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Multiclass job scheduling on a single machine: updating optimal control strategies when due-dates change in real-time

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***Abstract*— The problem of scheduling jobs, belonging to different classes, on a single machine, can be dealt with under a control-theoretic framework, with the aim of determining optimal (closed-loop) control strategies, instead of optimal (open-loop) scheduling decisions. In the model considered by the authors, optimal control strategies can be determined through a constructive procedure, based on the application of dynamic programming. However, in the case that one or more due-dates change in real-time, the strategies (determined off-line and used in real-time to find, at each decision instant, the optimal actions to be adopted) may become invalid. In this paper, sufficient conditions about the validity of the optimal control strategies are provided, in connection with some specific cases of change of due-dates; moreover, the algorithm to be used to determine the new strategies, when these conditions are violated, is also provided in the paper.**

I. INTRODUCTION

The problem of sequencing and timetabling jobs assigned to a resource (machine, manufacturing cell, etc.) of a manufacturing system is of uppermost importance within the so-called "operational decision level", where decisions have to be taken with reference to discrete entities (jobs and resources) and within a discrete-event framework. Scheduling problems in manufacturing systems [1], [2] may be considered within many different modelling settings, concerning the model structure, the constraints, and the objectives.

In [3], the authors considered the problem of scheduling jobs on a single machine, characterized by the following features: *i)* two jobs belonging to the same class are considered as completely equivalent; *ii)* a sequence of due-dates is specified for each class of jobs, and the serviced jobs, for each class, are assigned to the due-dates according to the earliest due-date (EDD) rule (generalized due-dates model [4], [5]); *iii)* the service time of any job of a given class has to be selected within an interval of possible values (controllable processing times). The decision variables are those concerning job sequencing and service times, and the performance index to be minimized includes both the total weighted tardiness and the total weighted deviation from the nominal service times. The approach proposed in [3] allows determining optimal control strategies (capable of providing, at each decision step, the optimal decisions as functions of the current system state) through a constructive procedure, based on the application of dynamic programming.

In this paper, the on-line application of the optimal control strategies provided in [3] is considered, and the case in which one or more due-dates change during real-time is specifically taken into account. Due-dates may change, for example, because a specific job becomes urgent. The change of a due-date may invalidate some of the control strategies determined off-line; thus, in this paper, a method to determine the set of system states that are interested by the change of due-dates is presented and, with reference to those states, sufficient conditions about the validity of the optimal control strategies are developed. Moreover, in case these conditions are violated, an algorithm providing the new optimal control strategies is provided.

It is worth noting that, in the considered model, the change of a due-date is a critical issue, owing to the assumption of generalized due-dates. As a matter of fact, the modification of a due-date may cause a significant change in the EDD sequence relevant to the jobs of a certain class. As an example, consider a class of jobs whose due-dates are 12, 21, 23, 31, 42, 50, 62, and 75; if the sixth due-date changes and becomes 30, the new EDD sequence to be considered is 12, 21, 23, 30, 31, 42, 62, and 75; it is evident that three due-dates (the fourth, the fifth, and the sixth) are actually different with respect to the original EDD sequence. Then, from the point of view of the system, the due-dates that changed are three, not only one. This case, in which the change of one due-date modifies the EDD sequence, is specifically taken into account in this paper.

This work was inspired by an interesting application of the model proposed in [6], [7] (in which, in addition to the features which caracterize the model proposed in [3], a sequence-dependent setup is required between the execution of jobs of different classes). The application is relevant to a specific "Inventory/Routing Problem" (IRP) [8], [9], in which both transportation and inventory policies are determined, whose generic statement may be the following: "a vehicle has to deliver goods (of the same class) from a deposit (from which it starts and to which it returns) to a set of retailers distributed over the territory; the amount of goods delivered to each retailer and the sequence according to which the retailers are visited have to be determined on the basis of the stock of goods (inventory level) at the retailers". Such a problem can be solved through the methodology developed for the multiclass job scheduling problem, as *i)* each retailer to be served corresponds to the (unique) job of a certain class to be executed, *ii)* the travel time between retailers corresponds to the setup between jobs of different classes, and *iii)* the stopover time at a retailer (to unload goods) corresponds to the service time of the job.

Moreover, in the considered IRP application, the due-

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date associated with the (unique) job of a certain class corresponds to the time instant at which the inventory of the retailer, relative to that class, becomes empty. The due-dates can be determined off-line (when the algorithm providing the optimal control strategies is executed) on the basis of the inventory levels that are available at that time instant and assuming, for each retailer, a certain fixed selling rate which removes goods from the inventory. However, the dynamics of removing goods from the real inventories may be different and thus, during real-time, an inventory may become empty before or after the previously (off-line) determined due-date. Then, the due-dates can be updated, during real-time, on the basis of the current levels of inventories (obviously assuming the availability of technologies which allows transmitting/receiving in real-time the inventory levels). As a consequence, the control strategies (that have been pre-determined) that the vehicle is using to choose, at each decision instant, which is the next retailer to be visited may be no longer effective because they have been determined on the basis of due-dates that are no longer valid. For this reason, before taking any decision about the next destination, it is important to determine if the "old" strategies are still valid and, in the negative case, the "new" strategies must be calculated.

This is the motivation of the problem considered in this paper, that is, how to check if the "old" strategies remain valid (after the recognition of changes in due-dates) and how to determine the "new" optimal control strategies. For the sake of simplicity, in this paper, only the original model proposed in [3] will be taken into account. In any case, the extension to the case with setup is straightforward.

II. THE SYSTEM MODEL AND THE SOLUTION PROCEDURE

Consider a single machine where N_1 jobs of class P_1, N_2 jobs of class P_2, ..., and N_K jobs of class P_K have to be executed. All jobs belonging to the same class are completely equivalent. No precedence constraint has to be fulfilled, all jobs are available at time instant 0, and the execution of any job has to be nonpreemptive. No setup is required between the execution of jobs of different classes. A sequence of N_k due-dates for jobs of class P_k, namely $dd_{k,1}, dd_{k,2}, \dots, dd_{k,N_k}$, $k = 1, \dots, K$, is specified, as well as a set of coefficients, namely $\alpha_{k,1}, \alpha_{k,2}, \dots, \alpha_{k,N_k}$, $k = 1, \dots, K$, specifying the unitary tardiness penalty for each job. It is assumed that $dd_{k,i} \le dd_{k,i+1}$ for $k = 1, \dots, K$ and $i = 1, \dots, N_k - 1$, and that jobs are assigned to due-dates according to the EDD rule. Thus the model is characterized by the generalized due-dates assumption, as mentioned in the introduction.

The processing time of the i-th job of class P_k is considered a continuous variable $pt_{k,i}$, $k = 1, \dots, K$, $i = 1, \dots, N_k$, whose value ranges from a lower bound pt_k^{low} up to a higher bound, namely pt_k^{nom}, which is also the nominal value of such a service time. The assumption that the highest possible service time coincides with the nominal one is justified if one supposes that there is no economic advantage in slowing down the service time of a job. Moreover, it is assumed that, once the service time of a certain job has been chosen, it cannot be changed during the service.

The cost function to be minimized is the sum, for each job of each class, of the weighted tardiness and the weighted deviation of the actual service time from the nominal value. It is

$$\sum_{k=1}^{K} \sum_{i=1}^{N_k} [\alpha_{k,i} \cdot \max\{ct_{k,i} - dd_{k,i}, 0\} + \beta \cdot (pt_k^{\text{nom}} - pt_{k,i})] \qquad (1)$$

where β is the weighting coefficient stating that the extra-cost paid for the reduction of the service time is simply proportional to this reduction (note that coefficient β is neither dependent on the class nor on the job); moreover, $ct_{k,i}$, $k = 1, \dots, K$, $i = 1, \dots, N_k$, is the completion time of the i-th job of class P_k that is completed. It is assumed that $\alpha_{k,i} > \beta$, for any pair (k, i), $i = 1, \dots, N_k$, $k = 1, \dots, K$; this assumption states that any unitary tardiness cost is greater than the unitary cost related to the deviation from the nominal service time. The overall cost function is of regular type [10], as there is no advantage in delaying any completion time if the other ones remain unchanged. Hence, there is an optimal solution of the scheduling problem where no idle time is inserted between the execution of two subsequent jobs. Besides, in this optimal solution, the execution of the first job starts immediately at time instant 0. For these reasons, in the following, attention will be restricted only to solutions without idle times.

A. The state-space model and the optimal control problem

In the following, the case $K = 2$ (two-classes job scheduling) will be taken into account. The reason for this choice is purely technical, as, in the case of two classes, it is easier to show the results about the determination of the new optimal control strategies when due-dates change in real-time. In any case, the extension to the general case is conceptually not too complicated (the reader can find in [3] both the formalization of the optimal control problem and the procedure which allows determining optimal control strategies, when K classes of jobs are considered).

The system model can be represented through a state-space model, where the system state, when a new decision has to be taken, i.e., at time instant $t = 0$ or at any instant t at which a job has been completed (but for the last one), is the vector $\underline{x}(t) = [n_1(t), n_2(t), t]^{\mathrm{T}}$, being $n_k(t)$, $k = 1, 2$, the number of jobs of class P_k already served at time instant t. It is apparent that the system state does not change between two successive decision instants.

As decision instants are discrete in time (although not equally spaced, and thus not generally integer), they can be denoted as t_j, $j = 0, 1, \dots, N - 1$, being N the overall number of jobs to be executed, namely $N = N_1 + N_2$. Thus $t_0 = 0$. At time instant t_j, j jobs have already been completed. Then, the "action" $\underline{u}(t_j) = \underline{f}[\underline{x}(t_j)]$ that has to be taken at time instant t_j corresponds to the choice of the class and the value of the service time of the next job to be served. Let $\delta_k(t_j) \in \{0, 1\}$, $k = 1, 2$, denote a binary decision variable whose value is 1 if a job of class P_k is selected, for the $(j+1)$-th service, and 0 otherwise, and let $\tau(t_j)$ indicate the service time for the selected job. Obviously, $\delta_1(t_j)+\delta_2(t_j)=1\ \forall\, t_j$. Thus, $\underline{u}(t_j)=[\delta_1(t_j),\delta_2(t_j),\tau(t_j)]^{\mathrm{T}}$. For the sake of brevity, notation $\underline{x}_j$ will be used instead of $\underline{x}(t_j)$. Similarly defined notations $n_{k,j}$, $\underline{u}_j$, $\delta_{k,j}$ and τ_j will also be used.

The state equation of the system can be written as follows

$$\underline{x}_{j+1}=\begin{bmatrix} n_{1,j+1}\\ n_{2,j+1}\\ t_{j+1}\end{bmatrix}=\begin{bmatrix} n_{1,j}+\delta_{1,j}\\ n_{2,j}+\delta_{2,j}\\ t_j+\tau_j\end{bmatrix} \tag{2}$$

$j=0,1,\dots,N-1$, being $\underline{x}_0=[0,0,0]^{\mathrm{T}}$ the initial state vector. Decision variables $\delta_{k,j}$, $k=1,2$, and τ_j are constrained by

$$\textstyle\sum_{k=1}^{2}\delta_{k,j}=1 \tag{3}$$

$$\textstyle\sum_{k=1}^{2} pt_k^{\mathrm{low}}\,\delta_{k,j}\le\tau_j\le\sum_{k=1}^{2} pt_k^{\mathrm{nom}}\,\delta_{k,j} \tag{4}$$

for any $j=0,1,\dots,N-1$. Moreover, let h_j be a variable which specifies the class of the job whose service begins at decision instant t_j, namely $h_j=\sum_{k=1}^{2}k\,\delta_{k,j}$. The cost function can be rewritten as

$$\begin{aligned}\textstyle\sum_{j=0}^{N-1}\big[&\alpha_{h_j,n_{h_j,j+1}}\cdot\max\{t_{j+1}-dd_{h_j,n_{h_j,j+1}},0\}+\\ &+\beta\cdot(pt_{h_j}^{\mathrm{nom}}-\tau_j)\big]\end{aligned} \tag{5}$$

Note that, in writing (5), the state equation (2) has been explicitly considered and the generalized due-dates model has been taken into account. On these bases, it is possible to formalize the following optimal control problem.

Problem 1: With reference to the dynamic system described by (2), and taking into account constraints (3) and (4), find control strategies $\delta_{k,j}^{\circ}(n_{1,j},n_{2,j},t_j)$, $k=1,2$, and $\tau_j^{\circ}(n_{1,j},n_{2,j},t_j)$ to be applied at any state $[n_{1,j},n_{2,j},t_j]^{\mathrm{T}}$, $j=0,\dots,N-1$, with $0\le n_{k,j}\le N_k$, $k=1,2$, and t_j non-negative real, that minimize the objective function (5).

In the rest of the paper, a "compact" representation of the system state vector, namely $[n_1,n_2,t_j]^{\mathrm{T}}$, in which subscript j appears only at the last state variable, will be adopted. It is apparent that this compact representation is equivalent to $[n_{1,j},n_{2,j},t_j]^{\mathrm{T}}$, i.e., it contains the same amount of information, as $j=n_1+n_2$ (both represent the state reached at time t_j at which n_k jobs of class P_k, $k=1,2$ have been executed).

B. The constructive procedure which provides the optimal control strategies

The following Theorem allows determining a solution to Problem 1 (the reader can find its proof in [3]).

Theorem 1: The optimal control strategies solving Problem 1 can be obtained through the following four-steps procedure.

Step *a)* – Determine the two sets of coefficients $\{\lambda^1_{n_1,n_2},0\le n_1\le N_1-1,0\le n_2\le N_2\}$, and $\{\lambda^2_{n_1,n_2},0\le n_1\le N_1,0\le n_2\le N_2-1\}$, through the backward recursions:

– if $n_1\le N_1-1$ and $n_2\le N_2-1$:

$$\lambda^1_{n_1,n_2}=\begin{cases}\min\{dd_{1,n_1+1},\max\{\lambda^1_{n_1+1,n_2}-pt_1^{\mathrm{nom}},\\ \quad\lambda^2_{n_1+1,n_2}-pt_2^{\mathrm{nom}}\}\},\quad \text{if } n_1<N_1-1\\ \min\{dd_{1,n_1+1},\lambda^2_{n_1+1,n_2}-pt_2^{\mathrm{nom}}\},\\ \quad\text{if } n_1=N_1-1\end{cases} \tag{6}$$

$$\lambda^2_{n_1,n_2}=\begin{cases}\min\{dd_{2,n_2+1},\max\{\lambda^1_{n_1,n_2+1}-pt_1^{\mathrm{nom}},\\ \quad\lambda^2_{n_1,n_2+1}-pt_2^{\mathrm{nom}}\}\},\quad \text{if } n_2<N_2-1\\ \min\{dd_{2,n_2+1},\lambda^1_{n_1,n_2+1}-pt_k^{\mathrm{nom}}\},\\ \quad\text{if } n_2=N_2-1\end{cases} \tag{7}$$

– if $n_1=N_1$ and $n_2<N_2-1$:

$$\lambda^2_{N_1,n_2}=\min\{dd_{2,n_2+1},\lambda^2_{N_1,n_2+1}-pt_2^{\mathrm{nom}}\} \tag{8}$$

– if $n_2=N_2$ and $n_1<N_1-1$:

$$\lambda^1_{n_1,N_2}=\min\{dd_{1,n_1+1},\lambda^1_{n_1+1,N_2}-pt_1^{\mathrm{nom}}\} \tag{9}$$

with initial conditions

$$\lambda^1_{N_1-1,N_2}=dd_{1,N_1} \tag{10}$$

$$\lambda^2_{N_1,N_2-1}=dd_{2,N_2} \tag{11}$$

Step *b)* – Build the two sets of functions $\{\tau_j^1(n_1,n_2,t_j),0\le n_1\le N_1-1,0\le n_2\le N_2\}$, and $\{\tau_j^2(n_1,n_2,t_j),0\le n_1\le N_1,0\le n_2\le N_2-1\}$, as follows:

$$\tau_j^k(n_1,n_2,t_j)=\begin{cases} pt_k^{\mathrm{nom}},\quad t_j\le\lambda^k_{n_1,n_2}-pt_k^{\mathrm{nom}}\\ -t_j+\lambda^k_{n_1,n_2},\\ \quad\lambda^k_{n_1,n_2}-pt_k^{\mathrm{nom}}<t_j<\lambda^k_{n_1,n_2}-pt_k^{\mathrm{low}}\\ pt_k^{\mathrm{low}},\quad t_j\ge\lambda^k_{n_1,n_2}-pt_k^{\mathrm{low}}\end{cases}\qquad k=1,2 \tag{12}$$

where $\tau_j^k(n_1,n_2,t_j)$ indicates the optimal duration of the service time of the job activated in state $[n_1,n_2,t_j]^{\mathrm{T}}$, provided that a job of class k is selected, $k=1,2$.

Step *c)* – Determine, for each pair (n_1,n_2) such that $0\le n_1\le N_1$, $0\le n_2\le N_2$, $n_1+n_2\ne N$, the function $\mathcal{J}^{\circ}_{n_1,n_2}(t_j)$ representing the optimal cost-to-go as dependent on the current time instant, as follows:

– if $n_1\le N_1-1$ and $n_2\le N_2-1$:

$$\begin{aligned}\mathcal{J}^{\circ}_{n_1,n_2}(t_j)=\min\{&\mathcal{J}^{\circ}_{n_1,n_2}(t_j\mid\delta_{1,j}=1),\\ &\mathcal{J}^{\circ}_{n_1,n_2}(t_j\mid\delta_{2,j}=1)\}\end{aligned} \tag{13}$$

where the two "conditioned costs-to-go" inside the $\min$ operator in the r.h.s. of (13) are determined by means of the following backward recursive relation

$$\begin{aligned}&\mathcal{J}^{\circ}_{n_1,n_2}(t_j\mid\delta_{k,j}=1)=\\ &\quad=\alpha_{k,n_k+1}\cdot\max\{t_j+\tau_j^k(n_1,n_2,t_j)-dd_{k,n_k+1},0\}+\\ &\quad+\beta\cdot(pt_k^{\mathrm{nom}}-\tau_j^k(n_1,n_2,t_j))+\\ &\quad+\mathcal{J}^{\circ}_{n_1+\sigma_{1,k},n_2+\sigma_{2,k}}(t_j+\tau_j^k(n_1,n_2,t_j))\\ &k=1,2\end{aligned} \tag{14}$$

being $\sigma_{h,k} = 1$ when $k = h$, and 0 otherwise;
– if $n_1 = N_1$:

$$\begin{aligned}
&\mathcal{J}^{\circ}_{N_1,n_2}(t_j) = \\
&\quad = \alpha_{2,n_2+1} \cdot \max\left\{t_j + \tau^2_{Nj}(N_1, n_2, t_j) - dd_{2,n_2+1}, 0\right\} + \\
&\qquad + \beta \cdot \left(pt^{\text{nom}}_2 - \tau^2_j(N_1, n_2, t_j)\right) + \\
&\qquad + \mathcal{J}^{\circ}_{N_1,n_2+1}\left(t_j + \tau^2_j(N_1, n_2, t_j)\right)
\end{aligned} \tag{15}$$

– if $n_2 = N_2$:

$$\begin{aligned}
&\mathcal{J}^{\circ}_{n_1,N_2}(t_j) = \\
&\quad = \alpha_{1,n_1+1} \cdot \max\left\{t_j + \tau^1_j(n_1, N_2, t_j) - dd_{1,n_1+1}, 0\right\} + \\
&\qquad + \beta \cdot \left(pt^{\text{nom}}_1 - \tau^1_j(n_1, N_2, t_j)\right) + \\
&\qquad + \mathcal{J}^{\circ}_{n_1+1,N_2}\left(t_j + \tau^1_j(n_1, N_2, t_j)\right)
\end{aligned} \tag{16}$$

initialized by $\mathcal{J}^{\circ}_{N_1,N_2}(t_{N_1+N_2}) = 0$; note that in (14)÷(16) the expression of strategies (12) has to be used.

Step *d)* – The optimal control strategies are obtained as:
– if $n_1 \leq N_1 - 1$ and $n_2 \leq N_2 - 1$:

$$\delta^{\circ}_{1,j}(n_1, n_2, t_j) = \begin{cases} 1, \text{ if } \mathcal{J}^{\circ}_{n_1,n_2}\left(t_j \mid \delta_{1,j} = 1\right) \leq \\ \qquad \leq \mathcal{J}^{\circ}_{n_1,n_2}\left(t_j \mid \delta_{2,j} = 1\right) \\ 0, \text{ otherwise} \end{cases} \tag{17}$$

$$\delta^{\circ}_{2,j}(n_1, n_2, t_j) = 1 - \delta^{\circ}_{1,j}(n_1, n_2, t_j) \tag{18}$$

$$\begin{aligned} \tau^{\circ}_j(n_1, n_2, t_j) = \delta^{\circ}_{1,j}(n_1, n_2, t_j) \cdot \tau^1_j(n_1, n_2, t_j) + \\ + \delta^{\circ}_{2,j}(n_1, n_2, t_j) \cdot \tau^2_j(n_1, n_2, t_j) \end{aligned} \tag{19}$$

– if $n_1 = N_1$:

$$\delta^{\circ}_{1,j}(N_1, n_2, t_j) = 0 \tag{20}$$

$$\delta^{\circ}_{2,j}(N_1, n_2, t_j) = 1 \tag{21}$$

$$\tau^{\circ}_j(N_1, n_2, t_j) = \tau^2_j(N_1, n_2, t_j) \tag{22}$$

– if $n_2 = N_2$:

$$\delta^{\circ}_{1,j}(n_1, N_2, t_j) = 1 \tag{23}$$

$$\delta^{\circ}_{2,j}(n_1, N_2, t_j) = 0 \tag{24}$$

$$\tau^{\circ}_j(n_1, N_2, t_j) = \tau^1_j(n_1, N_2, t_j) \tag{25}$$

□

III. REACTING TO DUE-DATE CHANGES IN REAL-TIME

The optimal control strategies, which are determined off-line through the four-steps procedure in Theorem 1, are used in real-time to determine, at each decision time instant, the class of the next job to execute and the relative processing time. The optimal control strategies have been determined on the basis of the due-dates of each class of jobs, ordered according the EDD rule. Then, in case one or more due-dates change, the optimal control strategies may become invalid. It is here assumed that the changes of due-dates occur at certain time instants during the execution of jobs, that is, in real-time.

Let t^{c} be the current time instant, at which the machine is processing the $(p+q-1)$-th job and that the end of such job

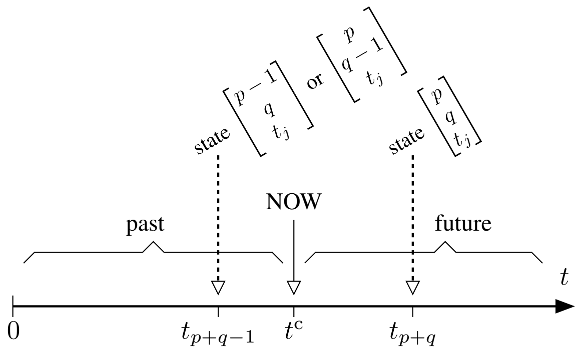

Fig. 1. Schematization of system state at the current time instant.

processing will lead the system to state $[p, q, t_j]^{\text{T}}$ (this means that, at t^{c}, the single machine is processing either the p-th job of class 1 or the q-th job of class 2). Fig. 1 schematizes this situation. Then, $[p, q, t_j]^{\text{T}}$ is the first state to be considered for possible modifications of the optimal control strategies, as it is assumed that job services are nonpreemptive and that processing times cannot be modified during service. Let us consider separately the case in which modifications in the due-dates of jobs that are still to be processed do not induce modifications in the EDD sequence, and the case in which such modifications take place.

A. *Change of one due-date without modifications in the EDD sequence*

Consider the case in which only one due-date changes. Let the due-date that changes be the i-th of class 1, and let $dd^{\star}_{1,i}$ (of course, $i > p$) be the new value of such due-date (obviously, all the following considerations can be easily modified in the case the due-date that changes is the i-th of class 2). Moreover, assume that

$$dd_{1,i-1} \leq dd^{\star}_{1,i} \leq dd_{1,i+1} \tag{26}$$

which means that the original EDD sequence for jobs of class 1 is not modified except for the value of the i-th due-date.

With reference to the backward procedure which provides the optimal control strategies, proposed in Theorem 1, the first time that the i-th due-date of class 1 is used is in state $[i-1, N_2, t_j]^{\text{T}}$. In fact, such a due-date is necessary to compute the coefficient λ^1_{i-1,N_2} at step *a)* of the procedure, being $\lambda^1_{i-1,N_2} = \min\{dd_{1,i}, \lambda^1_{i,N_2} - pt^{\text{nom}}_1\}$. If the value of such a coefficient changes, the optimal control strategy $\tau^{\circ}_j(i-1, N_2, t_j)$ changes as well. This modifications are propagated backward to the preceding states until the first state to be considered, namely $[p, q, t_j]^{\text{T}}$, is reached. Then, the part of the system state diagram which is interested by the change of due-date $dd_{1,i}$ is the "thick" part in Fig. 2.

The "interested part" of the system state diagram can be re-drawn as in Fig. 3. First of all, the number of nodes interested by the change of due-date $dd_{1,i}$ is $(i-p)\times(N_2-q+1)$. Moreover, it is worth noting that only strategies relevant to nodes belonging to column $i-1$ directly depend on the i-th due-date of class 1, since $\lambda^1_{i-1,N_2} = \min\{dd_{1,i}, \lambda^1_{i,N_2} - pt^{\text{nom}}_1\}$ and $\lambda^1_{i-1,n_2} = \min\{dd_{1,i}, \max\{\lambda^1_{i,n_2} - pt^{\text{nom}}_1, \lambda^2_{i,n_2} - pt^{\text{nom}}_2\}\}$ for any $q \leq n_2 < N_2$, in accordance with step *a)* of the

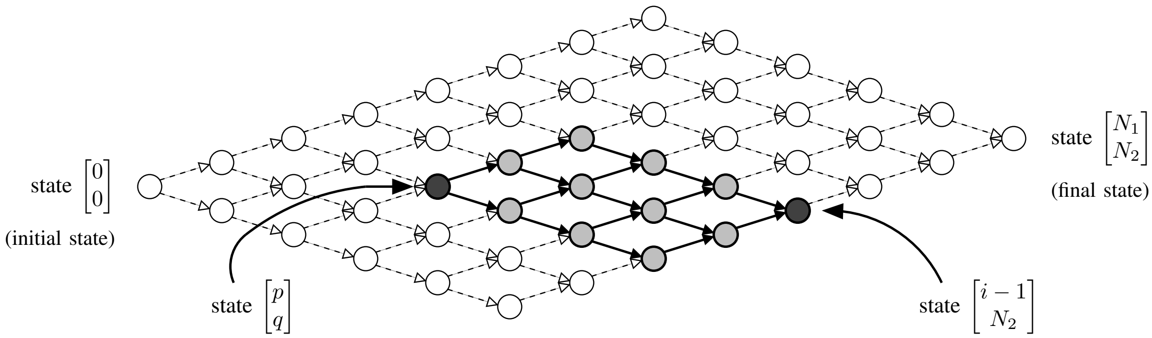

Fig. 2. The system state diagram (the first two state variables are indicated only) and the part of it interested by the change of due-date $dd_{1,i}$ ("thick" part of the diagram, between states $[p, q, t_j]^{\mathrm{T}}$ and $[i-1, N_2, t_j]^{\mathrm{T}}$).

procedure in Theorem 1. The strategies of all the other nodes depend on the i-th due-date of class 1 indirectly, through coefficients $\lambda^1_{n_1,n_2}$ and $\lambda^2_{n_1,n_2}$, with $p < n_1 \le i-1$ and $q \le n_2 < N_2$. In Fig. 3, states of column i do not belong to the part of the system state diagram that is interested by the change of due-date $dd_{1,i}$, but they are depicted because their coefficients and optimal costs-to-go are necessary to compute coefficients and optimal costs-to-go of states belonging to column $i-1$.

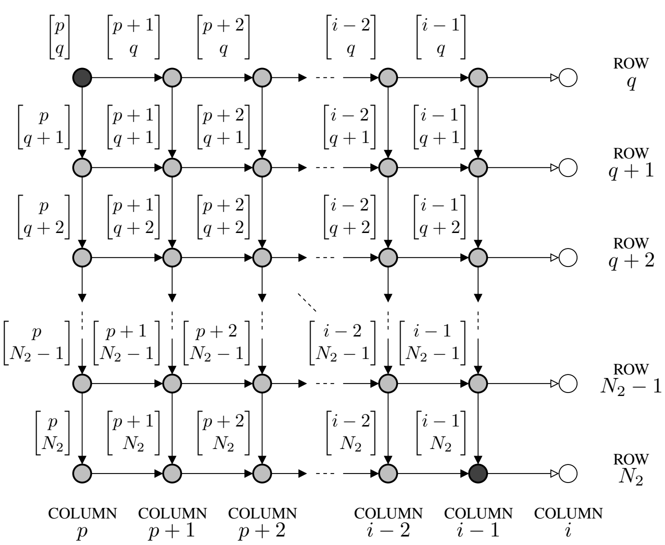

Fig. 3. Diagram of the part of the system state diagram that is interested by the change of due-date $dd_{1,i}$ (the first two state variables are indicated only).

The following theorem provides a sufficient condition about the validity of the optimal control strategies, after the change of the i-th due-date of class 1, for the generic state $[n_1, n_2, t_j]^{\mathrm{T}}$ (belonging to the part of the system state diagram that is interested by the due-date change).

Theorem 2: With regards to the change of the i-th due-date of class 1, the optimal control strategies in state $[n_1, n_2, t_j]^{\mathrm{T}}$, with $p \le n_1 \le i-1$ and $q \le n_2 \le N_2$, do not change if the following conditions hold

$$\min\{dd_{1,i}, dd^\star_{1,i}\} \ge \lambda^1_{i,N_2} - pt^{\mathrm{nom}}_1 \tag{27a}$$

$$\begin{aligned}&\min\{dd_{1,i}, dd^\star_{1,i}\} \ge \\ &\quad \ge \max\{\lambda^1_{i,\nu_2} - pt^{\mathrm{nom}}_1, \lambda^2_{i,\nu_2} - pt^{\mathrm{nom}}_2\} \\ &\quad \nu_2 = N_2 - 1, N_2 - 2, \ldots, n_2\end{aligned} \tag{27b}$$

being $dd_{1,i}$ and $dd^\star_{1,i}$, with $dd_{1,i-1} \le dd^\star_{1,i} \le dd_{1,i+1}$, respectively the "original" and the "new" due-dates. □

Proof: First of all, it is worth observing that, according to the four-steps algorithm included in Theorem 1, the optimal control strategies for the generic state $[n_1, n_2, t_j]^{\mathrm{T}}$ may be modified in consequence of a change of coefficients $\lambda^1_{n_1,n_2}$ and $\lambda^2_{n_1,n_2}$, which characterize the functions $\tau^1_j(n_1, n_2, t_j)$ and $\tau^2_j(n_1, n_2, t_j)$ and, in turn, the conditioned costs-to-go $\mathcal{J}^\circ_{n_1,n_2}(t_j \mid \delta_{1,j} = 1)$ and $\mathcal{J}^\circ_{n_1,n_2}(t_j \mid \delta_{2,j} = 1)$, or in consequence of a modification of the two optimal costs-to-go $\mathcal{J}^\circ_{n_1+1,n_2}(t_j)$ and $\mathcal{J}^\circ_{n_1,n_2+1}(t_j)$ relative to the two states that can be reached from $[n_1, n_2, t_j]^{\mathrm{T}}$, which characterize the conditioned costs-to-go. As a matter of fact, the optimal control strategies depend on functions $\tau^k_j(n_1, n_2, t_j)$, $k = 1, 2$, and on conditioned costs-to-go $\mathcal{J}^\circ_{n_1,n_2}(t_j \mid \delta_{k,j} = 1)$, $k = 1, 2$, as stated by (17), (18), and (19). It is also worth observing that if $\lambda^1_{n_1,n_2}$, $\lambda^2_{n_1,n_2}$, $\mathcal{J}^\circ_{n_1+1,n_2}(t_j)$, and $\mathcal{J}^\circ_{n_1,n_2+1}(t_j)$ do not change then the optimal cost-to-go $\mathcal{J}^\circ_{n_1,n_2}(t_j)$ in the considered generic state does not change as well. Obviously, when $n_1 = N_1$ (respectively, $n_2 = N_2$) the optimal control strategies and the optimal cost-to-go are not modified when only $\lambda^2_{N_1,n_2}$ (resp., $\lambda^1_{n_1,N_2}$) and $\mathcal{J}^\circ_{N_1,n_2+1}(t_j)$ (resp., $\mathcal{J}^\circ_{n_1+1,N_2}(t_j)$) do not change.

With reference to Fig. 3, consider state $[i-1, N_2, t_j]^{\mathrm{T}}$. If (27a) holds, then λ^1_{i-1,N_2} (which is the only coefficient to be computed in this state, since all jobs of class 2 have been completed) does not change; moreover, optimal cost-to-go $\mathcal{J}^\circ_{i,N_2}(t_j)$ is definitely unchanged as it belongs to a state which is not interested by the considered change of due-date; then, the optimal control strategies in state $[i-1, N_2, t_j]^{\mathrm{T}}$ do not change, as well as the optimal cost-to-go $\mathcal{J}^\circ_{i-1,N_2}(t_j)$. Consider now state $[i-2, N_2, t_j]^{\mathrm{T}}$; if λ^1_{i-1,N_2} does not change, then also λ^1_{i-2,N_2} is not modified; since λ^1_{i-2,N_2} and $\mathcal{J}^\circ_{i-1,N_2}(t_j)$ are unchanged, then the optimal control strategies in state $[i-2, N_2, t_j]^{\mathrm{T}}$ and the relevant cost-to-go $\mathcal{J}^\circ_{i-2,N_2}(t_j)$ are not modified. Proceeding backward along row N_2, it is easy to show that (27a) is sufficient to ensure that the optimal control strategies do not change in any state $[n_1, N_2, t_j]^{\mathrm{T}}$, $p \leq n_1 \leq i-1$, of this row.

Consider now row $N_2 - 1$ and, in particular, state $[i-1, N_2-1, t_j]^{\mathrm{T}}$. If (27b) holds when $\nu_2 = N_2 - 1$, then λ^1_{i-1,N_2-1} does not change; the optimal cost-to-go $\mathcal{J}^{\circ}_{i,N_2-1}(t_j)$ does not change as well, because it belongs to a state which is not interested by the considered change of due-date; moreover, it has been shown, in row N_2, that if (27a) holds then λ^1_{i-1,N_2} and $\mathcal{J}^{\circ}_{i-1,N_2}(t_j)$ do not change and, as a consequence, λ^2_{i-1,N_2-1} is not modified; this means that (27a) and (27b), the latter in the case $\nu_2 = N_2 - 1$, ensure that, in state $[i-1, N_2-1, t_j]^{\mathrm{T}}$, the optimal control strategies and the optimal cost-to-go $\mathcal{J}^{\circ}_{i-1,N_2-1}(t_j)$ do not change. Consider now state $[i-2, N_2-1, t_j]^{\mathrm{T}}$; if λ^1_{i-1,N_2-1} and λ^2_{i-1,N_2-1} do not change, as discussed, then also λ^1_{i-2,N_2-1} is not modified; moreover, it has been previously shown that (27a) guarantees that both λ^1_{i-2,N_2} and $\mathcal{J}^{\circ}_{i-2,N_2}(t_j)$ do not change, which means that if (27a) holds then λ^2_{i-2,N_2-1} is not modified; then, also in state $[i-2, N_2-1, t_j]^{\mathrm{T}}$, (27a) and (27b), the latter in the case $\nu_2 = N_2 - 1$, ensure that the optimal control strategies and the optimal cost-to-go $\mathcal{J}^{\circ}_{i-2,N_2-1}(t_j)$ do not change. Proceeding backward along row $N_2 - 1$, it is easy to show that (27a) and (27b), the latter in the case $\nu_2 = N_2 - 1$, are sufficient to ensure that the optimal control strategies do not change in any state $[n_1, N_2-1, t_j]^{\mathrm{T}}$, $p \leq n_1 \leq i-1$, of this row.

Consider now row $N_2 - 2$ and assume that (27b) holds when $\nu_2 = N_2 - 2$. Following the same reasoning line that has been previously adopted for rows N_2 and $N_2 - 1$, it is easy to show that (27a) and (27b), the latter in the two cases $\nu_2 = N_2 - 1$ and $\nu_2 = N_2 - 2$, ensure that the optimal control strategies for all the states belonging to row $N_2 - 2$, namely $[i-1, N_2-2, t_j]^{\mathrm{T}}$, $[i-2, N_2-2, t_j]^{\mathrm{T}}$, …, $[p, N_2-2, t_j]^{\mathrm{T}}$, do not change. Then, proceeding backward along rows, it turns out that conditions (27) are sufficient to ensure that the optimal control strategies do not change in any state of row n_2 and, in particular, in state $[n_1, n_2, t_j]^{\mathrm{T}}$. This concludes the proof. ■

The following theorem, whose proof is not reported (it is immediate, on the basis of Theorem 2), provides a sufficient condition about the validity of the whole set of the optimal control strategies determined in the off-line phase.

Theorem 3: Optimal control strategies are not modified in consequence of the change of the i-th due-date of class 1 if

$$\min\{dd_{1,i}, dd^{\star}_{1,i}\} \geq \lambda^1_{i,N_2} - pt^{\mathrm{nom}}_1 \tag{28a}$$

$$\begin{aligned} &\min\{dd_{1,i}, dd^{\star}_{1,i}\} \geq \\ &\qquad \geq \max\{\lambda^1_{i,\nu_2} - pt^{\mathrm{nom}}_1, \lambda^2_{i,\nu_2} - pt^{\mathrm{nom}}_2\} \\ &\quad \nu_2 = N_2 - 1, N_2 - 2, \ldots, q \end{aligned} \tag{28b}$$

being $dd_{1,i}$ and $dd^{\star}_{1,i}$, with $dd_{1,i-1} \leq dd^{\star}_{1,i} \leq dd_{1,i+1}$, respectively the "original" and the "new" due-dates, and q the number of jobs of class 2 which have been completed at the decision time instant at which the change of due-date is taken into account. □

It is worth noting that if conditions (28) do not hold, then the optimal control strategies may change or not. As a matter of fact, (28) are sufficient but not necessary conditions to ensure that optimal control strategies do not change. Necessary and sufficient conditions can be defined only for states belonging to row N_2. However, if conditions (28) do not hold, one can solve the following algorithm to determine the (possible) new optimal control strategies.

if $\min\{dd_{1,i}, dd^{\star}_{1,i}\} < \lambda^1_{i,N_2} - pt^{\mathrm{nom}}_1$ **then**
  GETSTRATEGIES($p, q, i-1, N_2$)
  Exit
end if
for $\nu_2 = N_2 - 1$ **down to** q **do**
  if $\min\{dd_{1,i}, dd^{\star}_{1,i}\} < \max\{\lambda^1_{i,\nu_2} - pt^{\mathrm{nom}}_1, \lambda^2_{\nu_2} - pt^{\mathrm{nom}}_2\}$ **then**
    GETSTRATEGIES($p, q, i-1, \nu_2$)
    Exit
  end if
end for

Remark 1: The previous algorithm could be redefined in order to avoid the execution of unnecessary operations, that is, the determination of coefficients λ and functions τ which do not change their value and structure. The result of such a redefinition would be an extremely more complicated algorithm. However, since all the optimal costs-to-go of the states from $[p, q, t_j]^{\mathrm{T}}$ to $[i-1, \nu_2, t_j]^{\mathrm{T}}$ change (being ν_2 the first row of the diagram in Fig. 3 for which either (28a) or (28b) is not satisfied), then the algorithm must always visit all the states from $[p, q, t_j]^{\mathrm{T}}$ to $[i-1, \nu_2, t_j]^{\mathrm{T}}$. Then, the execution of some simple operations, such as those relevant to the determination of λ and τ, does not increase the complexity of the algorithm, which depends on the number of states that the algorithm visits.

In the previous algorithm, the procedure GETSTRATEGIES($n^{\mathrm{o}}_1, n^{\mathrm{o}}_2, n^{\mathrm{d}}_1, n^{\mathrm{d}}_2$) calculates, on the basis of

- $\lambda^1_{\nu_1, n^{\mathrm{d}}_2+1}$, $\lambda^2_{\nu_1, n^{\mathrm{d}}_2+1}$, $\mathcal{J}^{\circ}_{\nu_1, n^{\mathrm{d}}_2+1}(t_j)$, for all $\nu_1 = n^{\mathrm{o}}_1, \ldots, n^{\mathrm{d}}_1$ (only if $n^{\mathrm{d}}_2 < N_2$),
- $\lambda^1_{n^{\mathrm{d}}_1+1, \nu_2}$, $\lambda^2_{n^{\mathrm{d}}_1+1, \nu_2}$, $\mathcal{J}^{\circ}_{n^{\mathrm{d}}_1+1, \nu_2}(t_j)$, for all $\nu_2 = n^{\mathrm{o}}_2, \ldots, n^{\mathrm{d}}_2$ (only if $n^{\mathrm{d}}_1 < N_1$),

the new optimal control strategies backwardly from state $[n^{\mathrm{d}}_1, n^{\mathrm{d}}_2, t_j]^{\mathrm{T}}$ down to state $[n^{\mathrm{o}}_1, n^{\mathrm{o}}_2, t_j]^{\mathrm{T}}$. Such a procedure (which is not reported due to the lack of space) executes the equations which characterize the four-steps procedure in Theorem 1, between two arbitrary states.

B. Change of one due-date with modifications in the EDD sequence

Consider now the case in which the change of one due-date causes a modification of the EDD sequence, as previously discussed. Let the due-date whose value changes be again the i-th of class 1 and let $dd^{\star}_{1,i}$ its new value (remember that, it must be $i > p$). Two cases are possible.

In the first case, it is assumed that

$$dd_{1,i-r} \leq dd^{\star}_{1,i} < dd_{1,i-r+1} \tag{29}$$

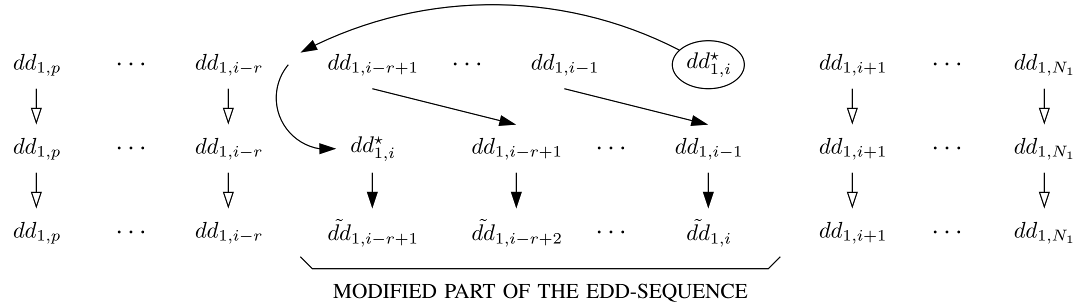

Fig. 4. Modification of EDD sequence when the new due-date is in accordance with (29).

if $p \geq 0$, with $2 \leq r \leq i-p$, or

$$dd^\star_{1,i} < dd_{1,1} \tag{30}$$

if $p = 0$. This is the case in which a job (the i-th) becomes urgent. A new EDD sequence is then necessary, in which r elements (when (29) holds), that is those from the $(i-r+1)$-th to the i-th, or i elements (when (30) holds), that is those from the first to the i-th, are different than before.

The situation which refers to assumption (29) is illustrated in Fig. 4, in which the new due-date $dd^\star_{1,i}$ is "put" between $dd_{1,i-r}$ and $dd_{1,i-r+1}$. Then, in the new EDD sequence, the $(i-r+1)$-th position of the sequence is assigned to $dd^\star_{1,i}$ whereas positions from the $(i-r+2)$-th to the i-th are assigned to $dd_{1,i-r+1}, \ldots, dd_{1,i-1}$, that is

$$\begin{array}{l} \tilde{d}d_{1,i-r+1} = dd^\star_{1,i} \quad \tilde{d}d_{1,i-r+2} = dd_{1,i-r+1} \quad \cdots \\ \cdots \quad \tilde{d}d_{1,i-1} = dd_{1,i-2} \quad \tilde{d}d_{1,i} = dd_{1,i-1} \end{array} \tag{31}$$

(the positions from the p-th to the $(i-r)$-th and from the $(i+1)$-th to the N_1-th do not change). It is worth noting that, owing to (29) and (31), it turns out $\tilde{d}d_{1,i-r+1} < dd_{1,i-r+1}$ and $\tilde{d}d_{1,\nu_1} \leq dd_{1,\nu_1}$, with $\nu_1 = i-r+2, \ldots, i$.

The situation which refers to assumption (30) can be handled analogously. In this case, in the new EDD sequence, it turns out

$$\begin{array}{l} \tilde{d}d_{1,1} = dd^\star_{1,i} \quad \tilde{d}d_{1,2} = dd_{1,1} \quad \cdots \\ \cdots \quad \tilde{d}d_{1,i-1} = dd_{1,i-2} \quad \tilde{d}d_{1,i} = dd_{1,i-1} \end{array} \tag{32}$$

(the positions from the $(i+1)$-th to the N_1-th do not change). Moreover, owing to (30) and (32), it turns out $\tilde{d}d_{1,1} < dd_{1,1}$ and $\tilde{d}d_{1,\nu_1} \leq dd_{1,\nu_1}$, with $\nu_1 = 2, \ldots, i$.

Consider the change of the i-th due-date of class 1, and assume $p > 0$ and that the new value $dd^\star_{1,i}$ satisfies (29), for some $r \in \{2, \ldots, i-p\}$. In this case, the part of the system state diagram that is interested by the change of the due-date is the same illustrated in Figs. 2 and 3. The difference with respect to the case discussed in Subsection III-A is that now it is necessary to consider r "new" due-dates (those from the $(i-r+1)$-th to the i-th), as discussed above (see (31)); as a consequence, the nodes of the diagram in Fig. 3 which directly depend on the new due-dates are those belonging to columns $i-r, i-r+1, \ldots, i-2, i-1$. This means that r conditions for the validity of the previously determined optimal control strategies must be considered for each row of the diagram in Fig. 3. Such conditions are stated in the following theorem (again, the proof is not reported).

Theorem 4: Consider the change of the i-th due-date of class 1, and assume $p > 0$ and that the new value $dd^\star_{1,i}$ satisfies (29), for some $r \in \{2, \ldots, i-p\}$; in this case, the optimal control strategies are not modified if

$$\tilde{d}d_{1,i-r+1} \geq \lambda^1_{i-r+1,N_2} - pt^{\text{nom}}_1 \tag{33a}$$

$$\begin{array}{l} \left(\tilde{d}d_{1,\nu_1} \geq \lambda^1_{\nu_1,N_2} - pt^{\text{nom}}_1 \ \wedge \ \tilde{d}d_{1,\nu_1} < dd_{1,\nu_1}\right) \vee \\ \quad \vee \ \tilde{d}d_{1,\nu_1} = dd_{1,\nu_1} \\ \quad \nu_1 = i-r+2, \ldots, i \end{array} \tag{33b}$$

$$\begin{array}{l} \tilde{d}d_{1,i-r+1} \geq \\ \quad \geq \max\left\{\lambda^1_{i-r+1,\nu_2} - pt^{\text{nom}}_1, \lambda^2_{i-r+1,\nu_2} - pt^{\text{nom}}_2\right\} \\ \quad \nu_2 = N_2-1, N_2-2, \ldots, q \end{array} \tag{33c}$$

$$\begin{array}{l} \left(\tilde{d}d_{1,\nu_1} \geq \max\left\{\lambda^1_{\nu_1,\nu_2} - pt^{\text{nom}}_1, \lambda^2_{\nu_1,\nu_2} - pt^{\text{nom}}_2\right\} \wedge \right. \\ \quad \left. \wedge \ \tilde{d}d_{1,\nu_1} < dd_{1,\nu_1}\right) \ \vee \ \tilde{d}d_{1,\nu_1} = dd_{1,\nu_1} \\ \quad \nu_1 = i-r+2, \ldots, i \\ \quad \nu_2 = N_2-1, N_2-2, \ldots, q \end{array} \tag{33d}$$

being dd_{1,ν_1} and $\tilde{d}d_{1,\nu_1}$, $\nu_1 = i-r+1, \ldots, i$, respectively the "original" and the "new" due-dates (from the $(i-r+1)$-th to the i-th), that is, the due-dates in the original EDD sequence and in the new one (the new due-dates are provided by (31)), and q the number of jobs of class 2 which have been completed at the decision time instant at which the change of the due-date is taken into account. □

It is worth noting that, in (33a)÷(33d) the term $\min\{dd_{1,\nu_1}, \tilde{d}d_{1,\nu_1}\}$, which was present in (28a) and (28b) (for the specific case $\nu_1 = i$), does not appear. Now, only the value $\tilde{d}d_{1,\nu_1}$ matters. The reason is that, in the case considered by Theorem 4, $\tilde{d}d_{1,i-r+1} < dd_{1,i-r+1}$ and $\tilde{d}d_{1,\nu_1} \leq dd_{1,\nu_1}$, with $\nu_1 = i-r+2, \ldots, i$, as discussed after (31). Then, it turns out $\min\{dd_{1,\nu_1}, \tilde{d}d_{1,\nu_1}\} = \tilde{d}d_{1,\nu_1}$ for all $\nu_1 = i-r+1, \ldots, i$.

If conditions (33) do not hold, one can solve the following algorithm to determine the (possible) new optimal control strategies.

```
for ν1 = i down to i − r + 1 do
    if d̃d_{1,ν1} < dd_{1,ν1} then
        if d̃d_{1,ν1} < λ^1_{ν1,N2} − pt^nom_1 then
            GetStrategies(p, N2, ν1 − 1, N2)
            GetStrategies(p, q, i − 1, N2 − 1)
```

Exit
end if
end if
end for
for $\nu_2 = N_2 - 1$ **down to** q **do**
  for $\nu_1 = i$ **down to** $i - r + 1$ **do**
    if $\tilde{d}d_{1,\nu_1} < dd_{1,\nu_1}$ **then**
      if $\tilde{d}d_{1,\nu_1} < \max\{\lambda^1_{\nu_1,\nu_2} - pt_1^{\text{nom}}, \lambda^2_{\nu_1,\nu_2} - pt_2^{\text{nom}}\}$ **then**
        GETSTRATEGIES$(p, \nu_2, \nu_1 - 1, \nu_2)$
        GETSTRATEGIES$(p, q, i - 1, \nu_2 - 1)$
        Exit
      end if
    end if
  end for
end for

The situation in which $p = 0$ and the new value $dd^\star_{1,i}$ satisfies (30) is very similar. In this case, it is necessary to consider the new due-dates from the first to the i-th, in accordance with (32).

The second case, in which it is assumed $dd_{1,i+s-1} < dd^\star_{1,i} \leq dd_{1,i+s}$, with $2 \leq s \leq N_1 - i$, or $dd^\star_{1,i} > dd_{1,N_1}$ (the i-th job is deferred), is not reported here due to the lack of space. In any case, it is easy to derive both sufficient conditions and the algorithm to determine the new strategies, by following a reasoning line that is strictly analogous to that followed with reference to the first case.

C. Change of due-dates belonging to jobs of different classes

Assume that the due-dates that change are the i-th of class 1 and the h-th of class 2. The new values of such due-dates are respectively $dd^\star_{1,i}$ and $dd^\star_{2,h}$. Moreover, assume that $dd_{1,i-1} \leq dd^\star_{1,i} \leq dd_{1,i+1}$ and $dd_{2,h-1} \leq dd^\star_{2,h} \leq dd_{2,h+1}$, which mean that the original EDD sequences (for both classes) are not modified except for the value of the i-th due-date in the sequence of class 1 and the value of the h-th due-date in the sequence of class 2. It is obvious that it must be $i > p$ and $h > q$, otherwise the considered problem is not meaningful.

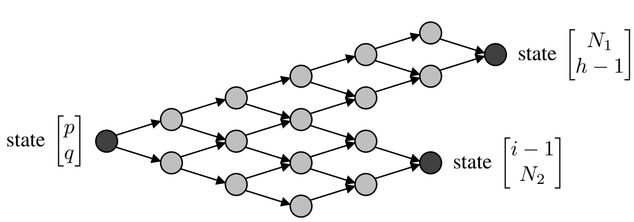

Fig. 5. Part of the system state diagram that is interested by the change of due-dates $dd_{1,i}$ and $dd_{2,h}$ (the first two state variables are indicated only).

In this case, the part of the system state diagram that is interested by the change of due-dates $dd_{1,i}$ and $dd_{2,h}$ is different than that illustrated in Fig. 2, as also the states between $[p, q, t_j]^{\mathrm{T}}$ and $[N_1, h-1, t_j]^{\mathrm{T}}$ are included (see Fig. 5). The following theorem provides a sufficient condition about the validity of the optimal control strategies determined in the off-line phase.

Theorem 5: Optimal control strategies are not modified in consequence of the change of the i-th due-date of class 1 and the change of the h-th due-date of class 2 if

$$\min\{dd_{1,i}, dd^\star_{1,i}\} \geq \lambda^1_{i,N_2} - pt_1^{\text{nom}} \tag{34a}$$

$$\begin{aligned} &\min\{dd_{1,i}, dd^\star_{1,i}\} \geq \\ &\quad \geq \max\{\lambda^1_{i,\nu_2} - pt_1^{\text{nom}}, \lambda^2_{i,\nu_2} - pt_2^{\text{nom}}\} \\ &\quad \nu_2 = N_2 - 1, N_2 - 2, \ldots, h \end{aligned} \tag{34b}$$

$$\min\{dd_{2,h}, dd^\star_{2,h}\} \geq \lambda^2_{N_1,h} - pt_2^{\text{nom}} \tag{34c}$$

$$\begin{aligned} &\min\{dd_{2,h}, dd^\star_{2,h}\} \geq \\ &\quad \max\{\lambda^1_{\nu_1,h} - pt_1^{\text{nom}}, \lambda^2_{\nu_1,h} - pt_2^{\text{nom}}\} \\ &\quad \nu_1 = N_1 - 1, N_1 - 2, \ldots, i \end{aligned} \tag{34d}$$

being $dd_{1,i}$ and $dd^\star_{1,i}$, with $dd_{1,i-1} \leq dd^\star_{1,i} \leq dd_{1,i+1}$, respectively the "original" and the "new" due-dates for the i-job of class 1, and $dd_{2,h}$ and $dd^\star_{2,h}$, with $dd_{2,h-1} \leq dd^\star_{2,h} \leq dd_{2,h+1}$, respectively the "original" and the "new" due-dates for the h-job of class 2. □

IV. CONCLUSIONS

The problem of updating, when necessary, optimal control strategies has been considered in this paper, in connection with the solution of a specific scheduling problem that has been recently proposed by the authors. In the paper, three significant cases have been considered; however, further cases (as the one in which two or more due-dates relative to the same class change) can be dealt with by following the same reasoning lines which have been followed to solve the considered ones. As introduced at the beginning, these results will be used in the IRP application of the adopted scheduling model.

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