Hence, to complete the proof we need to show that the Q_δ's are indeed polyhedral sets, but this follows directly from Property 3) of Proposition 1. ∎

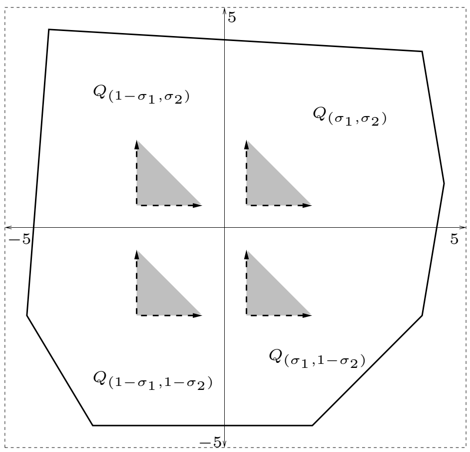

Fig. 3. The figure illustrates the set W in Proposition 2. The set W, bounded by the thick line, is inside $V = \psi_\sigma(U_\sigma)$, the square, in the x_1x_2-space. Here, $(X_i^l, X_i^u) = (0, 5)$. The direction of the vector field f_δ satisfying Condition (14) in Proposition 4 is indicated by the dark shaded triangles.

We shall call a triple $(W, \mathcal{C}, \mathcal{F})$ as in Proposition 2 a local switched system generated by the chart (U_σ, ψ_σ).

We calculate the vector fields in $\mathcal{F}$ explicitly. The vector fields $f_\delta \in \mathcal{F}$ are given by

$$f_\delta(y) = J_\delta \xi_\delta(x), \qquad \text{where } x = J_\delta^{-1} y + r_\sigma.$$

and in coordinates

$$\langle f_\delta(y), e_i\rangle = \begin{cases}(-1)^{\delta_i}\langle \xi_\delta(x), e_i\rangle & \text{for } 1 \le i \le m\\ \langle \xi_\delta(x), e_i\rangle & \text{for } m < i \le n,\end{cases} \tag{11}$$

where $e_i, i \in \{1, \ldots, n\}$, are canonical basis vectors in $\mathbb{R}^n$, $e_1 = (1, 0, \ldots, 0), \ldots, e_n = (0, \ldots, 0, 1)$. In particular, ξ_δ is transversal to a facet F of P_δ at $x \in F$ if and only if f_δ is transversal to the facet $\psi_\sigma(F)$, provided x is in the domain of ψ_σ.

In the next proposition, we characterize solutions of a local switched system generated by a chart. A crucial question is whether such a solution is the same as a solution of the original switched system. Indeed, the next proposition shows that in any chart, a trajectory of a system with hystereses is exactly the solution of the local switched system generated by this chart. Thus, the analysis of the system with hystereses can be carried out in the charts covering the space X.

Proposition 3: Let $(W, \mathcal{C}, \mathcal{F})$ be a local switched system generated by the chart (U_σ, ψ_σ) on X (of the switched system $(\mathcal{P}, \mathcal{S}, \mathcal{R})$), and let $(\mathcal{T}_k, \gamma)$ be a trajectory of $(\mathcal{P}, \mathcal{S}, \mathcal{R})$ with $\gamma(t_0; 1) = x_0$ such that $\gamma(t; i) \in \psi_\sigma^{-1}(W)$ for all $(t; i) \in \mathcal{T}_k$. Then

$$y(t) = \psi_\sigma(\gamma(t; i)) \tag{12}$$

is a solution of the Cauchy problem

$$\dot{y}(t) \in F(y(t)) \text{ a.e.},\ y(0) = \psi_\sigma(x_0). \tag{13}$$

Conversely, if $y(t)$ is a solution of the Cauchy problem (13) in W then there is a trajectory $(\mathcal{T}_k, \gamma)$ of the switched system $(\mathcal{P}, \mathcal{S}, \mathcal{R})$ such that (12) holds.

Proof: Let $\{t_i\}_{i \in \{0,\ldots,k\}}$ be the switching sequence corresponding to $\mathcal{T}_k$. For each $i \in \{1, ..., k\}$ there exists $\delta \in \mathbf{2}^m$ such that $\dot{\gamma}(t; i) = \xi_\delta(\gamma(t; i))$ for all $t \in [t_{i-1}, t_i]$. Recall that

$$\psi^\delta = \psi_\sigma|_{\psi_\sigma^{-1}(Q_\delta)},\ \delta \in \mathbf{2}^m.$$

The vector fields $f_\delta \in \mathcal{F}$ and $\xi_\delta \in \mathcal{S}$ are ψ^δ-related, i.e.,

$$f_\delta = D\psi^\delta \xi_\delta \circ (\psi^\delta)^{-1}.$$

Thus γ and the solution y of the Cauchy problem $\dot{y}(t) = f_\delta(y(t)), y(t_{i-1}) = \psi_\sigma(\gamma(t_{i-1}; i))$ commute in the following sense $\psi_\sigma(\gamma(t; i)) = y(t)$ for $t \in [t_{i-1}, t_i]$. This completes the first part of the proof since $\psi_\sigma(\gamma(t_i; i)) = \psi_\sigma(\gamma(t_i; i+1))$ by Property 3) in Definition 2.

To prove the second statement, let y be the solution of the Cauchy problem (13) on $[0, T]$ with $0 < T < \infty$, and define $\bar{\delta} : [0, T] \to \mathbf{2}^m$ a.e. by

$$\bar{\delta}(t) = \delta \text{ if and only if } \dot{y}(t) = D\psi^\delta \xi_\delta \circ (\psi^\delta)^{-1}(y(t)).$$

Let $\{t_i\}_{i \in \{0,..,k\}}$ with $k \in \mathbb{N}_+ \cup \{\infty\}$ be the increasing sequence of points in $[0, T]$ where $\bar{\delta}$ is not well defined. Hence, $\bar{\delta}$ is constant on $I_{i+1} =]t_i, t_{i+1}[$, and so it may be (trivially) extended to a continuous map on $\mathrm{cl}(I_i)$. Now, for each $i \in \{1, \ldots, k\}$, we define the map γ_i on I_i by $\gamma_i(t) = (\psi^{\bar{\delta}(t)})^{-1}(y(t))$ and extend it to a continuous map on $\mathrm{cl}(I_i)$. Hence, the time derivative of γ_i on I_i is

$$\dot{\gamma}_i(t) = \xi_{\bar{\delta}(t)}(\gamma_i(t)).$$

The trajectory $(\mathcal{T}_k; \gamma)$ is now defined by $\mathcal{T}_k = \bigcup_{i \in \{1,\ldots,k\}} \mathrm{cl}(I_i)$ and $\gamma(t; i) = (\gamma_i(t), \bar{\delta}(t))$. ∎

In the next proposition, we show that under the following condition

$$\begin{cases}\langle \xi(x; \delta), e_i\rangle\langle \xi(x; l(i, \delta)), e_i\rangle < 0\\ \forall i \in \{1, \ldots, m\},\ \forall \delta \in \mathbf{2}^m, \forall x \in F_i^{l_i(i,\delta)} \times \{\delta\}\end{cases} \tag{14}$$

the solution of the Cauchy problem in (13) is unique. Condition (14) means that at any interior point of each facet of any polyhedral set P_δ, after gluing polyhedral sets together, there is exactly one vector field pointing into and one vector field pointing out of P_δ. We find this condition natural for systems with hystereses as switching provoked by

x_i crossing the upper limit X_i^l or the lower limit X_i^u should change the direction of the flow to the opposite. Specifically, if x_i were increasing before the switching then it would decrease after the switching, and vice versa.

Proposition 4: Suppose that Condition (14) holds. Let $(W, \mathcal{C}, \mathcal{F})$ be a local switched system generated by the chart (U_σ, ψ_σ). Then, for any $y_0 \in \mathrm{int}(W)$, there exists a unique solution at y_0. That is, there exist $0 < T < \infty$ and a unique absolutely continuous function $y : [0, T] \to W;\ t \mapsto y(t)$ which solves the Cauchy problem

$$\dot{y}(t) \in F(y(t)) \text{ a.e.},\ y(0) = y_0.$$

Proof: If $y_0 \in \mathrm{int}(Q_\delta)$ for some δ, then there is an open neighborhood O of y_0 such that $F(y)$ is a singleton for any $y \in O$ thus the proposition follows from the Picard-Lindelöf Theorem.

If $y_0 \notin \mathrm{int}(Q_\delta)$, for any δ, then $y_0 \in \{(a, b) \in \mathbb{R}^m \times \mathbb{R}^{n-m} \mid \exists i \in \{1, \ldots, m\} \text{ such that } a_i = 0\}$. Hence, from Condition (14) and equation (11), we conclude that $F(y_0) \cap T_{Q_\delta}(x_0) = \{f_\delta(y_0)\}$ for some δ, where, in generic notation, $T_S(x)$ denotes the contingent cone to the convex set S at $z \in S$. In other words, $T_S(x)$ is the closure of the cone spanned by $S \setminus \{z\}$. This implies the existence of a unique solution at y_0. ∎

We have the following corollary form the proof of Proposition 4.

Corollary 1: If F in Proposition 4 is replaced by $F^c : W \to 2^{\mathbb{R}^n};\ y \mapsto \mathrm{co}(F(y))$, where $\mathrm{co}(F(y))$ is the convex hull of $F(y)$, then the statement of the proposition still holds. Furthermore, the unique solutions of the two Cauchy problems $\dot{y}(t) \in F(y(t))$ and $\dot{y}(t) \in F^c(y(t))$ with $y(0) = y_0$ coincide.

By combining Proposition 4 with Proposition 3, we conclude under inherent assumptions that the trajectories of a system with hystereses can be represented uniquely as solutions of a local switched system generated by a chart. This is formalized in the theorem below.

Theorem 2: Suppose that Condition (14) holds. Let $(W, \mathcal{C}, \mathcal{F})$ be a local switched system generated by the chart (U_σ, ψ_σ) on X. Then $(\mathcal{T}_k, \gamma)$ is a trajectory of $(\mathcal{P}, \mathcal{S}, \mathcal{R})$ with $\gamma(t_0; 1) = x_0$ such that $\gamma(t; i) \in \psi_\sigma^{-1}(W)$ for all $(t; i) \in \mathcal{T}_k$ if and only if

$$y(t) = \psi_\sigma(\gamma(t; i))$$

is the unique solution of the Cauchy problem

$$\dot{y}(t) \in F(y(t)) \text{ a.e.},\ y(0) = \psi_\sigma(x_0).$$

IV. Conclusion

We have shown that the state space of an n-dimensional system with m hystereses can be modeled as a smooth manifold, the Cartesian product of an m-torus and an $(n-m)$-dimensional Euclidean space. The charts have been constructed explicitly, making the results of this work ready for use in concrete applications. Moreover, the dynamics of the system with hystereses have been shown to be equivalent to a dynamical system defined in a chart by means of a differential inclusion.

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