2011 50th IEEE Conference on Decision and Control and
European Control Conference (CDC-ECC)
Orlando, FL, USA, December 12-15, 2011

Iterative methods to compute center and center-stable manifolds with application to the optimal output regulation problem

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***Abstract*— This paper presents iterative methods for computing center and center-stable manifolds. The methods are based on the contraction mapping theorem and compute flows on the invariant manifolds. An important application includes the design of optimal output regulators. It will be shown that the center manifold algorithm solves the regulator equation and the center-stable manifold algorithm computes controllers for optimal output regulation.**

I. INTRODUCTION

Invariant manifolds play important roles in analyzing dynamical systems and designing control systems. Center manifold theory not only provides stability analysis of nonlinear systems with purely imaginary eigenvalues, but also derives conditions for the nonlinear output regulation problem [4], [3]. Stable manifold theory, together with Hamiltonian mechanics or symplectic geometry, has been applied to the theory of Hamilton-Jacobi equations such as optimal control and H^∞ control problems [12], [13], [10]. Despite its importance, computation methods for invariant manifolds are not well-developed. For center manifolds, Taylor expansion had been the only method for actual computation for many years. Recently, Suzuki *et. al.* [11] proposed an iterative method for center manifold computation. For stable manifolds, the Taylor expansion method [8] has been used for years as well. On the other hand, Krauskopf *et. al.* [6] review several recent numerical approaches and Sakamoto and van der Schaft [10] propose an iterative method based on the contraction mapping theorem.

The output regulation problem is one of central problems in control theory. It is also called servo mechanism and its design is a fundamental technology in engineering. Isidori and Byrnes [4] obtained a necessary and sufficient condition for the solvability of local nonlinear output regulation problems, which consists of a set of a partial differential equation and an algebraic equation (*the regulator equation*). Also, it was shown that the nonlinear regulator equation is closely related to a center manifold of an extended system with exosystem that generates the signals to be tracked. The crucial part of this problem is finding the zero-error manifold, which is a manifold such that, provided that the control is suitable (this is to be determined as well), it is forward invariant and, moreover, the tracking error equals zero on the manifold. These conditions are described in the regulator equation. In the linear case, its solution can be found in terms of matrix equations while the solution is a rather difficult problem in the nonlinear case. The classical method described in [2] and other papers relies on expansion of all the involved functions into Taylor series and the approximation of the solution is sought in a form of Taylor polynomials. This method uses only basic calculus but requires rather laborious and difficult to algorithmize the computations. Methods based on the finite-elements were presented in [9]. These methods, unfortunately, require specialized software for the solution of the finite-elements. The method based on approximation of center manifolds is proposed in [11]. It is an iterative method and if the plant and the exosystem are described by polynomials, approximate solutions of the regulator equation are produced without solving any algebraic equations which is required in the Taylor expansion approach. However, this method is limited only to the case where zero dynamics are hyperbolic and the computational load grows exponentially according to the order of approximation.

In this paper, we propose iterative algorithms that compute center and center-stable manifolds in a system of ordinary differential equations. The algorithms are defined using time flows approximately on the center or center-stable manifolds and can be handled in both analytic and numerical ways. The idea of algorithms is similar to those in [10], [11] which is based on the contraction mapping theorem. The center manifold algorithm in the paper is different form that in [11] in that the new algorithm consists of flows on the center manifold while polynomial functions of initial values (or, parameters y_0) are produced in [11]. Numerical treatment of flows in the algorithms is known to be advantageous from the viewpoints of computational load and larger domains of validity [1]. It should be noted that Taylor series solutions are valid, in general, in small domains in which their convergence is guaranteed. We apply these computation theories to the design problem of optimal output regulation. We prove that the center manifold algorithm, when applied to the Hamiltonian system associated with the optimal control problem, approximately solves the regulator equation under the mild assumption that relative degrees are well-defined. We also prove that the optimal controller is computed from the center-stable manifold of the Hamiltonian system and that one does not need to solve the regulator equation. A similar problem is addressed in [7], but, the results in the present paper provide a constructive design procedure and do not assume the solution of regulator equation.

This work was supported by the JSPS Postdoctoral Fellowship Program.
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The paper is organized as follows: after this introduction the section containing description of approximation theory for center and center-stable manifolds is presented. The third section describes its applications to optimal output regulation. A numerical example is given in the fourth section, after which concluding remarks follow.

II. Center and Center-Stable Manifolds

Consider the system of ordinary differential equations of the form

$$\dot{x} = Ax + X(x, y, z) \tag{1a}$$

$$\dot{y} = By + Y(x, y, z) \tag{1b}$$

$$\dot{z} = Cz + Z(x, y, z). \tag{1c}$$

It is assumed $A \in \mathbb{R}^{n_x \times n_x}$ has eigenvalues with negative real part, $B \in \mathbb{R}^{n_y \times n_y}$ has eigenvalues with zero real part and $C \in \mathbb{R}^{n_z \times n_z}$ has eigenvalues with positive real parts. The functions X, Y, Z are smooth and their values as well as the values of their first derivatives at the origin equal zero. It is well-known (see, e.g., [5]) that there exist center and center-stable manifolds in (1a)-(1c). These invariant manifolds can be reformulated into the form of integral equations. The equations describing the center manifold are as follows:

$$x(t) = \int_{-\infty}^{t} e^{A(t-s)} X(x(s), y(s), z(s)) ds \tag{2a}$$

$$y(t) = e^{Bt} y_0 + \int_0^t e^{B(t-s)} Y(x(s), y(s), z(s)) ds \tag{2b}$$

$$z(t) = -\int_t^{\infty} e^{C(t-s)} Z(x(s), y(s), z(s)) ds \tag{2c}$$

In the case of the center-stable manifold, the equations to be solved are:

$$x(t) = e^{At} x_0 + \int_0^t e^{A(t-s)} X(x(s), y(s), z(s)) ds \tag{3a}$$

$$y(t) = e^{Bt} y_0 + \int_0^t e^{B(t-s)} Y(x(s), y(s), z(s)) ds \tag{3b}$$

$$z(t) = -\int_t^{\infty} e^{C(t-s)} Z(x(s), y(s), z(s)) ds \tag{3c}$$

In [5], the existence of center and center-stable manifolds is proven by showing that certain operators on a functional space are contraction and use the contraction mapping theorem. However, this approach is not constructive and computational methods are limited to the one that uses Taylor expansion for equivalent partial differential equations describing center and center-stable manifolds. In this paper, we employ different operators and propose a constructive and iterative method to obtain those invariant manifolds.

First, we note that, from the smoothness of the system, there exists a nonnegative function $\eta : [0, +\infty) \to [0, +\infty)$ such that for all $x_i \in \mathbb{R}^{n_x}, y_i \in \mathbb{R}^{n_y}, z_i \in \mathbb{R}^{n_z}$, $i = 1, 2$, if $|(x_1, y_1, z_1) - (x_2, y_2, z_2)| \leq \delta$ holds, then, it also holds that

$$\begin{aligned} &|X(x_1, y_1, z_1) - X(x_2, y_2, z_2)| \\ &\quad \leq \eta(\delta)|(x_1, y_1, z_1) - (x_2, y_2, z_2)|, \\ &|Y(x_1, y_1, z_1) - Y(x_2, y_2, z_2)| \\ &\quad \leq \eta(\delta)|(x_1, y_1, z_1) - (x_2, y_2, z_2)|, \\ &|Z(x_1, y_1, z_1) - Z(x_2, y_2, z_2)| \\ &\quad \leq \eta(\delta)|(x_1, y_1, z_1) - (x_2, y_2, z_2)|. \end{aligned} \tag{4}$$

We assume there exist positive constants a, K such that $\|e^{At}\| \leq Ke^{-at}$, $e^{Bt} \leq K$, $\|e^{Ct}\| \leq Ke^{at}$ for all $t \geqslant 0$. Here, $(C^{\infty}(\mathbb{R}_+))^{n_x+n_y+n_z}$ is a set of smooth functions defined on $[0, \infty)$.

We define the sequences $(x_k(t), y_k(t), z_k(t))$ as

$$(x_1(t) = 0,\ y_1(t) = e^{Bt} y_0,\ z_1(t) = 0$$

$$\begin{pmatrix} x_{k+1} \\ y_{k+1} \\ z_{k+1} \end{pmatrix}(t) = \begin{pmatrix} \int_{-\infty}^{t} e^{A(t-s)} X(x_k(s), y_k(s), z_k(s))\, ds \\ e^{Bt} y_0 + \int_0^t e^{B(t-s)} Y(x_k(s), y_k(s), z_k(s))\, ds \\ -\int_t^{\infty} e^{C(t-s)} Z(x_k(s), y_k(s), z_k(s))\, ds \end{pmatrix}$$

with $y_0 \in \mathbb{R}^{n_y}$ being initial condition in the case if the center manifold is sought or

$$(x_1(t) = e^{At} x_0,\ y_1(t) = e^{Bt} y_0,\ z_1(t) = 0$$

$$\begin{pmatrix} x_{k+1}(t) \\ y_{k+1}(t) \\ z_{k+1}(t) \end{pmatrix}(t) = \begin{pmatrix} e^{At} x_0 + \int_0^t e^{A(t-s)} X(x_k(s), y_k(s), z_k(s))\, ds \\ e^{Bt} y_0 + \int_0^t e^{B(t-s)} Y(x_k(s), y_k(s), z_k(s))\, ds \\ -\int_t^{\infty} e^{C(t-s)} Z(x_k(s), y_k(s), z_k(s))\, ds \end{pmatrix}$$

with $x_0 \in \mathbb{R}^{n_x}, y_0 \in \mathbb{R}^{n_y}$ taken as initial condition in the case if the center-stable manifold is to be found. This initial condition is regarded as a parameter. For the sake of simplicity this dependence is not reflected in the notation. The fixed point whose existence will be shown then solves the equations (2), resp. (3).

Theorem 2.1: Center manifold case: There exists $\delta_0 > 0$ such that, for each initial condition y_0 such that $|y_0| < \delta_0$ the sequence $(x_k(t), y_k(t), z_k(t))$ converges locally uniformly in R to a limit function $(x^*(t, y_0), y^*(t, y_0), z^*(t, y_0))$ which satisfies (2a-2c). Also, in the center-stable manifold case, the sequence $(x_k(t), y_k(t), z_k(t))(t)$ converges locally uniformly in R to a limit function $(x^*(t, x_0, y_0), y^*(t, x_0, y_0), z^*(t, x_0, y_0))$ which satisfies (3a-3c).

The proof is rather lengthy, therefore it is omitted.

Next, we demonstrate that the functions x^*, z^* in the center case and the function z^* in the center-stable case are tangent to the other axes.

Theorem 2.2: For sufficiently small $|y_0|$,

$$|(x^*(0, y_0), z^*(0, y_0)| \leq C_1 |y_0|^2, \tag{5}$$

or, for sufficiently small $|(x_0, y_0)|$,

$$|z^*(0, x_0, y_0)| \leq C_2 |(x_0, y_0)|^2$$

with positive constants C_1, C_2.

Proof: The proof is done by induction using the iterations above. Again, it is omitted. ∎

III. Optimal Output Regulation

This section describes an application of the center, center-stable manifold algorithms to nonlinear output regulation theory with optimality. It will be shown that the center manifold algorithm approximately solves the nonlinear regulator equation which is a necessary and sufficient condition for output regulation. It will be also shown, however, that solving the nonlinear regulator equation is not necessary for the design of optimal output regulator and that center-stable manifold of an associated Hamiltonian system provides the controller. The formulation of optimal output regulation is as follows.

System: $\dot{x} = f(x) + g(x)u, \quad x(t) \in \mathbb{R}^n,\ f(0) = 0$

Exosystem: $\dot{w} = s(w), \quad w(t) \in \mathbb{R}^p,\ s(0) = 0$

Error (output) equation: $e = h(x, w)$

Denote

$$A = \frac{\partial f}{\partial x}x(0), \quad B = g(0), \quad C = \frac{\partial h}{\partial x}(0,0),$$
$$S = \frac{\partial s}{\partial w}(0), \quad Q = \frac{\partial h}{\partial w}(0,0).$$

Assumption 3.1: i) The exosystem is Lyapunov stable at $w = 0$, Poisson stable around $w = 0$ and all eigenvalues of S are purely imaginary.

ii) The pair $(A,\ B)$ is stabilizable.

iii) The pair (C, A) is detectable.

iv) The system is square, that is, the number of inputs and outputs are both r.

v) The system has relative degree 1, that is, $L_g h(0,0)$ is nonsingular.

Assumptions 3.1-iv), 3.1-v) are for the sake of brevity in notation and can be easily extended to the general cases where systems have different numbers of inputs and outputs and general relative degrees.

We look for the control such that

$$J = \frac{1}{2}\int_0^\infty |e|^2 + |\dot{e}|^2 \, dt$$

is minimized. J can be written as

$$J = \tfrac{1}{2}\textstyle\int_0^\infty |h(x,w)|^2 + |L_f h(x,w) + (L_g h(x,w))u + L_s h(x,w)|^2 \ dt.$$

We first apply dynamic programming and derive a Hamilton-Jacobi equation for this optimal control problem. The Hamiltonian H_D for dynamic programming is

$$H_D = p_x^T(f + gu) + p_w^T s(w) + \frac{1}{2}\left|h(x,w)\right|^2 + \frac{1}{2}\left|L_f h(x,w) + (L_g h(x,w))u + L_s h(x,w)\right|^2,$$

and the control vector $\bar{u}$ that minimizes H_D is

$$\bar{u} = -(L_g h)^{-1}\left\{(L_g h)^{-T} g(x)^T p_x + L_f h + L_s h\right\}.$$

Note that $L_g h(x, w)$ is nonsingular around $(0, 0)$ from Assumption 1-v). Therefore, the Hamilton-Jacobi equation for our optimal control problem is

$$p_x^T\left\{f - g(L_g h)^{-1}(L_f h + L_s h)\right\} + p_w^T s(w) - \frac{1}{2}p_x^T g (L_g h)^{-1}(L_g h)^{-T} g^T p_x + \frac{1}{2}|h(x,w)|^2. \quad (6)$$

Noting that

$$\begin{aligned}
L_g h(x,w) &= CB + O(|x| + |w|),\\
L_s h(x,w) &= QSw + O((|x| + |w|)^2),\\
f - g(L_g h)^{-1} L_f h &= (A - B(CB)^{-1}CA)x + O((|x| + |w|)^2),\\
g(L_g h)^{-1} L_s h &= B(CB)^{-1}QSw + O((|x| + |w|)^2),\\
&\cdots,
\end{aligned}$$

the Hamiltonian system associated with (6) is

$$\begin{cases}
\dot{x} = (A - B(CB)^{-1}CA)x - B(CB)^{-1}QSw \\
\qquad - B\left(B^T C^T CB\right)^{-1} B^T p_x + N_1(x, w, p_x) \\
\dot{w} = Sw + N_2(w) \\
\dot{p}_x = C^T Cx - C^T Qw \\
\qquad - (A - B(CB)^{-1}CA)^T p_x + N_3(x, w, p_x) \\
\dot{p}_w = -Q^T Cx - Q^T Qw + S^T Q^T (B^T C^T)^{-1} B^T p_x \\
\qquad - S^T p_w + N_4(x, w, p_x, p_w).
\end{cases} \quad (7)$$

The nonlinear terms N_1, N_2 and N_3 are appropriately calculated. For example,

$$N_1 = f - g(L_f h)^{-1}(L_f h + L_s h) - g(L_g h)^{-1}(L_g h)^{-T} g^T p_x - (A - B(CB)^{-1}CA)x + B(CB)^{-1}QSw + B\left(B^T C^T CB\right)^{-1} B^T p_x. \quad (8)$$

Here, it will be important that N_1, N_2 and N_3 do not depend on p_w.

Assumption 3.2: The linear regulator equation

$$\Pi S = A\Pi + B\Sigma, \quad C\Pi + Q = 0$$

has a solution $\Pi \in \mathbb{R}^{n\times p}$, $\Sigma \in \mathbb{R}^{r\times p}$.

Define the matrix describing the linear part in (7) as H. Let also $\bar{A} = A - B(CB)^{-1}CA$. Then, using Assumptions 3.1-iv), 3.1-v), it follows that

$$T_1^{-1} H T_1 = \begin{pmatrix} \bar{A} & 0 & -B(B^T C^T CB)^{-1} B^T & 0 \\ 0 & S & 0 & 0 \\ -C^T C & 0 & -\bar{A}^T & 0 \\ 0 & 0 & 0 & -S^T \end{pmatrix},$$

where

$$T_1 = \begin{pmatrix} I & \Pi & 0 & 0 \\ 0 & I & 0 & 0 \\ 0 & 0 & I & 0 \\ 0 & 0 & -\Pi^T & I \end{pmatrix}.$$

To see this, one uses $-QS = CA\Pi + CB\Sigma$ from Assumption 3.2. Assumptions 3.1-ii) and 3.1-iii) enable the diagonalization:

$$U^{-1}\begin{pmatrix} A-B(CB)^{-1}CA & -B(B^TC^TCB)^{-1}B^T \\ -C^TC & -(A-B(CB)^{-1}CA)^T \end{pmatrix} U = \begin{pmatrix} A_c & 0 \\ 0 & -A_c{}^T \end{pmatrix}, \quad U = \begin{pmatrix} I & V \\ P & PV+I \end{pmatrix}$$

where $A_c = A - B(CB)^{-1}CA - B(B^TC^TCB)^{-1}B^TP$ is a stable matrix, P is a solution of a Riccati equation associated with the Hamiltonian matrix in the left

$$P\bar{A} + \bar{A}^TP - PR_BP + C^TC = 0;$$
$$\bar{A} = A - B(CB)^{-1}CA, \quad R_B = B(B^TC^TCB)^{-1}B^T$$

and V is a solution of Lyapunov equation $VA_c + A_c^TV = B(B^TC^TCB)^{-1}B^T$. Thus, after the linear transformation T_1T_2 with

$$T_2 = \begin{pmatrix} I & 0 & V & 0 \\ 0 & I & 0 & 0 \\ P & 0 & PV+I & 0 \\ 0 & 0 & 0 & I \end{pmatrix},$$

the Hamiltonian system (7) has the form (1a)-(1c) in the previous section in the new coordinates.

Remark 3.1: Linear optimal input is

$$\begin{aligned}\bar{u} =& -(B^TC^TCB)^{-1}\{B^TP(x-\Pi w) + B^TC^TCAx \\ &+ B^TC^TQSw\} \\ =& -(B^TC^TCB)^{-1}B^TP(x-\Pi w) \\ &-(CB)^{-1}CAx + (CB)^{-1}CA\Pi w + \Sigma w \\ =& -\{(CB)^{-1}CA + (B^TC^TCB)^{-1}B^TP\}(x-\Pi w) + \Sigma w,\end{aligned}$$

which corresponds to the standard form in the theory of output regulation $u = K(x - \Pi w) + \Sigma w$ with a stabilizing gain K.

In the new coordinates $[x'^T, w'^T, p_x'{}^T, p_w'{}^T]^T = T_1^{-1}[x^T, w^T, p_x{}^T, p_w{}^T]^T$, the Hamiltonian system (7) is written as

$$\begin{cases} \dot{x}' &= \bar{A}x' - R_B^{-1}p_x' + \tilde{N}_1(x', w', p_x') \\ \dot{w}' &= Sw' + \tilde{N}_2(w') \\ \dot{p}_x' &= -C^TCx' - \bar{A}^Tp_x' + \tilde{N}_3(x', w', p_x') \\ \dot{p}_w' &= -S^Tp_w' + \tilde{N}_4(x', w', p_x', p_w'), \end{cases} \tag{9}$$

and, further by $[x''^T, w''^T, p_x''^T, p_w''{}^T]^T = T_2^{-1}[x'^T, w'^T, p_x'{}^T, p_w'{}^T]^T$,

$$\begin{cases} \dot{x}'' &= A_cx'' + \bar{N}_1(x'', w'', p_x'') \\ \dot{w}'' &= Sw'' + \bar{N}_2(w'') \\ \dot{p}_x'' &= -A_c^Tp_x'' + \bar{N}_3(x'', w'', p_x'') \\ \dot{p}_w'' &= -S^Tp_w'' + \bar{N}_4(x'', w'', p_x'', p_w''). \end{cases} \tag{10}$$

The center manifold theory states that there exist center manifolds $x'' = \bar{\pi}_1(w')$, $p_x'' = \bar{\pi}_2(w')$ for (10) around $w'' = 0$ and center-stable manifold $p_x'' = \pi_3(x'', w'')$. Here, we note that the other center part corresponding to p_w'' does not affect the manifolds since x'' and p_x'' do not depend on p_w''. In the coordinates $[x'^T, w'^T, p_x'{}^T, p_w'{}^T]^T$, the center manifold is

$$x' = \pi_1(w') := \bar{\pi}_1(w') + V\bar{\pi}_2(w')$$
$$p_x' = \pi_2(w') := P\bar{\pi}_1(w') + (PV+I)\bar{\pi}_2(w').$$

In the original coordinates, then,

$$x = \pi(w) = \Pi w + \pi_1(w) = \Pi w + \bar{\pi}_1(w) + V\bar{\pi}_2(w),$$
$$p_x = \pi_2(w) = P\bar{\pi}_1(w) + (PV+I)\bar{\pi}_2(w).$$

The next theorem asserts that the center manifold algorithm in the previous section approximately solves the nonlinear regulator equation

$$\frac{\partial \pi}{\partial w}s(w) = f(\pi(w)) + g(\pi(w))\sigma(w), \quad h(\pi(w)) = q(w). \tag{11}$$

Theorem 3.3: The solution of the regulator equation (11) is given by

$$\begin{aligned}\pi(w) &= \Pi w + \pi_1(w), \\ \sigma(w) &= -(L_gh(\pi(w), w))^{-1}\left\{(L_gh(\pi(w), w))^{-T}g(\pi(w))^T \right. \\ &\quad \left. \times\pi_2(w) + L_fh(\pi(w), w) + L_sh(\pi(w), w)\right\}.\end{aligned}$$

Proof: Considering the relations of (7), (9) and (10) with T_1, T_2, it can be shown that $x = \pi(w)$, $p_x = \pi_2(w)$ is the center manifold in (7). The pde that describes its invariance property is

$$\begin{aligned}\frac{\partial \pi}{\partial w}s(w) &= (A - B(CB)^{-1}CA)\pi(w) - B(CB)^{-1}QSw \\ &- B(B^TC^TCB)^{-1}B^T\pi_2(w) + N_1(\pi(w), w, \pi_2(w)).\end{aligned}$$

From (8), the right side is

$$\begin{aligned}&f(\pi(w)) - g(\pi(w))(L_gh(\pi(w), w))^{-1} \\ &\quad \times\{L_fh(\pi(w), w) + L_sh(\pi(w), w)) \\ &\quad\quad + (L_gh(\pi(w), w))^{-T}g(\pi(w))^T\pi_2(w)\},\end{aligned}$$

from which one concludes that $\pi(w)$ and $\sigma(w)$ above are the solution of the regulator equation. The proof that π, σ satisfy the algebraic part of the regulator equation can be done in the standard way using Assumption 3.1-i) (see, e.g., [3], [4]). ■

Remark 3.2: When the system and exosystem are all linear, $\pi(w) = \Pi w$ and $\sigma(w) = \Sigma w$ because the center manifolds $\pi_1(w)$, $\pi_2(w)$ do not include first order terms (Theorem 2.2) and

$$\begin{aligned}\sigma(w) &= -B(B^TC^TCB)^{-1}(B^TC^TCA\Pi + B^TC^TQS)w \\ &= -(CB)^{-1}CA\Pi w - (CB)^{-1}QSw \\ &= -(CB)^{-1}CA\Pi w + (CB)^{-1}(CA\Pi + CB\Sigma)w \\ &= \Sigma w.\end{aligned}$$

On the other hand, the center-stable manifold $p_x'' = \pi_3(x'', w'')$ is, in the original coordinates, written as

$$-P(x - \Pi w) + p_x = \pi_3((VP+I)(x - \Pi w) - Vp_x, w).$$

The implicit function theorem and Theorem 2.2 assure that it can be re-written as

$$p_x = p_x(x, w)$$

around $(x, w) = (0, 0)$.

Theorem 3.4: The optimal output regulation controller is given by

$$u = -(L_g h)^{-1} \times \left\{ (L_g h)^{-T} g(x)^T p_x(x, w) + L_f h(x, w) + L_s h(x, w) \right\},$$

where $p_x(x, w)$ represents the center-stable manifold of (10) around the origin.

Proof: We prove the theorem by showing that $p_x(x, w) = \partial V / \partial x^T$ with a solution $V(x, w)$ of the Hamilton-Jacobi equation (6). To do that, we show that there is a Lagrangian submanifold L on which $p_x = p_x(x, w)$ and the canonical projection to (x, w) space is surjective.

First, we show that (10) is also a Hamiltonian system, by showing that the coordinate transformation $T_1 T_2$ is symplectic transforms. From the expression of T_1, T_2, it follows that

$$sT_1^T J T_1 = J, \quad T_2^T J T_2 = J; \quad J = \begin{pmatrix} 0 & 0 & I_n & 0 \\ 0 & 0 & 0 & I_p \\ -I_n & 0 & 0 & 0 \\ 0 & -I_p & 0 & 0 \end{pmatrix}.$$

The group property of symplectic transforms proves that $T_1 T_2$ is symplectic. We next prove that there exists an $(n + p)$-dimensional Lagrangian submanifold L on which $p_x = p_x(x, w)$ holds. Using the fact that $\bar{N}_1$, $\bar{N}_2$ and $\bar{N}_3$ do not depend on p''_w and that $\bar{N}_4(0, p''_x, w'', 0) = 0$, it is possible to find a function $p''_w(w'')$ such that

$$\{(x'', p''_x, w'', p''_w) \,|\, p''_x = \pi_3(x'', w''),\ p''_w = p''_w(w'')\}$$

is an invariant manifold of (10) of dimension $(n+p)$ and that the Hamiltonian flow on it converges to $x'' = 0$, $p''_w = 0$. Considering the symplectic 2-form in this coordinates and its invariance along the Hamiltonian flow, one can show that the symplectic 2-form vanishes on

$$L = \{(x'', p''_x, w'', p''_w) \,|\, p''_x = \pi_3(x'', w''),\ p''_w = p''_w(w'')\},$$

which shows that L is a Lagrangian submanifold on which $p''_x = \pi_3(x'', w'')$. In the original coordinates, $p_x = p_x(x, w)$ holds on L. From linear analysis of (6), namely, solving the approximating $(n+p)$-dimensional Riccati equation, one knows that

$$V(x, w) = (x - \Pi)^T P (x - \Pi)/2 + O((|x| + |w|)^2),$$

which guarantees that L is surjective to the base space (x, w) around the origin. This proves that there exists a generating function $V(x, w)$ of L such that

$$\frac{\partial V}{\partial x}^T = p_x(x, w), \quad \frac{\partial V}{\partial x}^T = p_w(x, w),$$

satisfying the Hamilton-Jacobi equation (6). ■

Remark 3.3: From Theorem 3.4, one sees that it is sufficient to compute center-stable manifold to design nonlinear optimal output regulators and it is not necessary to solve the nonlinear regulator equation (11). The actual computation of $p_x(x, w)$ we carry out in this paper is as follows. Applying the algorithm to (10), we have parameterized solutions

$$\begin{aligned} x''(t) &= x''(t, x_0), \quad x''(0) = x_0 \\ w''(t) &= w''(t, w_0), \quad w''(0) = w_0 \\ p''_x(t) &= p''_x(t, x_0, w_0), \end{aligned}$$

where parameters x_0 and w_0 are sufficiently small so that the convergence of the algorithm is guaranteed. In the original coordinates,

$$\begin{aligned} &x(t) = x''(t) + V p''_x(t) + \Pi(P x''(t) + (PV + I) p''_x(t)) \\ &w(t) = w''(t) \\ &p_x(t) = P x''(t) + (PV + I) p''_x(t) \end{aligned}$$

give the parameterized center-stable manifold. The implicit function theorem assures that p_x can be uniquely represented, on the center-stable manifold around the origin, as a function of x, w. In the next section, we use polynomial fitting to get the function.

IV. Numerical Example

Consider the example with unstable linearization

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \end{pmatrix} = \begin{pmatrix} 1 & 0.5 \\ 2 & 1.6 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} + \begin{pmatrix} x_1^3 \\ 2x_1^3 + x_2^3 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \end{pmatrix} u$$

with exosystem

$$\dot{w} = 0.$$

The goal is to achieve $x_1 = w$ in the limit. To do this, as in Sec.III, the cost functional is defined as

$$J = \frac{1}{2} \int_0^{+\infty} (x_1 - w)^2 + (\dot{x}_1)^2 dt.$$

Defining the matrix H as

$$H = \begin{pmatrix} 0 & 0 & 0 & -1 & -1 & 0 \\ 2 & 1.1 & 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 & -2 & 0 \\ 0 & 0 & 0 & 0 & -1.1 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \end{pmatrix}$$

and using the approach described above, one infers the equation (7) attains the form

$$\begin{pmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dot{w} \\ \dot{p}_1 \\ \dot{p}_2 \\ \dot{p}_w \end{pmatrix} = H \begin{pmatrix} x_1 \\ x_2 \\ w \\ p_1 \\ p_2 \\ p_w \end{pmatrix} + \begin{pmatrix} x_1^3 \\ x_1^3 + x_2^3 \\ 0 \\ -3x_1^2 p_2 \\ -3x_2^2 p_2 \\ 0 \end{pmatrix}.$$

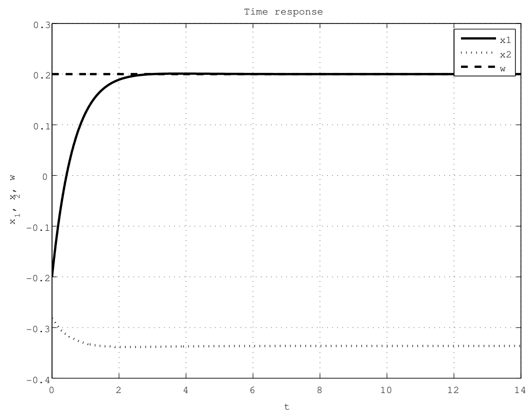

Fig. 1. The states of the system and the exosystem

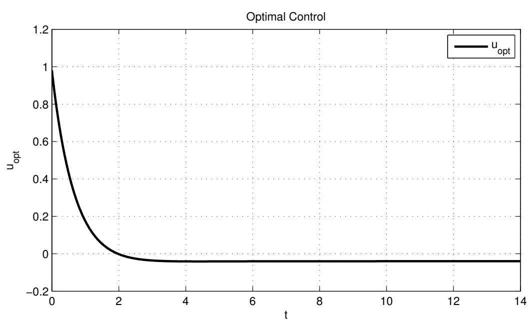

Fig. 2. The optimal control

The simulation results can be seen in the figures Fig. 1 and 2. Here, the initial conditions are $x_1(0) = -0.05$, $x_2(0) = 0$ and $w(0) = 0.2$ which is the value to be approached by the state x_1. Fig. 1 shows the states of the system and the state of the exosystem. The state x_1 is represented by the solid line, the dotted line shows the second state x_2 and, finally, the state of the exosystem is demonstrated by the dashed line. Fig. 2 shows the optimal control.

To compare the results with the standard LQ controller the same system with the same initial conditions and cost functional was used. The results are in Fig. 3, the meaning of all lines is the same as in the Fig. 1. One can see that the steady state error is much larger due to presence of nonlinearities that cannot be taken into account in the process of designing the LQ controller.

V. Concluding Remarks

In this paper, we proposed new iterative and constructive methods to compute center and center-stable manifolds using the contraction mapping theorem. The algorithms are written in a suitable way for computer implementation. One of the important applications of this approximation theory is the design of optimal output regulators. It has been shown that the center manifold algorithm approximately solves the regulator equation and the center-stable manifold algorithm computes the optimal output regulation controllers.

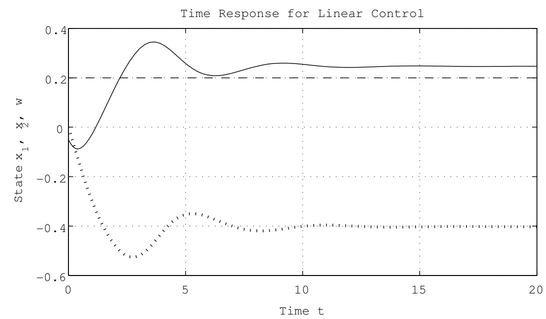

Fig. 3. The states of the system under LQ control

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