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Experiments of Inverse Optimal Control Problem for Inverted Pendulum with Horizontal and Vertical Inputs

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***Abstract*— In this paper, inverse optimal control problem for an inverted pendulum with horizontal and vertical movements is considered. This inverted pendulum can be transformed into a bilinear system by using input transformation and coordinate transformation focused on the center of percussion of the pendulum. For bilinear systems, an optimal control algorithm which minimizes a new quadratic cost function was proposed. However, there are no experiments by using this controller. Thus this control algorithm may not be able to be applied to real examples. And so, in this paper, this control algorithm is applied to the bilinear system of the inverted pendulum. Furthermore, the effectiveness of this controller is shown by experiments.**

I. Introduction

An inverted pendulum has been a well-known example of nonlinear mechanical systems. Many control researchers has treated this system as benchmark for control algorithms.

The well-known inverted pendulum system is constructed by a cart and a pendulum. The cart move only horizontally, and the pendulum is attached to the cart by a rotational joint. The major control problem is that stabilization of an inverted pendulum can be realized only by moving the cart horizontally. However, when humans try to balance a rod on their palms intuitively, they probably move their hands not only in the horizontal direction but also in the vertical direction (Fig. 1) It can be seen that they probably move their hands in the horizontal direction when the rod is nearly at the upright position; in contrast, they probably move their hands in the vertical direction when the angle of the rod is large. From this viewpoint, there may be possibility to increase control performance of stabilizing the inverted pendulum by using redundant movement in the vertical direction. In addition, there is no doubt that humans properly use both the horizontal movement and the vertical movement based on their optimal performance indices. Hence, the objective of this paper is to derive an optimal controller for the inverted pendulum with horizontal and vertical movements.

In this paper, we regard the inverted pendulum with horizontal and vertical movements as a bilinear system. The inverted pendulum system can be transformed into a bilinear system by input transformation and coordinate transformation focused on the Center of Percussion (COP) of the pendulum. Optimal control problems for bilinear systems have been studied by numerous authors [2],[7]. However, no explicit expression for optimal feedback control of bilinear systems has been reported because the solution of the bilinear optimal control problem is characterized by Hamilton-Jacobi Bellman equation (HJBE) which has no analytical solution at present.

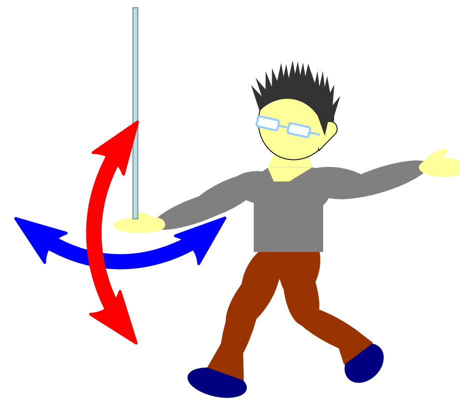


Fig. 1. image of the inverted pendulum with horizontal and vertical movements

And so, we proposed the optimal control algorithm for bilinear systems by using the inverse optimal design [8]. This algorithm minimizes a new quadratic cost function. However, in real world, the usability of this controller has not been confirmed because no one has made an experiment using this algorithm.

In this paper, we apply this optimal feedback controller to the system of the inverted pendulum. And we carry out experiments of this control system and confirm the effectiveness of this control algorithm.

This paper is organized as follows. In Section II, the model of the inverted pendulum with horizontal and vertical movement is introduced. It is also shown that the system of the pendulum can be transformed into the bilinear system. In Section III, the inverse optimal design for the bilinear system of the inverted pendulum is presented and the meanings of the cost function are discussed. In Section IV, in order to confirm the effectiveness of this control algorithm experiments of this system are carried out. Finally, Section V concludes this paper.

II. Mathematical Model

In this section, the mathematical model of the inverted pendulum with horizontal and vertical inputs is built. This pendulum can be consider as a cart pendulum with additional vertical force input. So, in this paper, we consider this pendulum system shown in Fig. 2. The horizontal and vertical force inputs effect to a base which connect a pendulum by a passive joint.

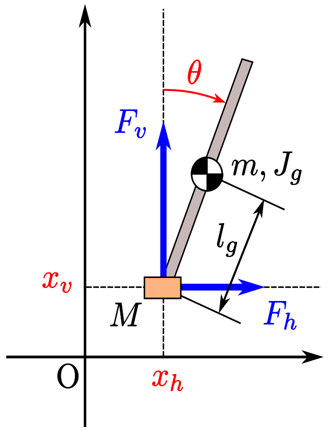


Fig. 2. model of the inverted pendulum with horizontal and vertical inputs

TABLE I

VARIABLES OF SIMPLIFIED MODEL

x_h	:	horizontal position of the base
x_v	:	vertical position of the base
θ	:	angular between vertical line and the pendulum
F_h	:	horizontal force input
F_v	:	vertical force input

Next, let us define notations of physical parameters. Let M be the mass of the base, m be the mass of the pendulum, l_g be the distance between the base and the center of gravity of the pendulum, J_g be the moment of inertia with respect to the center of gravity of the pendulum, and g be the acceleration of gravity.

In this notations, equations of motion can be written as

$$M(q)\ddot{q} + C(q,\dot{q}) + G(q) = F, \tag{1}$$

where

$$q = \begin{bmatrix} x_h & x_v & \theta \end{bmatrix}^T, \ F = \begin{bmatrix} F_h & F_v & 0 \end{bmatrix}^T,$$

$$M(q) = \begin{bmatrix} M+m & 0 & ml_g\cos\theta \\ 0 & M+m & -ml_g\sin\theta \\ ml_g\cos\theta & -ml_g\sin\theta & ml_g^2 + J_g \end{bmatrix},$$

$$C(q,\dot{q}) = \begin{bmatrix} -ml_g\dot{\theta}^2\sin\theta \\ -ml_g\dot{\theta}^2\cos\theta \\ 0 \end{bmatrix}, \ G(q) = \begin{bmatrix} 0 \\ (M+m)g \\ -mgl_g\sin\theta \end{bmatrix}.$$

These equations of motion can be transformed into a bilinear system by utilising coordinate transformation and input transformation. This transformation can be obtained by modifying of transformation proposed by Imura [4].By utilising next

Theorem 1: **Transformation into the bilinear system**

Let $\lambda = \frac{ml_g}{ml_g^2+J_g}$. Then, $\frac{1}{\lambda}$ means the distance between the base and the center of percussion (COP) of the pendulum. Suppose that θ is confined $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$. And let u_1 and u_2 be new inputs defined in Table II. By using coordinate transformation (2) and input transformation (3)

$$z = \begin{bmatrix} z_v \\ z_\theta \\ z_h \end{bmatrix} = \begin{bmatrix} x_v + \frac{1}{\lambda}\cos\theta - \frac{1}{\lambda} \\ \tan\theta \\ x_h + \frac{1}{\lambda}\cos\theta \end{bmatrix}, \tag{2}$$

$$\begin{aligned}\begin{bmatrix} F_h \\ F_v \end{bmatrix} &= \begin{bmatrix} \frac{m^2l_g^2\cos\theta\sin\theta}{ml_g^2+J_g}g - ml_g\dot{\theta}^2\sin\theta \\ (M+m)g - ml_g\dot{\theta}^2\cos\theta - \frac{m^2l_g^2\sin^2\theta}{m^2l_g^2+J_g}g \end{bmatrix} \\ &+ \begin{bmatrix} M+m-\frac{m^2l_g^2\cos^2\theta}{ml_g^2+J_g} & \frac{m^2l_g^2\sin\theta\cos\theta}{ml_g^2+J_g} \\ \frac{m^2l_g^2\sin\theta\cos\theta}{ml_g^2+J_g} & M+m-\frac{m^2l_g^2\sin^2\theta}{ml_g^2+J_g} \end{bmatrix} \\ &\left(\begin{bmatrix} \sin\theta & \cos\theta \\ \cos\theta & -\sin\theta \end{bmatrix}\begin{bmatrix} \frac{u_1+g}{\cos\theta} + \frac{1}{\lambda}\dot{\theta}^2 \\ -\frac{1}{\lambda}(u_2\cos^2\theta - 2\dot{\theta}^2\tan\theta) \end{bmatrix} + \begin{bmatrix} 0 \\ -g \end{bmatrix}\right),\end{aligned} \tag{3}$$

the equations of motion (1) is transformed into the bilinear system (4).

$$\begin{bmatrix} \ddot{z}_v \\ \ddot{z}_\theta \\ \ddot{z}_h \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ z_\theta u_1 + z_\theta g \end{bmatrix} \tag{4}$$

Proof : Consider new inputs $u_h = \ddot{x}_h, u_v = \ddot{x}_v$ and the input transformation given by

$$\begin{aligned}\begin{bmatrix} F_h \\ F_v \end{bmatrix} &= \begin{bmatrix} \frac{m^2l^2\cos\theta\sin\theta}{ml^2+J_g}g - ml\dot{\theta}^2\sin\theta \\ (M+m)g - ml\dot{\theta}^2\cos\theta - \frac{m^2l^2\sin^2\theta}{ml^2+J_g}g \end{bmatrix} \\ &+ \begin{bmatrix} M+m-\frac{m^2l^2\cos^2\theta}{(ml^2+J_g)} & \frac{m^2l^2\sin\theta\cos\theta}{ml^2+J_g} \\ \frac{m^2l^2\sin\theta\cos\theta}{ml^2+J_g} & M+m-\frac{m^2l^2\sin^2\theta}{ml^2+J_g} \end{bmatrix}\begin{bmatrix} u_h \\ u_v \end{bmatrix}.\end{aligned}$$

Then, we get the following system:

$$\begin{bmatrix} \ddot{x}_h \\ \ddot{x}_v \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} u_h \\ u_v \\ \lambda(g\sin\theta - u_h\cos\theta + u_v\sin\theta) \end{bmatrix}.$$

In addition, by using the coordinate transformation focused on the COP obtained by

$$\begin{bmatrix} z_h \\ z_v \\ \theta \end{bmatrix} = \begin{bmatrix} x_h + \frac{1}{\lambda}\sin\theta \\ x_v + \frac{1}{\lambda}\cos\theta - \frac{1}{\lambda} \\ \theta \end{bmatrix} \tag{5}$$

and the input transformation given by

$$\begin{bmatrix} u_h \\ u_v \end{bmatrix} = \begin{bmatrix} \sin\theta & \cos\theta \\ \cos\theta & -\sin\theta \end{bmatrix}\begin{bmatrix} a_1 + \frac{1}{\lambda}\dot{\theta}^2 \\ -\frac{1}{\lambda}a_2 \end{bmatrix} - \begin{bmatrix} 0 \\ g \end{bmatrix}, \tag{6}$$

we have

$$\begin{bmatrix} \ddot{z}_h \\ \ddot{z}_v \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} a_1\sin\theta \\ a_1\cos\theta - g \\ a_2 \end{bmatrix}. \tag{7}$$

Finally, by using the coordinate transformation given by

$$\begin{bmatrix} z_v \\ z_\theta \\ z_h \end{bmatrix} = \begin{bmatrix} z_v \\ \tan\theta \\ z_h \end{bmatrix} \tag{8}$$

and the input transformation given by

$$\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} \frac{g+u_1}{\cos\theta} \\ u_2\cos^2\theta - 2\dot{\theta}^2\tan\theta \end{bmatrix}, \tag{9}$$

we obtain

$$\begin{bmatrix} \ddot{z}_v \\ \ddot{z}_\theta \\ \ddot{z}_h \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 \\ z_\theta u_1 + z_\theta g \end{bmatrix}. \tag{10}$$

■

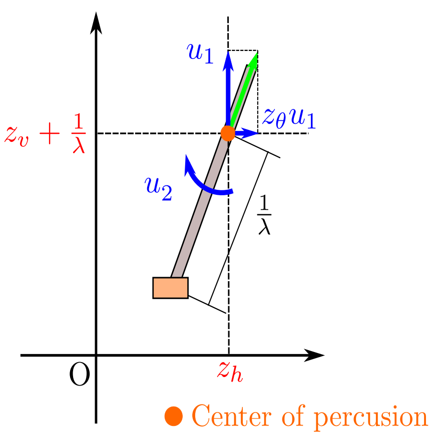


Fig. 3. bilinear model of the inverted pendulum

TABLE II

VARIABLES OF THE BILINEAR SYSTEM

z_h	:	horizontal position of the COP
z_v	:	vertical position of the COP with offset $(-1/\lambda)$
z_θ	:	tangent of θ
u_1	:	vertical acceleration input for the COP
u_2	:	rotational acceleration input around the COP

This bilinear system looks like Fig. 3, then z_h is the horizontal position of the COP and z_v is the vertical position of the COP with offset $-\frac{1}{\lambda}$. By this offset, when the original coordinate $[x_h, x_v, \theta]^T = 0$, the new coordinate $[z_v, z_\theta, z_h]^T = 0$. The bilinear system (4) can be written as state equations

$$\dot{x} = Ax + B(x)u, \tag{11}$$

where

$$x := \begin{bmatrix} z_v \\ \dot{z}_v \\ z_\theta \\ \dot{z}_\theta \\ z_h \\ \dot{z}_h \end{bmatrix}, A := \begin{bmatrix} A_{s1} & 0 \\ 0 & A_{s2} \end{bmatrix}, \; B(x) := \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \\ z_\theta & 0 \end{bmatrix},$$

$$u := \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, A_{s1} := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, A_{s2} := \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ g & 0 & 0 & 0 \end{bmatrix}.$$

Let us divide the system into the subsystem Σ_{s1} with respect to the vertical movement and the subsystem Σ_{s2} with respect to the horizontal and rotational movements. The subsystems are given by

$$\Sigma_{s1}: \; \dot{x}_{s1} = A_{s1}x_{s1} + b_{s1}u_1, \tag{12}$$

$$\Sigma_{s2}: \; \dot{x}_{s2} = A_{s2}x_{s2} + b_{s2}u_2 + N_{s2}(x_{s2})u_1, \tag{13}$$

where

$$x_{s1} := \begin{bmatrix} z_v \\ \dot{z}_v \end{bmatrix}, b_{s1} = \begin{bmatrix} 0 \\ 1 \end{bmatrix},$$

$$x_{s2} := \begin{bmatrix} z_\theta \\ \dot{z}_\theta \\ z_h \\ \dot{z}_h \end{bmatrix}, b_{s2} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, N_{s2}(x_{s2}) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ z_\theta \end{bmatrix}.$$

Now, we consider the effectiveness of the vertical acceleration input u_1 in Σ_{s2}.

- When $z_\theta \neq 0$, u_1 affects the state x_{s2}. So it is possible to control horizontal and rotational movements by utilising vertical acceleration input u_1.
- When $z_\theta = 0$, that is, the pendulum is at the upright position, any u_1 does not affect the state. So, it is impossible to control horizontal and rotational movements by using u_1.

As $z_\theta = 0$, the bilinear system (4) is equivalent to the linear approximation system of (4). Thus vertical input u_1 can not be used effectively by the optimal control for the linear approximation system. In the next section, we will propose the new optimal control. This optimal control is constructed for the bilinear system.

III. INVERSE OPTIMAL CONTROL FOR THE INVERTED PENDULUM

A. Inverse Optimal Control Problem for Bilinear System of the Inverted Pendulum

In this section, we consider the nonlinear optimal control for the bilinear system of the inverted pendulum with horizontal and vertical inputs. The present evidence of the nonlinear optimal feedback control is that the optimal controller is based on the solution of the Hamiltom-Jacobi-Bellman equation (HJBE). Nevertheless the HJBE for this system cannot be solved analytically. Therefore we consider the inverse optimal control design for the bilinear system of this pendulum.

Theorem 2: **Inverse optimal design for the pendulum with horizontal and vertical inputs**

Let P_{s1} be the solution of the following algebraic Riccati equation for the subsystem Σ_{s1}

$$A_{s1}^T P_{s1} + P_{s1}A_{s1} - P_{s1}b_{s1}r_{11}^{-1}b_{s1}^T P_{s1} + Q_{s1} = 0, \tag{14}$$

and P_{s2} be the solution of the following algebraic Riccati equation for the linear approximation system of subsystem Σ_{s2}

$$A_{s2}^T P_{s2} + P_{s2}A_{s2} - P_{s2}b_{s2}r_{22}^{-1}b_{s2}^T P_{s2} + Q_{s2} = 0, \tag{15}$$

where $Q_{s1}, Q_{s2}, r_{11}, r_{22}$ are weighting factors for the states and inputs. For the bilinear system (4), the control input

$$u^* = -R^{-1}B(x)^T Px \tag{16}$$

makes the following cost function minimized.

$$J = \int_0^\infty q(x) + u^T R u dt, \tag{17}$$

where

$$\begin{aligned}
q(x) &:= \begin{bmatrix} x_{s1} \\ x_{s2} \\ \mu_1(x) \end{bmatrix}^T Q \begin{bmatrix} x_{s1} \\ x_{s2} \\ \mu_1(x) \end{bmatrix}, \\
Q &:= \begin{bmatrix} Q_{s1} & 0 & P_{s1}b_{s1}r_{11}^{-1} \\ 0 & Q_{s2} & 0 \\ (P_{s1}b_{s1}r_{11}^{-1})^T & 0 & r_{11}^{-1} \end{bmatrix}, \\
R &:= \begin{bmatrix} r_{11} & 0 \\ 0 & r_{22} \end{bmatrix}, \quad P := \begin{bmatrix} P_{s1} & 0 \\ 0 & P_{s2} \end{bmatrix} \\
\mu_1(x) &:= N_{s2}(x_{s2})^T P_{s2} x_{s2}.
\end{aligned}$$

The proof of this theorem is based on the proof of Theorem 2 in [8]

Proof : First, we show that the control input and the cost function are satisfies with the HJBE.

Consider the nonlinear affine system

$$\dot{x} = f(x) + g(x)u, \tag{18}$$

where $f(0) = 0$. The nonlinear regulation problem for (18) is defined by the following cost function

$$J = \int_0^\infty \alpha(x) + u^T R(x) u \ \mathrm{dt} \tag{19}$$

where $\alpha(x) \geq 0, R(x) > 0$ for all x and the associated HJBE

$$\left(\frac{\partial V}{\partial x}\right)^T f - \frac{1}{4}\left(\frac{\partial V}{\partial x}\right)^T g(x)R(x)^{-1}g(x)^T \frac{\partial V}{\partial x} + \alpha(x) = 0. \tag{20}$$

Then the optimal feedback controller can be obtained as following u^* from the solution $V(x)$ of (20)

$$u^* = -\frac{1}{2}R(x)^{-1}g(x)^T \frac{\partial V}{\partial x} \tag{21}$$

Now, we consider this theorem applied to the bilinear system (4). Define

$$f(x) := Ax, g(x) := B(x), \alpha(x) := q(x), R(x) := R. \tag{22}$$

The HJBE is written by

$$\left(\frac{\partial V}{\partial x}\right)^T Ax - \frac{1}{4}\left(\frac{\partial V}{\partial x}\right)^T B(x)R^{-1}B(x)^T \frac{\partial V}{\partial x} + q(x) = 0. \tag{23}$$

We will prove the optimality of the cost function (17) by showing that there exists the solution $V(x)$ which satisfies (23). By using the solution P, let us define

$$V(x) := x^T P x. \tag{24}$$

Then, we have

$$\begin{aligned}
&\left(\frac{\partial V}{\partial x}\right)^T Ax - \frac{1}{4}\left(\frac{\partial V}{\partial x}\right)^T B(x)R^{-1}B(x)^T \frac{\partial V}{\partial x} + q(x) \\
=&x^T(A^T P + PA - PB(x)R^{-1}B(x)^T P)x \\
&+ \begin{bmatrix} x_{s1} \\ x_{s2} \\ \mu_1(x) \end{bmatrix}^T \begin{bmatrix} Q_{s1} & 0 & P_{s1}b_{s1}r_{11}^{-1} \\ 0 & Q_{s2} & 0 \\ (P_{s1}b_{s1}r_{11}^{-1})^T & 0 & r_{11}^{-1} \end{bmatrix} \begin{bmatrix} x_{s1} \\ x_{s2} \\ \mu_1(x) \end{bmatrix} \\
=&x^T(A^T P + PA - PB(x)R^{-1}B(x)^T P)x \\
&- x_{s1}^T(A_{s1}^T P_{s1} + P_{s1}A_{s1} - P_{s1}b_{s1}r_{11}^{-1}b_{s1}^T P_{s1})x_{s1} \\
&- x_{s2}^T(A_{s2}^T P_{s2} + P_{s2}A_{s2} - P_{s2}b_{s2}r_{22}^{-1}b_{s2}^T P_{s2})x_{s2} \\
&+ 2x_{s2}^T P_{s2} N_{s2}(x_{s2}) r_{11}^{-1} b_{s1}^T P_{s1} x_{s1} \\
&+ x_{s2}^T P_{s2} N_{s2}(x_{s2}) r_{11}^{-1} N_{s2}(x_{s2})^T P_{s2} x_{s2} \\
=&0.
\end{aligned}$$

Therefore, the optimality of the cost function J given by (17) is proved.

Next, we have to prove the origin of the closed-loop system is asymptotically stable. The closed-loop system is

$$\begin{aligned}
\dot{x} &= Ax + B(x)u \\
&= (A - B(x)R^{-1}B(x)^T P)x.
\end{aligned} \tag{25}$$

The linear approximation system of this closed-loop system is

$$\dot{x} = (A - B(0)R^{-1}B(0)^T P)x. \tag{26}$$

The origin of (26) is exponentially stable. Therefore, the origin of the closed-loop system (25) is asymptotically stable. Because the origin of the nonlinear system is exponentially stable if the origin of the linear approximation system is exponentially stable[9]. The proof is completed. ■

In this theorem, the solutions P_{s1}, P_{s2} of the Riccati equations for the linear approximation system of the bilinear system is used. If this goes on, the controller does not have the optimality for the bilinear system. So the nonlinearity is revived in (16). For this nonlinear controller, the quadratic cost function for linear system is changed into (17).

Remark 1 : Since the proof of the optimality of this controller is based on the satisfaction of the HJBE, the optimality is satisfied only when the solution $x^*(t)$ of (25) stays in the following domain:

$$X_c := \{x^*(t) | q(x^*(t)) \geq 0\}. \tag{27}$$

B. The Meanings of the cost function J

In this section, the meanings of the cost function J given by (17) is discussed. Fig. 4 shows the image of the meanings of the cost function J. Because $\mu_1(x)$ have the second order term with respect to z_θ, we have the following approximations.

- $||x|| \gg ||\mu(x)|| \cong 0$ (in the neighborhood of $z_\theta = 0$)
- $||x|| \ll ||\mu(x)||$ (in the area apart from $z_\theta = 0$)

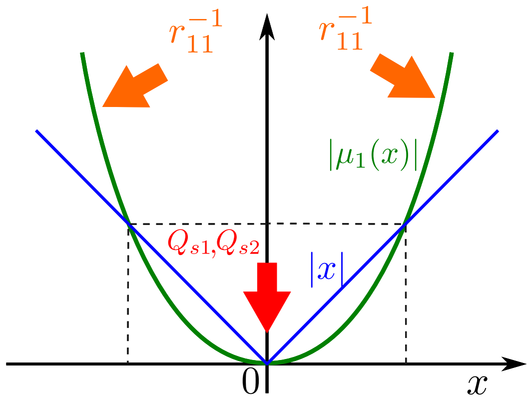


Fig. 4. The cost function J mainly depend on the quadratic term $x_{s1}^T Q_{s1} x_{s1} + x_{s2}^T Q_{s2} x_{s2}$ which means a cost function of a linear approximation system, when $|x|$ is small. While the fourth-order term $\mu_1(x)^T r_{11}^{-1} \mu_1(x)$ dominate the cost function J when $|x|$ is large.

Therefore, when x stays in the neighborhood of the $z_\theta = 0$, we have

$$\begin{bmatrix} x_{s1} \\ x_{s2} \\ \mu_1(x) \end{bmatrix}^T \begin{bmatrix} Q_{s1} & 0 & P_{s1} b_{s1} r_{11}^{-1} \\ 0 & Q_{s2} & 0 \\ (P_{s1} b_{s1} r_{11}^{-1})^T & 0 & r_{11}^{-1} \end{bmatrix} \begin{bmatrix} x_{s1} \\ x_{s2} \\ \mu_1(x) \end{bmatrix} \cong \begin{bmatrix} x_{s1} \\ x_{s2} \end{bmatrix}^T \begin{bmatrix} Q_{s1} & 0 \\ 0 & Q_{s2} \end{bmatrix} \begin{bmatrix} x_{s1} \\ x_{s2} \end{bmatrix}. \quad (28)$$

This means the second-order term with respect to x is dominant in the neighborhood of z_θ. It is clear that this is the natural expansion of the case of the linear system.

On the other hand, when x stays in the area apart from $z_\theta = 0$, we have

$$\begin{bmatrix} x_{s1} \\ x_{s2} \\ \mu_1(x) \end{bmatrix}^T \begin{bmatrix} Q_{s1} & 0 & P_{s1} b_{s1} r_{11}^{-1} \\ 0 & Q_{s2} & 0 \\ (P_{s1} b_{s1} r_{11}^{-1})^T & 0 & r_{11}^{-1} \end{bmatrix} \begin{bmatrix} x_{s1} \\ x_{s2} \\ \mu_1(x) \end{bmatrix} \cong \mu_1(x)^T r_{11}^{-1} \mu_1(x). \quad (29)$$

This means the fourth-order term is dominant in the area apart from the origin and the weighting factor for $\mu_1(x)$ is r_{11}^{-1}. It is clear that the smaller r_{11} induces the smaller $\|\mu_1(x)\|$ in the area apart from $z_\theta = 0$. This means $\|\mu_1(x)\|$ is smaller when the vertical acceleration input $\|u_1\|$ is larger.

IV. Experiments

In this section, experiments has been carried out for the inverted pendulum with horizontal and vertical inputs. A equipment of the experiments is shown in Fig. 5.

This equipment is constructed by 2 links manipulator with a pendulum. It have two rotation motors which can apply torque inputs. The mathematical model of this equipment is different from Fig. 2. However x_h, x_v in Fig. 2 is written by

$$x_h = l_1 \cos(\phi_1 - \phi_{1o}) + l_2 \cos(\phi_1 + \phi_2 - \phi_{1o} - \phi_{2o})$$
$$x_v = l_1 \sin(\phi_1 - \phi_{1o}) + l_2 \sin(\phi_1 + \phi_2 - \phi_{1o} - \phi_{2o})$$

where l_1, l_2 are the length of link 1 and link 2 respectively, ϕ_1 is an absolute angle of link 1, ϕ_2 is a relative angle between

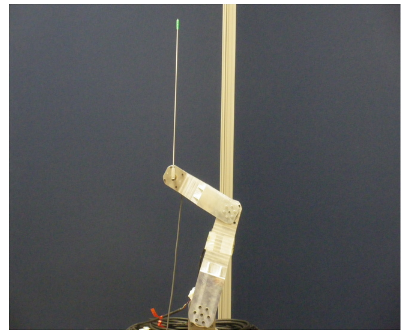


Fig. 5. equipment of the experiments

link 1 and link 2, and $(x_h, x_v) = (0, 0)$ when $(\phi_1, \phi_2) = (\phi_{1o}, \phi_{2o})$.

So we can transform the torque inputs into the acceleration inputs u_h, u_v in (5). Thus the mathematical model of this equipment can be transformed into the same bilinear system (4).

However, the domain of the horizontal and vertical movements is limited because length of links is fixed and rotational motors are used. The domain is shown in Fig. 6. So we have to choose initial values and weighting matrices which the trajectory of the case is contained in the domain.

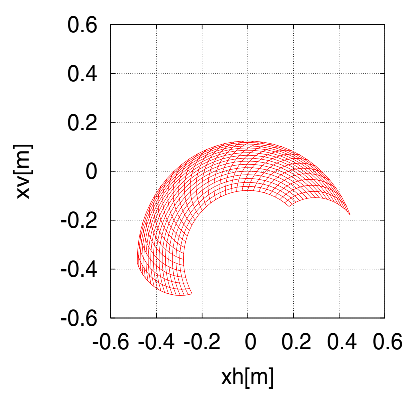


Fig. 6. domain of the movements

Let the weighting matrices Q_{s1}, Q_{s2}, R be diagonal.

$$Q_{s1} = \mathrm{diag}(q_{11}, q_{22})$$
$$Q_{s2} = \mathrm{diag}(q_{33}, q_{44}, q_{55}, q_{66})$$
$$R = \mathrm{diag}(r_{11}, r_{22})$$

The experiments has been carried out using initial values shown in Table III. Designing parameters are shown in Table IV. The results of the experiments are shown in Fig. 7

We will confirm whether the weighting matrices of the cost function (17) acts as intended. First, the trajectories of the position of COP are compared in Fig. 7(a) between Param 1 and Param 2. The vertical movement of Param 2 is larger than that of Param 1. Thus the effectiveness that r_{11} , the weight parameter for the vertical acceleration input , of Param 2 is smaller than that of Param 2 is shown.

TABLE III
INITIAL VALUES

x_H	-0.11 [m]	z_H	-0.00444 [m]
$\dot{x}_H$	0.0 [m/s]	$\dot{z}_H$	-0.0370 [m/s]
θ	10.1 [deg]	z_θ	0.178 []
$\dot{\theta}$	0.736 [rad/s]	$\dot{z}_\theta$	0.759 [1/s]
x_V	0.0 [m]	z_V	-0.0597 [m]
$\dot{x}_V$	0.0 [m/s]	$\dot{z}_V$	0.208 [m/s]

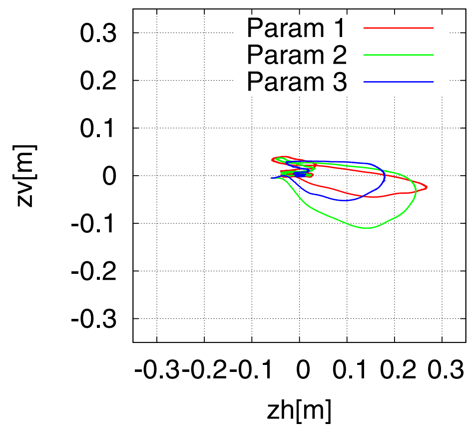
TABLE IV
WEIGHT MATRICES FOR EXPERIMENTS OF THE BILINEAR SYSTEM

	q_{11}	q_{22}	q_{33}	q_{44}	q_{55}	q_{66}	r_{11}	r_{22}
Param 1	1	1	1	1	10	1	0.5	0.05
Param 2	1	1	1	1	10	1	0.1	0.05
Param 3	1	1	1	1	10	1	0.1	0.01
Param 3'	5	5	5	5	50	5	0.5	0.05

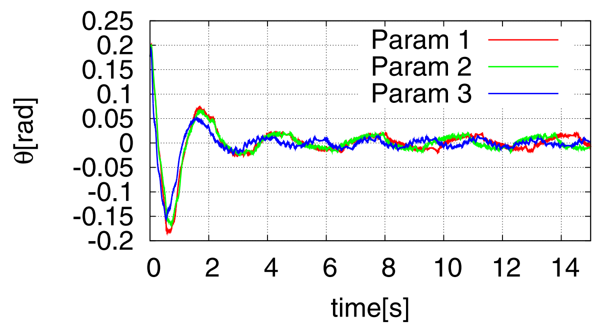
Second the trajectories of θ are compared in Fig. 7(b) between Param 2 and Param 3. The peeks of the trajectory of Param 2 are smaller than those of Param 3. Thus the effectiveness that r_{22} , the weight parameter for the rotational acceleration input, of Param 2 is smaller than that of Param 2 is shown.

Finally, the trajectories of the position of COP are compared in Fig. 7(a) between Param 1 and Param 3. The weighting factors of Param 3 can be multiplied by a scalar. So the same movement can be obtained by Param 3'. Thus it is considered that the weighting factors for state variables of Param 3 is larger than those of Param 1. Hence the trajectory of Param 3 is supposed to be closer to the origin than that of Param 1. This is confirmed in Fig. 7(a).

Hence, the effectiveness of the weighting matrices is confirmed.



(a) center of percusion



(b) angle θ

Fig. 7. Experimental Results utilizing Inverse Optimal Control for the Bilinear System

V. CONCLUSION

The optimal control algorithm which minimizes the new quadratic cost function[8] is very useful for many bilinear systems, such as the system of the semi-active suspension for automobiles[6]. However, this algorithm has not been applied to a real system.

In this paper, we dealt with the inverted pendulum with horizontal and vertical inputs as one of the example of bilinear systems. The algorithm was applied to this pendulum system and experiments were carried out. In the experiments, it was shown that the smaller the weighting factor r_{11} was, the larger the vertical movement was, and that the smaller the weighting factor r_{22} was, the smaller the peaks of the angle θ was. Thus the optimality of the cost function was confirmed. Hence, we showed the effectiveness of this optimal control algorithm in reality.

The results of this paper are important because they make a possibility that this optimal controller is applied to other bilinear systems which exist in reality.

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